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Connected Edge Geodetic Domination Number Of A Graph

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Abstract: *In this paper the concept of connected edge geodetic domination number of a graph is introduced. A set of vertices S of a graph G is a connected edge geodetic domination set (CEGD set) if it is edge geodetic set, a domination set of G and the induced sub graph <S> is connected. The connected edge geodetic domination number (CEGD number) of G,* $\gamma_{g_{ce}}$ *(G) is the cardinality of a minimum CEGD set. CEGD number of some connected graphs is realized. Connected graphs of order p with CEGD number p are characterised. It is shown that for every pair of integers m and n such that* $3 \le m \le n$, *there exist a connected graph G of order n with* $\gamma_{g_{ce}}$ *(G) = m. Also, for any positive integers p, q and r there is a connected graph G such that g(G) = p,* $g_e(G) = q$ *and* $\gamma_{g_{ce}}$ *(G)= r. Again, for any connected graph G,* $\gamma_{g_{ce}}$ *(G) lies between* $\frac{p}{1+\Delta(G)}$ *and p.*

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I. Introduction

By a graph $G = (V, E)$ we consider a finite undirected graph without loops or multiple edges. The order and size of a graph are denoted by p and q respectively. For the basic graph theoretic notations and terminology we refer to Buckley and Harary [3]. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A geodetic set of G is a set $S \subseteq V(G)$ such that vertex of G is contained in a geodesic joining some pair of vertices in S. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets.

The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an extreme vertex if the sub graph induced by its neighbourhood is complete. A vertex v in a connected graph G is a cut vertex of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if $\Delta(<N(v)>) = |N(v)| - 1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. An acyclic connected graph is called a tree [3].A vertex v in a connected graph G is a cut-vertex if $G - v$ is disconnected.

A dominating set in a graph G is a subset of vertices of G such that every vertex outside the subset has neighbour in it. The size of a minimum dominating set in a graph G is called the domination number of G and is denoted by $\gamma(G)$. A geodetic domination set of G is a subset of $V(G)$ which is both geodetic and dominating set of G. The minimum cardinality of a geodetic domination set is denoted by $\gamma_{g_e}(G)$. A detailed study of geodetic domination set is available in [7]. An edge geodetic set of G is a subset $S \subset V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_e(G)$ of G is the minimum order of its edge geodetic sets. Edge geodetic set of a connected graph is studied in [9].
A set of vertices of G is said to be edge geodetic domination set or EGD set if it is both edge geodetic set and a domination set of G. The minimum cardinality among all the EGD sets of G is called edge geodetic domination number and is denoted by $\gamma_{g_e}(G)$. If G is a connected graph of order $p \ge 3$ and G contains exactly one universal vertex, then $g_e(G) = p - 1$.

II. Connected Edge Geodetic Domination Number Of A Graph

2.1 Definition: A set S of vertices of a graph G is connected edge geodetic domination set (abbreviated as CEGD set) if it is (i) an edge geodetic set of G (ii) a domination set of G and (iii) the induced sub graph of S ,<S> is connected. The minimum cardinality among all the CEGD set of G is called CEGD number and is denoted by $\gamma_{g_{ce}}(G)$.

2.2 Example: Consider the graph given in Figure 01.
Here $S_1 = \{v_1, v_3, v_4\}$ is an edge geodetic domination set and $S_2 = \{v_1,v_2,v_3,v_4,v_6\}$ is a minimum CEGD set. Therefore $\gamma_{g_{ce}}(G) = 5$.

2.3 Theorem: Let G be a connected graph. Then $2 \leq \gamma_{g_e}(G) \leq \gamma_{g_{ce}}(G) \leq p$.

Proof: Since any geodetic set contains at least two vertices, $2 \leq \gamma_{g_e}(G)$. Again, every CEGD set is an edge geodetic domination set, $\gamma_{g_e}(G) \leq \gamma_{g_{ce}}(G)$. Since the set of all vertices of G is always a CEGD set, $\gamma_{g_{ce}}(G) \leq p$.

2.4 Remark: The bounds in Theorem 2.3 are sharp. In the example given in figure 01, $p = 6$, $\gamma_{g_e}(G) = 3$, $\gamma_{g_{ce}}(G) = 5$.

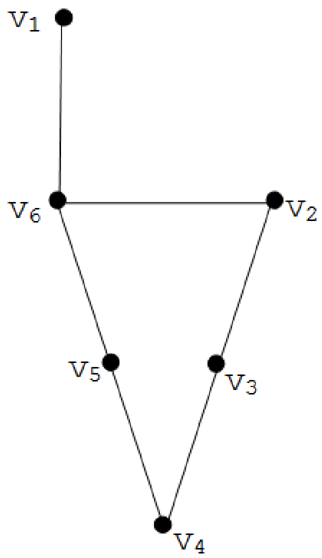

Figure:01

2.5 Theorem: For any connected graph of order p, $2 \leq g_{ce}(G) \leq \gamma_{g_{ce}}(G) \leq p$.

Proof: Every connected edge geodetic set has at least two vertices. Also, every CEGD set is a connected edge geodetic set.

2.6 Theorem: Each extreme vertex belongs to every CEGD set.

Proof: Since each extreme vertex belongs to every geodetic set (see Theorem 2.3 [10]), these extreme vertices also belongs to every CEGD set.

2.6.1 Corollary: Each end vertices of a connected graph G belongs to every CEGD set. This is due to end vertices are also extreme vertices.

2.7 Theorem: Each semi-extreme vertex belongs to every CEGD set.

Proof: Each semi-extreme vertex belongs to every connected edge geodetic set of G. Also every CEGD set is connected edge geodetic set, the result follows.

2.8 Theorem: For complete graph K_p, $\gamma_{g_{ce}}(G) = p$.

Proof: In a complete graph G every vertex is an extreme vertex and results follow from Theorem 2.6.

2.9 Theorem: For complete bipartite graph $K_{m,n}$,

$$\gamma_{g_{ce}}(G) = \begin{cases} 2, & if \quad m = n = 1 \\ n, & if \ n \geq 2, m = 1 \\ \min\{m, n\} + 1, & if \ m, n \geq 2 \end{cases}$$

Proof: Case (i) is trivial. Here the graph is K_2. Case (ii): Here the graph is a tree. Every vertex is either an extreme vertex or a cut-vertex. For case (iii), take $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ be a partition of G. Assume $m \leq n$. Consider $S_1 = X$. Then S_1 is a minimum edge geodetic set (By Theorem 2.11 of [1]). But its induced sub graph is not connected. Take $S_2 = S_1 \cup \{y_i\}$ for $1 \leq i \leq n$. Then S_2 is a minimum CEGD set. Therefore $\gamma_{g_{ce}}(G) = |S_2| = \min\{m,n\} + 1$.

2.10 Theorem: For cycle graph C_p, $\gamma_{g_{ce}}(C_p) = p - 2$, for $p \geq 5$.

Proof: Take any consecutive $p - 2$ vertices in C_p. These vertices dominate C_p. Also that set is a connected edge geodetic set. Therefore $\gamma_{g_{ce}}(G) \leq 2$. Now if the vertices are not consecutive vertices not dominate C_p. Thus $\gamma_{g_{ce}}(G) \geq p - 2$.

2.11 Theorem: Each cut-vertex of a connected graph belongs to every CEGD set of G.

Proof: By Theorem 2.7 of [9], each cut-vertex of a connected graph G belongs to every minimum connected edge geodetic set of G. Since every CEGD set is an edge geodetic set, the result follows.

2.12 Theorem: For any non-trivial tree T of order p, $\gamma_{g_{ce}}(G) = p$.

Proof: Since every vertex of T is either a cut-vertex or an end vertex, the result follows from Theorem 2.11 and Corollary 2.6.1.

III. Realisation Results

3.1 Theorem: Let G be a connected graph. Then every vertex of G is either a cut-vertex of G if and only if $\gamma_{g_{ce}}(G) = p$.

Proof: Since every cut-vertex and semi extreme-vertex belongs to every CEGD set, the necessary part is true. Conversely, let $\gamma_{g_{ce}}(G) = p$. Suppose there exists a vertex v in G which is neither a cut-vertex nor a semi-extreme vertex. Since v is not semi-extreme, the neighbourhood of v , $N(v)$ does not induce a complete sub graph so that there exist two vertices x and y in $N(v)$ such that d $(x, y) = 2$. That is v lies on a x–y geodetic path in G. Since v is not a cut-vertex of G, $G– v$ is connected. Thus $V(G) – \{v\}$ is a connected edge geodetic set of G. Since every p– 1 vertices dominate $V(G)$, these vertices form a CEGD set which is a contradiction to the fact that $\gamma_{g_{ce}}(G) = p$. Hence v is either a cut-vertex or semi-extreme vertex.

3.2Theorem: For every pair m, n of integers with $3 \le m \le n$, there exist a connected graph G of order n such that $\gamma_{g_{ce}}(G) = m$.

Proof: Let P_m: $v_1,v_2,\dots,v_m$ be a path of m vertices. Take $n – m$ new vertices $x_1,x_2,\dots,x_{n–m}$ and join each $x_i(1 \le i \le n–m)$ with v_1 and v_3, we get the connected graph G(see the figure 02). Its order is $(n– m) + m = n$. Now let $S_1 = \{v_3,v_4,\dots,v_{m–1}\}$. All these vertices are cut vertices of G and belong to minimum CEGD set. Thus $\gamma_{g_{ce}}(G) \ge m–3$. Neither $S_1\cup \{x_i\}$ nor $S_1\cup \{v_2\}$ are EGD set and$S_1\cup \{v_1\}$ is not connected. Thus $S = S_1\cup \{v_1,v_2,v_m\}$ is a CEGD set and is minimum. Now $|S| = m– 3 + 3 = m$ as desired.

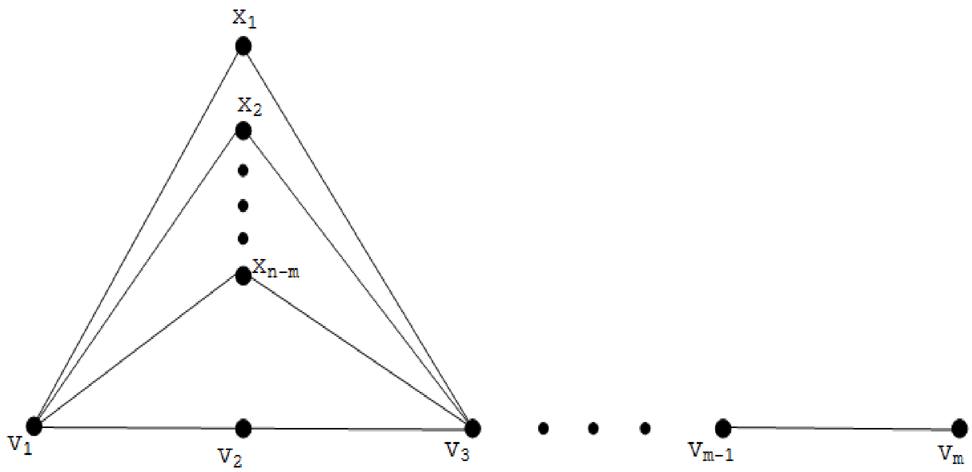

Figure: 02

3.3 Theorem: Let G be a connected graph of order p. Then $\frac{p}{1+\Delta(G)} \le \gamma_{g_{ce}}(G) \le p$, where, $\Delta(G)$ is the minimum degree of G.

Proof: First, for the connected graph G, the set of all vertices is a CEGD set. Therefore $\gamma_{g_{ce}}(G) \le p$. Suppose $\gamma_{g_{ce}}(G) = k$. Take $S = \{v_1,v_2,\dots,v_k\}$ be a minimum CEGD set. Next, any vertex u_i dominate all p vertices of G. Therefore $p \le \sum_{i=1}^{k} 1 + \deg v_i$. Now deg $v_i \le \Delta(G)$. Therefore 1 + deg $v_i \le 1 + \Delta(G)$. That is $p \le \sum_{i=1}^{k} 1 + \deg v_i \le \sum_{i=1}^{k} 1+\Delta(G) \le k(1 + \Delta(G))$. Therefore $k \ge \frac{p}{1+\Delta(G)}$ .Thus $\gamma_{g_{ce}}(G) \ge \frac{p}{1+\Delta(G)}$.

3.4 Theorem: For any positive integers a,b,c with $a \le b \le c$ there exist a connected graph G such that $g(G) = a$, $g_e(G) = b$ and $\gamma_{g_{ce}}(G) = c$.

Proof: Let G be the graph given in Figure 03, having a path $u_1,u_2,u_3,\dots,u_{c–b+2}$ and by adding b–2 vertices $v_1,v_2,\dots v_{b-a},w_1,w_2,\dots,w_{a–2}$ with this path, and join each v_i with u_1,u_2 and u_3 and join each w_i with u_2.Then $S_1 = \{u_1,w_1,w_2,\dots,w_{a–2},u_{c–b+2}\}$ is a minimum geodetic set so that $g(G)$ = a. Now $S_2= S_1\cup \{v_1,v_2,\dots,v_{b-a}\}$ is a minimum edge geodetic set. Thus $g_e(G) = |S_2| = b$. Take $S_3 = S_2\cup \{u_2,u_3,\dots,u_{c–b+1}\}$. Clearly S_3 is a CEGD set and is the minimum. Hence $\gamma_{g_{ce}}(G) = |S_3| = b + (c– b) =c$.

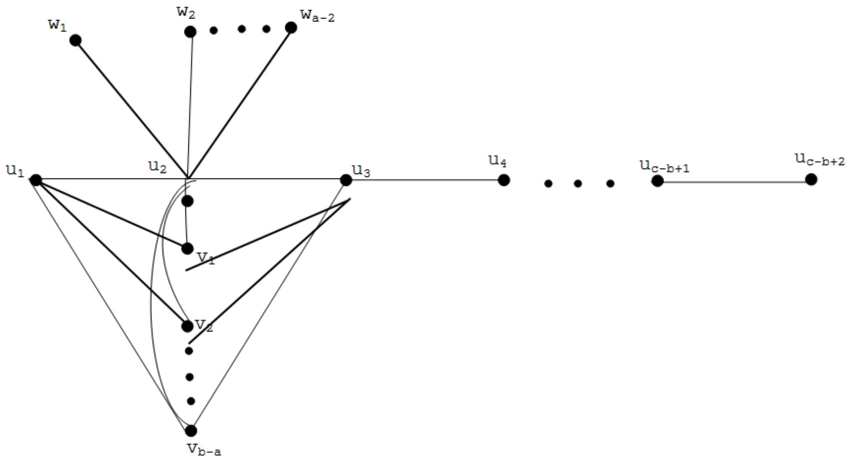

Figure : 03

IV. conclusion

We can extent connected edge geodetic number to find upper connected EGD set, forcing connected EGD set and CEGD number of join of graphs, CEGD number of composition of graphs and CEGD hull number of graphs and so on. It has so many applications in security of buildings and communication networks.

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