Numerical Technique to Increase the Accuracy of theADI Method in two Dimensional Equations

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Abstract: *we search for a new formula to increase the accuracy of the Alternative Direction Iteration method for solving Partial Differential Equation by using the forth order finite difference method in time and space which will be still consist and unconditional stable.*
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I. Introduction

Alternating Direction Implicit (ADI) method was suggestedfirst by Peaceman and Rachford in the mid-50s of the last century for solving systems of algebraic equations in two dimension, which results from the finite difference discretization method for solving PDEs [1]. From iterative method's perspective, the ADI method may be considered as a special relaxation method, where a big system is simplified into a number of smaller sub-systems, such that each of them may be solved efficiently and the solution of the whole system is got from the solutions of the sub-systems in an iterative method [2]. In classical (ADI) method, a second order (centered with time and backward with space) finite difference formulation usually used to obtain a scheme, the formulation obtained in this way will be unconditionally stable with a consistent condition which will be a sufficient for a method convergent**.**

II. Implementation of The ADI Method

Consider two dimensional diffusion equations[3]:
$$u_t = u_{xx} + u_{yy} \quad t \in (0,\infty] \quad , (x,y) \in D \qquad \ldots \ldots \ldots (1)$$

where$D = (0,l_1) \times (0,l_2)$, with initial condition:
$$u(x,y,0) = u_0(x,y) \quad , (x,y) \in D$$
and the boundary conditions are at along ∂D .

The finite difference formula:
$$u_t = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} \qquad \ldots\ldots\ldots (2)$$

$$u_{xx} = \frac{-u_{i+2}^n + 16u_{i+1}^n - 30u_i^n + 16u_{i-1}^n - u_{i-2}^n}{(\Delta x)^2} \ldots\ldots\ldots (3)$$

$$u_{yy} = \frac{-u_{j+2}^n + 16u_{j+1}^n - 30u_j^n + 16u_{j-1}^n - u_{j-2}^n}{(\Delta y)^2} \ldots\ldots\ldots (4)$$

This two formulation for u_{xx} and u_{yy} was the centeral differences with $O(x)^4$ and $O(y)^4$.
Replacing both sides in diffusion equation by their central difference approximation, to get:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{-u_{i+2,j}^n + 16u_{i+1,j}^n - 30u_{i,j}^n + 16u_{i-1,j}^n - u_{i-2,j}^n}{(\Delta x)^2} + \frac{-u_{i,j+2}^n + 16u_{i,j+1}^n - 30u_{i,j}^n + 16u_{i,j-1}^n - u_{i,j-2}^n}{(\Delta y)^2} \qquad \ldots\ldots\ldots (5)$$

Then, after considering$\Delta x = \Delta y = h$, and simplifying equation (5), it will become:

$$u_{i,j}^{n+1} - u_{i,j}^n = \frac{\Delta t}{h^2}[(-u_{i+2,j}^n + 16u_{i+1,j}^n - 30u_{i,j}^n + 16u_{i-1,j}^n - u_{i-2,j}^n)$$
$$+ (-u_{i,j+2}^n + 16u_{i,j+1}^n - 30u_{i,j}^n + 16u_{i,j-1}^n - u_{i,j-2}^n)]\ldots\ldots\ldots (6)$$

In the ADI method formulation, equation (6) will be rearranged by taking two time steps as follows:

$$u_{i,j}^{n+1} - u_{i,j}^n = r\left[(-u_{i+2,j}^n + 16u_{i+1,j}^n - 30u_{i,j}^n + 16u_{i-1,j}^n - u_{i-2,j}^n) \right.$$
$$\left. +(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} - 30u_{i,j}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1})\right] \qquad\qquad \text{……... (7)}$$

$$u_{i,j}^{n+2} - u_{i,j}^{n+1} = r\left[-u_{i+2,j}^{n+2} + 16u_{i+1,j}^{n+2} - 30u_{i,j}^{n+2} + 16u_{i-1,j}^{n+2} - u_{i-2,j}^{n+2}) \right.$$
$$\left. +(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} - 30u_{i,j}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1})\right] \qquad\qquad \text{…… ... (8)}$$

Simplify equations (7)and(8)with some rearrangement:

$$u_{i,j}^{n+1} - 30r\, u_{i,j}^{n+1} = u_{i,j}^n - 30r\, u_{i,j}^n + r(-u_{i+2,j}^n + 16u_{i+1,j}^n + 16u_{i-1,j}^n - u_{i-2,j}^n)$$
$$+r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \dots\dots\dots (9)$$

$$u_{i,j}^{n+2} - 30r\, u_{i,j}^{n+2} = u_{i,j}^{n+1} - 30r\, u_{i,j}^{n+1} + r(-u_{i+2,j}^{n+2} + 16u_{i+1,j}^{n+2} + 16u_{i-1,j}^{n+2} - u_{i-2,j}^{n+2})$$
$$+r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \dots\dots\dots (10)$$

$$(1 - 30r)u_{i,j}^{n+1} = (1 - 30r)u_{i,j}^n + r(-u_{i+2,j}^n + 16u_{i+1,j}^n + 16u_{i-1,j}^n - u_{i-2,j}^n)$$
$$+r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \dots\dots\dots (11)$$

$$(1 - 30r)u_{i,j}^{n+2} = (1 - 30r)u_{i,j}^{n+1} + r(-u_{i+2,j}^{n+2} + 16u_{i+1,j}^{n+2} + 16u_{i-1,j}^{n+2} - u_{i-2,j}^{n+2})$$
$$+r(-u_{i,j+2}^{n+1} + 16u_{i,j+1}^{n+1} + 16u_{i,j-1}^{n+1} - u_{i,j-2}^{n+1}) \dots\dots\dots (12)$$

For i, j = 1, 2,...,n equations (11) and (12) will used alternatively and the resulting systems will be tridiagonal that may be solved easily.

III. Conclusion
In this paper, we have obtained a new formula of the (ADI) method depending on central forth order finite difference formula which will be a forth order of time and space for more accuracy .

References
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