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From the Modular Properties of (a,b)-Fibonacci Sequences to those of Generalized Lucas Sequences

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Abstract: *It is shown that period, rank and order of the (a,b)-Fibonacci sequence remain invariant by extension to the generalized (a,b,c,d)-Lucas sequence provided a specific determinant is relatively prime to the modulus. Given an odd prime modulus p and the $p-1$ exceptional (a,b)-Lucas sequences, for which p divides the discriminant of these sequences, we determine the corresponding periods and relate them to those of the (a,b)-Fibonacci sequence. The results can be viewed as extended versions of classical results by Wall in case $(a,b)=(1,1)$.*

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I. Introduction

The topic of Fibonacci numbers and their generalizations is an interesting and important one with a wide range of applications (e.g. Dunlap [1], Koshy [2], Stakhov [3]). The classical Fibonacci sequence $F=(F_n)$ is defined by the linear recurrence of order two $F_0=0, F_1=1, F_n=F_{n-1}+F_{n-2}, n\ge 2$. The related classical Lucas sequence $L=(L_n)$ follows the same recursion $L_n=L_{n-1}+L_{n-2}, n\ge 2$, with the different initial values $L_0=2, L_1=1$. These sequences are special cases of generalized sequences already studied by Lucas. Consider the (a,b)-*Fibonacci sequence* $F=(F_n)$ defined for non-zero integers a,b, by the linear recurrence

$$F_0=0, \quad F_1=1, \quad F_n=aF_{n-1}+bF_{n-2}, n\ge 2, \tag{1.1}$$

and the (a,b)-*Lucas sequence* $L=(L_n)$ defined by

$$L_0=2, \quad L_1=a, \quad L_n=aL_{n-1}+bL_{n-2}, n\ge 2. \tag{1.2}$$

These two integer sequences are better known under the parameterization $a=p, b=-q$, for which a simultaneous study has been undertaken by Cerda-Morales [4] and Jeffery and Pereira [5]. The latter authors have called them *generalized Lucas sequences* of the first and second order. They contain some important sequences. The sequence (1.1) generates the classical Fibonacci, Jacobsthal, Pell and Mersenne numbers for $(a,b)\in\{(1,1),(1,2),(2,1),(3,-2)\}$, while (2.2) generates the classical Lucas and Jacobsthal-Lucas numbers for $(a,b)\in\{(1,1),(1,2)\}$. In all these examples the discriminant $\Delta=a^2+4b$ of (1.1) and (1.2) is non-zero. Without further mention, it will be assumed that this holds true. It is also remarkable that these sequences contain the special cases $(q,a)=(1,a)$ of the generalized Fibonacci (q,a)-sequences and the generalized Lucas (q,a)-sequences studied in the book by Stakhov [3]. They are obtained setting $b=1$ in the above and have applications in computer science (Fibonacci measurement algorithms, Fibonacci computers, cryptography, etc.). On the other hand, it is also well-known that Lucas sequences play an important role in primality testing (e.g. Riesel [6], Chap. 4, Bressoud [7], Chap. 12, Pomerance [8]).

One notes that the integer sequences (1.1) and (1.2) can be embedded into the *generalized* (a,b,c,d)-*Lucas sequence* $G=(G_n)$ defined for non-zero integers a,b,c,d, by the general linear recurrence of order two with arbitrary initial values

$$G_0=c, \quad G_1=d, \quad G_n=aG_{n-1}+bG_{n-2}, n\ge 2. \tag{1.3}$$

For a general exposé of the generalized Lucas sequences the reader is refereed to Kalman and Mena [9]. In the classical case $(a,b)=(1,1)$ it is known that the period length of (1.3) is often independent of the initial values (c,d) and depend solely on the period length of (1.1). In generalization to this, we ask whether and how the modular properties of the (a,b) -Fibonacci sequence (1.1) established in Renault [10] extend to the (a,b) -Lucas sequence (1.2) and more generally to the generalized (a,b,c,d) -Lucas sequence (1.3). The content is organized as follows.

Section 2 recalls the notions and some main properties of period, rank, multiplier and order of linear recurrences as they are required to study the modular properties of the generalized (a,b,c,d) -Lucas sequence. Moreover, to make the content more self-contained, a brief summary of some known main modular properties is included. Section 3 is the core of the new contribution. Theorem 3.1 shows that period, rank and order of the (a,b) -Fibonacci sequence remain invariant by extension to the generalized (a,b,c,d) -Lucas sequence provided a well-defined determinant is relatively prime to the modulus. Theorem 3.2 is an extended version of Theorem 8 in Wall [11] and refines Theorem 3.1 for the special case $b=1$. In Section 4, given an odd prime modulus p and the $p-1$ exceptional (a,b)-Lucas sequences, for which p divides the discriminant of these sequences, Theorem 4.1 determines the corresponding periods and relate them to those of the (a,b) -Fibonacci sequence. Numerous examples illustrate the obtained results and relate them to earlier findings of Wall and Renault.

For simplicity, the dependence upon the parameters will be omitted in the sequence notations $F=(F_n)$, $L=(L_n)$ and $G=(G_n)$ because this will be clear from the context. Moreover, when reducing one of these sequences modulo m, it will always be assumed that m is chosen such that $\gcd(b,m)=1$ and $\gcd(c,d,m)=1$. With this convention, the linear recurrences (1.1)-(1.3) are uniquely determined for both positive and negative values of $n=0,\pm1,\pm2,...$.

II. Preliminaries

The representation used in [11] for the classical case $(a,b)=(1,1)$ is a crucial step of the analysis.

Lemma 2.1. The generalized (a,b,c,d) -Lucas sequence $G=(G_n)$ satisfies the representation $G_n = dF_n + bcF_{n-1}$, where $F=(F_n)$ is the (a,b) -Fibonacci sequence.

Proof. This is shown by induction on n. With $bF_{-1}=1, F_0=0, F_1=1$ the initial step is satisfied because $G_0 = dF_0 + bcF_{-1} = c$, $G_1 = dF_1 + bcF_0 = d$. Now, assuming the formula holds for indices less than n and using the recursions (1.1) and (1.3) one easily sees that $G_{n+1} = aG_n + bG_{n-1} = dF_n + bcF_{n-1}$, which is the stated formula for the index $n+1$. ◊

First of all, one observes that modulo m any pair of residues completely determines $G \pmod m$, and since there are only finitely many of them, all of the sequences (1.1)-(1.3) are periodic. The *period* of $G \pmod m$, sometimes called Pisano number, is denoted by $\pi_G(m)$. In the special cases (1.1) and (1.2) it is denoted by $\pi_F(m)$ respectively $\pi_L(m)$. The period satisfies the trivial divisibility property $\pi_G(m) \mid \pi_F(m)$. This follows from Lemma 2.1, which shows that G repeats after $n=\pi_F(m)$ terms, which implies that $\pi_G(m)$ divides $\pi_F(m)$.

The rank of apparition, or simply *rank* of $F \pmod m$, is the least positive integer r such that mod m one has $F_r \equiv 0$. Sometimes, the rank is also called restricted period and it is denoted by $\alpha_F(m)$. The number s such that $F_{\alpha_F(m)+1} \equiv s \pmod m$ is called *multiplier* of $F \pmod m$ and denoted by $\mu_F(m)=s$. Note that the terms of F starting with index $r=\alpha_F(m)$ are exactly the initial terms of F multiplied by the factor s, i.e. one has $F_{r+k} \equiv s \cdot F_k, k \ge 0$. Moreover, since $F_{r-1} \equiv b^{-1}s$ one sees with Lemma 2.1 that $L_{r+k} \equiv s \cdot L_k, k \ge 0$, and $G_{r+k} \equiv s \cdot G_k, k \ge 0$. In particular, it follows that there must exist least positive integers r_L, r_G such that the terms of L, G starting with these indices are exactly the initial terms of L, G multiplied by some factors s_L, s_G, i.e. one has congruences $L_{r_L+k} \equiv s_L \cdot L_k, k \ge 0$, and $G_{r_G+k} \equiv s_G \cdot G_k, k \ge 0$. It is natural to call r_L, r_G the *ranks* of L, G and denote them by $r_L = \alpha_L(m), r_G = \alpha_G(m)$. In particular, the discussion shows that one must have the divisibility properties $\alpha_L(m) \mid \alpha_F(m)$ and $\alpha_G(m) \mid \alpha_F(m)$. The numbers s_L, s_G are called multipliers of $L \pmod m$, $G \pmod m$ and denoted by $\mu_L(m), \mu_G(m)$.

Furthermore, if X denotes any of the sequences F, L, G, the *order* of $X \pmod m$ is defined and denoted by $\omega_X(m) = \pi_X(m)/\alpha_X(m)$. If $X = F$ its name stems from the fact that it represents the multiplicative order of the multiplier modulo m, that is one has $\omega_F(m) = ord_F(\mu_F(m))$ (e.g. [10], p. 374).

As a general property, given the unique decomposition of the modulus into prime power factors $m = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, the calculation of the period and rank depend solely on the period and rank of the prime power factors. This fact follows from the theory of integer sequences $S = (S_n)$ satisfying a linear recurrence of arbitrary order of the form $S_n = a_1 S_{n-1} + \dots + a_n S_{n-k}, n \ge k$. Indeed, one knows that $\pi_S(m)$ respectively $\alpha_S(m)$ is equal to the least common multiple of the $\pi_S(p^e)$'s respectively $\alpha_S(p^e)$'s (e.g. Knuth [12], Exercise 3.2.2.11, Cull et al. [13], p 220). In the following, the bracket $[m_1,\dots,m_k]$ denotes the least common multiple of the numbers $m_1,\dots,m_k$.

Theorem 2.1. Let X denote any of the sequences F, L, G. If $m = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ then

$$\alpha_X(m) = [\alpha_X(p_1^{e_1}), \alpha_X(p_2^{e_2}), \dots, \alpha_X(p_k^{e_k})], \quad \pi_X(m) = [\pi_X(p_1^{e_1}), \pi_X(p_2^{e_2}), \dots, \pi_X(p_k^{e_k})]. \qquad (2.1)$$

Corollary 2.1. If $m_1 \mid m_2$ then $\alpha_X(m_1) \mid \alpha_X(m_2)$ and $\pi_X(m_1) \mid \pi_X(m_2)$.

Reduction formulas for prime powers are also very useful. This means that for a prime p, the calculation of $\pi_X(p^e), \alpha_X(p^e)$ can often be expressed in terms of $\pi_X(p), \alpha_X(p)$. For example, if $X = F$ is the (a,b)-Fibonacci sequence (1.1) then the following formulas hold (e.g. [10], Theorem 2).

Theorem 2.2. Given is an integer $e \ge 1$. Three cases are distinguished:

Case 1: The prime p is odd

(i) $\alpha_F(p^e) = p^{e-e'} \alpha_F(p)$, with $1 \le e' \le e$ maximal such that $\alpha_F(p^{e'}) = \alpha_F(p)$

(ii) $\pi_F(p^e) = p^{e-e'} \pi_F(p)$, with $1 \le e' \le e$ maximal such that $\pi_F(p^{e'}) = \pi_F(p)$

Case 2: The prime $p = 2$, and $e \ge 2$

(i) $\alpha_F(2^e) = 2^{e-e'} \alpha_F(4)$, with $2 \le e' \le e$ maximal such that $\alpha_F(2^{e'}) = \alpha_F(4)$

(ii) $\pi_F(2^e) = 2^{e-e'} \pi_F(4)$, with $2 \le e' \le e$ maximal such that $\pi_F(2^{e'}) = \pi_F(4)$

Case 3: The prime $p = 2$, and $e = 1$

If a is odd then $\alpha_F(2) = \pi_F(2) = 3$, and if a is even then $\alpha_F(2) = \pi_F(2) = 2$.

III. Invariance of period, rank and order of generalized lucas sequences

Several results in [11] show that for the special case $(a,b) = (1,1)$ the period is often independent of the initial values (c,d). Here, we investigate whether and how this property extends to generalized Lucas sequences, and we also ask if corresponding results hold for the notions of rank and period defined in Section 2. The notation $\left(\frac{a}{p}\right)$ stands for the Legendre symbol.

Theorem 3.1. Let $G = (G_n)$ be a generalized Lucas sequence with feasible parameter vector (a,b,c,d) and set $D = b(bc^2 + d(ac - d))$. Then, if $\gcd(D,m) = 1$ one has $\pi_G(m) = \pi_F(m)$, $\alpha_G(m) = \alpha_F(m)$ and $\omega_G(m) = \omega_F(m)$. In particular, if p is a prime satisfying $\gcd(D,p) = 1$ then the reduction properties for prime powers in Theorem 2.2 hold for the sequence $G = (G_n)$.

Proof. The congruences which indicate that the sequence $G \pmod m$ repeats with period $n = \pi_G(m)$ can be written using Lemma 2.1 as follows:

$$\begin{aligned} G_n - c &= dF_n + c(bF_{n-1} - 1) \equiv 0 \pmod{m}, \\ G_{n+1} - d &= (ad + bc)F_n + d(bF_{n-1} - 1) \equiv 0 \pmod{m}. \end{aligned}$$

In matrix notation this linear system of equations modulo m reads

$$\begin{pmatrix} bc & d \\ bd & ad+bc \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix}. \tag{3.1}$$

Its determinant equals $D = b(bc^2 + d(ac - d))$ and immediately shows that if $\gcd(D, m) = 1$ its unique solution satisfies the congruences $bF_{n-1} \equiv 1, F_n \equiv 0$, which characterize the (a,b)-Fibonacci sequence (1.1). This shows that $\pi_F(m) \mid \pi_G(m)$. But, as a consequence of Lemma 2.1, one has also $\pi_G(m) \mid \pi_F(m)$. Together, this implies that $\pi_G(m) = \pi_F(m)$. Similarly, if $n = \alpha_G(m)$ is the rank, a linear congruence system of the type (3.1) must hold with the column vector $(c,d)^T$ on the right replaced by $(cs, ds)^T$ for some s, which is not congruent to zero. If $\gcd(D, m) = 1$ the unique solution to this modified linear system satisfies $bF_{n-1} \equiv s, F_n \equiv 0$, and by definition of the rank one must have $\alpha_F(m) \mid \alpha_G(m)$ and since $\alpha_G(m) \mid \alpha_F(m)$ one must have $\alpha_G(m) = \alpha_F(m)$. Obviously, the orders will also coincide and the reduction properties of Theorem 2.2 hold for primes p satisfying $\gcd(D, p) = 1$. ◊

Example 3.1: Period of the (a,b)-Lucas sequence

The special case $(c,d) = (2,a)$ defines the (a,b)-Lucas sequence $L = (L_n)$ studied by Lucas [14]. In this situation, one has $D = b\Delta$, with $\Delta = a^2 + 4b$ the discriminant of both F, L. Since $\gcd(b,m) = 1$ one has $\pi_L(m) = \pi_F(m)$ provided $\gcd(\Delta, m) = 1$. If further $(a,b) = (1,1)$ then $\Delta = 5$, and if $\gcd(5, m) = 1$ the periods modulo m of the classical Lucas and Fibonacci sequences are equal. The latter result is due to Wall [11], Corollary to Theorem 8. Clearly, the same relationships hold for the ranks and orders.

As mentioned in the Introduction, the case $b = 1$ is of special interest. We show that Theorem 3.1 can be refined to the following extended version of Theorem 8 in [11].

Theorem 3.2. Let $G = (G_n)$ be the generalized (a,c,d)-Lucas sequence defined by $G_0 = c, G_1 = d, G_n = aG_{n-1} + G_{n-2}, n \geq 2$, such that $\gcd(c, d, p^e) = 1$, where p is an odd prime and $e \geq 1$ is an integer. Set $\Delta = a^2 + 4$ and assume that $\left(\frac{\Delta}{p}\right) = -1$. Then, one has $\pi_G(p^e) = \pi_F(p^e)$, $\alpha_G(p^e) = \alpha_F(p^e)$ and $\omega_G(p^e) = \omega_F(p^e)$, where $F = (F_n)$ is the $(a,1)$-Fibonacci sequence.

Proof. Set $m = p^e$ and let $n = \pi_G(m)$. By the proof of Theorem 3.1 one must solve the linear congruence system

$$\begin{pmatrix} c & d \\ d & ad+c \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix}. \tag{3.2}$$

with determinant $D = c^2 + acd - d^2$. If $D \equiv 0 \pmod p$ then $c^2 + acd \equiv d^2 \pmod p$ and

$$4c^2 + 4acd + a^2 d^2 = (2c + ad)^2 \equiv \Delta d^2 \pmod{p}.$$

One must have $\gcd(d,p)=1$. Otherwise, if $d\equiv 0 \pmod{p}$ then also $c\equiv 0 \pmod{p}$, which contradicts the assumption $\gcd(c,d,p^e)=1$. Therefore, if $D\equiv 0 \pmod{p}$ then $\left(\frac{\Delta}{p}\right)=1$. But, since $\left(\frac{\Delta}{p}\right)=-1$ by assumption, one must have $\gcd(D,m)=1$. One concludes similarly to the proof of Theorem 3.1. Similarly, if $n=\alpha_G(m)$ is the rank, a linear congruence system of the type (3.2) must hold with the column vector $(c,d)^T$ on the right replaced by $(cs,ds)^T$ for some s. If $\gcd(D,m)=1$ the unique solution to this modified linear system satisfies $bF_{n-1}\equiv s,\ F_n\equiv 0$. One concludes as in the proof of Theorem 3.1. ◊

IV. The Exceptional (a,b)-Lucas Sequences For Odd Primes

To complete somewhat the picture in Example 3.1 let us describe the periodic behaviour of the (a,b)-Lucas sequence for the odd primes that divide the discriminant. The following notion will be useful.

Definition 4.1. An (a,b)-Lucas sequence is called *exceptional for the prime* $p\nmid ab$ if $\Delta=a^2+4b\equiv 0 \pmod{p}$.

The existence question for exceptional (a,b)-Lucas sequences is answered as follows.

Lemma 4.1. An (a,b)-Lucas sequence is exceptional for the odd prime $p\nmid ab$ if, and only if, one has $(a,b)=(\bar{a}+jp,-(\frac{a}{2})^2+kp),\ \bar{a}=1,2,...,p-1,\ j,k=0,\pm1,\pm2,...$.

Proof. The described pairs solve the defining congruence $a^2+4b\equiv 0 \pmod{p}$. ◊

Clearly, the period and rank only depend upon residue classes modulo p. Therefore, it suffices to restrict the attention to the exceptional pairs $(a,-(\frac{a}{2})^2),\ a=1,2,...,p-1$.

Theorem 4.1 (*Exceptional* (a,b)*-Lucas sequences for odd primes*). For an odd prime p let $(a,-(\frac{a}{2})^2),\ a=1,2,...,p-1$, be the $p-1$ exceptional pairs (a,b) of residue classes modulo p that satisfy the congruence $\Delta=a^2+4b\equiv 0 \pmod{p}$. The periods of the exceptional (a,b)-Lucas sequences and the (a,b)-Fibonacci sequences are determined and related as follows:

Case I: $(a,b)=(2,p-1)$

One has $\pi_L(p)=1$ and $p\pi_L(p)=\pi_F(p)$

Case II: $(a,b)=(a,-(\frac{a}{2})^2),\ a=1,2,...,p-1,\ a\neq 2$

One has $\pi_L(p)=2\cdot ord_p((\frac{a}{2})^2)$ and $\dfrac{ord_p(\frac{a}{2})}{ord_p((\frac{a}{2})^2)}\,p\pi_L(p)=2\cdot\pi_F(p)$

Proof. A straightforward calculation shows that the first few terms of the (a,b)-Lucas sequence are given modulo p by

$$L_0\equiv 2,\quad L_1\equiv a,\quad L_2\equiv -2b,\quad L_3\equiv -ab,\quad L_4\equiv 2b^2.$$

It is immediately seen that

$$\pi_L(p)=1 \iff a\equiv 2,\ \ -2b\equiv a \iff a\equiv 2,\ \ b\equiv p-1.$$

For this choice one has $ord_p(\frac{a}{2})=1$ and $\pi_F(p)=p$ by [10], Theorem 3 (c). This shows Case I. Now, assume that $(a,b)=(a,-(\frac{a}{2})^2),\ a\neq 2$. By Case I one must have $\pi_L(p)\geq 2$. Since $L_2\equiv 2(-b),\quad L_3\equiv a(-b)$

one sees that $\pi_L(p) = 2 \cdot ord_p(-b) = 2 \cdot ord_p((\frac{a}{2})^2)$. On the other hand, one knows from [10], Theorem 3 (c) that $\pi_F(p) = p \cdot ord_p(\frac{a}{2})$. Cross multiplying both equations one obtains the equation

$$p \cdot ord_p(\tfrac{a}{2}) \cdot \pi_L(p) = 2 \cdot ord_p((\tfrac{a}{2})^2) \cdot \pi_F(p),$$

which is clearly equivalent with the desired relationship in Case II. ◊

One notes further that in general either $p \cdot \pi_L(p) = \pi_F(p)$ or $p \cdot \pi_L(p) = 2 \cdot \pi_F(p)$ because the ratio of $ord_p(\frac{a}{2})$ to $ord_p((\frac{a}{2})^2)$ is 1 or 2. The Table 4.1 illustrates for $p = 3,5,7$. It is worthwhile to mention that for $p = 5$ and the classical Lucas sequence with $(a,b) = (1,1)$, one has the relation $5 \cdot \pi_L(5) = \pi_F(5)$, which is part of Theorem 9 in [11]. More generally, in the special case $b \equiv 1 \pmod{p}$ one has for all odd primes and independently of the value a that $(\frac{a}{2})^2 \equiv -1 \pmod{p}$, hence $ord_p(\frac{a}{2}) = 2 \cdot ord_p((\frac{a}{2})^2) = 4$. It follows that $p \cdot \pi_L(p) = \pi_F(p)$. Moreover, by Theorem 3 (c) in [10] one must have $\pi_F(p) = 4p, \pi_L(p) = 4$, and necessarily $p \equiv 1 \pmod{4}$ because $\left(\frac{-1}{p}\right) = 1$. This yields an improvement of the result by Renault in this special case.

Table 4.1 Periods of (a,b)-Lucas and (a,b)-Fibonacci sequences for $p = 3,5,7$

	(a,b)			(a,b)			
	(2,2)	(1,2)		(2,4)	(1,1)	(3,4)	(4,1)
$\pi_L(3)$	1	2	$\pi_L(5)$	1	4	2	4
$\pi_F(3)$	3	6	$\pi_F(5)$	5	20	20	20
	(a,b)						
	(2,6)	(1,5)	(3,3)	(4,3)	(5,6)	(6,5)	
$\pi_L(7)$	1	6	6	6	2	6	
$\pi_F(7)$	7	21	42	21	14	42	

V. Conclusion

Modular properties of integer sequences are important in number theory and combinatorics, and have numerous applications, especially in computer science. In this respect, the theory of Fibonacci sequences plays a special role. We have shown in Theorem 3.1 that the period, rank and order of (a,b)-Fibonacci sequences remain invariant under extension to generalized (a,b,c,d)-Lucas sequences provided the determinant is relatively prime to the modulus. A refinement of this, which generalizes an older result by Wall, has been obtained in Theorem 3.2 for the special case b=1. Some new results for the exceptionl (a,b)-Lucas sequences have also been found. They are related to earlier findings by Wall and Renault. Extensions to other types of generalized Fibonacci sequences might be considered in future work.

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