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Simplified formula for the curvature

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Abstract: *A simplified formula for the calculation of the curvature is suggested. This formula uses the velocity and acceleration, but avoids differentiation of the speed and the calculation of a functional determinant.*
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I. Introduction

A curvature is an important characteristic of a curve that determines its shape at a point. It shows how fast the unit tangent vector rotates at a given point. The curvature k is defined as

$$k = \left| \frac{d\boldsymbol{T}}{ds} \right|, \tag{1}$$

where $\boldsymbol{T}$ is a unit tangent vector, ds is a differential of the curve's length, and straight brackets stand for the magnitude of a vector. If a curve is a path of a moving body that is parameterized with time t, then the following formulas may be used to calculate the curvature, see, for example, [2]:

$$k = \left| \frac{1}{|\boldsymbol{v}|} \frac{d\boldsymbol{T}}{dt} \right| = \frac{|\boldsymbol{v} \times \boldsymbol{a}|}{|\boldsymbol{v}|^3}, \tag{2}$$

where $\boldsymbol{v}$ is velocity, $\boldsymbol{a}$ is acceleration, and symbol "×" stands for the vector product. The first formula is recommended for the plane curves, while the second, for the curves in space.When calculating thecurvature manually, computational problems may arise stemming either from the necessity of differentiating a unit vector $\boldsymbol{T}$ that is in turn a ratio of the velocity vector and the speed, or from the need to calculate a functional determinant $|\boldsymbol{v}\times\boldsymbol{a}|$. In this paper, we suggest a simpler formula based on the tangential and normal components of the acceleration. Such approach is mentioned in [1], but that publication does not provide a formula.

II. Formula derivation

Consider a curve in space $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, $c \le t \le d$, where parameter t is time. Then the vectors of velocity and acceleration are as follows:

$$\boldsymbol{v}(t) = \langle x'(t), y'(t), z'(t)\rangle,$$
$$\boldsymbol{a}(t) = \langle x''(t), y''(t), z''(t)\rangle. \tag{3}$$

A vector of acceleration may be presented as a sum of two orthogonal components: one alongside the unit tangent vector $\boldsymbol{T}$, and another, alongside the principal normal vector $\boldsymbol{N}$, see [1] or [2] for details:

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}, \tag{4}$$

where

$$a_T = \frac{d}{dt}|\boldsymbol{v}|,$$
$$a_N = k|\boldsymbol{v}|^2,$$
$$|\boldsymbol{a}|^2 = a_T^2 + a_N^2. \tag{5}$$

Computation of the derivative of the speed in the first formula in (5) is usually time consuming. But differentiation may be avoided as follows. Since $|\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$,

$$a_T = \frac{\mathrm{d}}{\mathrm{dt}}|\boldsymbol{v}| = \frac{\mathrm{d}}{\mathrm{dt}}\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \frac{\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}}{2\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}} = \frac{\boldsymbol{a}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}. \tag{6}$$

It may be mentioned at this point that formula (6) provides a simpler way of the calculations of the tangential scalar component of the acceleration a_T. This result also follows from the observation that the tangential component of the acceleration $a_T\boldsymbol{T}$ is the projection of the acceleration vector $\boldsymbol{a}$ on the unit tangent vector $\boldsymbol{T}$.

The formula (6) combined with the formulas (5) allows for obtaining a simple formula for the curvature. To do this, substitute the terms α_T and α_N in the lastformula in (5) for their expressions given by (6) and the second formula in (5). By doing so, we get

$$|\boldsymbol{a}|^2 = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{v}}{|\boldsymbol{v}|}\right)^2 + \left(k|\boldsymbol{v}|^2\right)^2 \tag{7}$$

Solving this equation for the curvature k, we obtain

$$k = \frac{\sqrt{|\boldsymbol{a}|^2|\boldsymbol{v}|^2 - (\boldsymbol{a}\cdot\boldsymbol{v})^2}}{|\boldsymbol{v}|^3}. \tag{8}$$

Since $|\boldsymbol{v}|^2$ is simpler to calculate than $|\boldsymbol{v}|^3$, the formula (8) may be transformed to become

$$k = \frac{\sqrt{|\boldsymbol{a}|^2 - \dfrac{(\boldsymbol{a}\cdot\boldsymbol{v})^2}{|\boldsymbol{v}|^2}}}{|\boldsymbol{v}|^2}. \tag{9}$$

Both formulas (8) and (9) are much simpler than the formulas (1) or (2). They do not require either differentiation of the unit tangent vector $\boldsymbol{T}$ or computation of the functional determinant.

Example

Following [2], consider a curve

$$\boldsymbol{r}(t) = p\cdot\cos(t)\boldsymbol{i} + p\cdot\sin(t)\,\boldsymbol{j} + qt\boldsymbol{k}, \qquad p,q>0. \tag{10}$$

For this curve,

$$\begin{aligned}
&\boldsymbol{v}(t) = -p\cdot\sin(t)\boldsymbol{i} + p\cdot\cos(t)\,\boldsymbol{j} + q\boldsymbol{k},\\
&\boldsymbol{a}(t) = -p\cdot\cos(t)\boldsymbol{i} - p\cdot\sin(t)\,\boldsymbol{j} + 0\boldsymbol{k},\\
&|\boldsymbol{a}|^2 = p^2,\\
&|\boldsymbol{v}|^2 = p^2+q^2,\\
&\boldsymbol{a}\cdot\boldsymbol{v} = p^2\cdot\sin(t)\cos(t) - p^2\cdot\cos(t)\sin(t) = 0.
\end{aligned} \tag{11}$$

By using the formula (9), we get

$$k = \frac{\sqrt{p^2 - \dfrac{0^2}{(p^2+q^2)}}}{\left(\sqrt{p^2+q^2}\right)^2} = \frac{p}{p+q}. \tag{12}$$

The result is the same as that in [2], but is obtained with much less routine calculations.
In the general case of smooth parameterization, formula (8) takes the form

$$k = \frac{\sqrt{|\boldsymbol{r}''|^2|\boldsymbol{r}'|^2 - (\boldsymbol{r}''\cdot\boldsymbol{r}')^2}}{|\boldsymbol{r}'|^3}, \tag{13}$$

while formula (9) becomes

$$k = \frac{\sqrt{|\boldsymbol{r}''|^2 - \dfrac{(\boldsymbol{r}''\cdot\boldsymbol{r}')^2}{|\boldsymbol{r}'|^2}}}{|\boldsymbol{r}'|^2}. \tag{14}$$

A direct proof of the identity

$$|\boldsymbol{v}\times\boldsymbol{a}|^2 = |\boldsymbol{a}|^2|\boldsymbol{v}|^2 - (\boldsymbol{a}\cdot\boldsymbol{v})^2 \tag{15}$$

may be suggested as an exercise.

References

[1]. Smith, R., Minton, R. Calculus. Early Transcendental Functions. 3rd Ed., New York: McGraw Hill Higher Education; 2007.

[2]. Thomas, G., Weir, M., Hass, J. Thomas' Calculus: Early Transcendentals, 13th Ed., New York: Pearson; 2014.