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Beyond the Quadratic Equations and the N-D Newton-Raphson Method

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Abstract: *The objective of this technical paper was to propose an approach for solving polynomials of degree higher than two. The main concepts were the decomposition of a polynomial of a higher degree to the product of two polynomials of lower degrees and the n-D Newton-Raphson method for a system of nonlinear equations. The coefficient of each term in an original polynomial of order m will be equated to the corresponding term from the collected-expanded product of the two polynomials of the lower degrees based on the concept of undetermined coefficients. Consequently a system of nonlinear equations was formed. Then the unknown coefficients of the decomposed polynomial of the lower degree of the two decomposed polynomials would be eliminated from the system of the nonlinear equations. After that the unknown coefficients in the decomposed polynomial of the higher degree would be obtained by the n-D Newton-Raphson. Finally the unknown coefficients for the decomposed polynomial of the lower degree would be obtained by back substitutions. In this technical paper the formulations for the decomposed polynomials would be derived for the polynomials degree three to nine. Several numerical examples were also given to verify the applicability of the proposed approach.*

Keywords: *Roots of a Polynomial of a High Degree, the n-D Newton-Raphson Method, Undetermined Coefficients, Jacobian of the Functions, Matrix Inversion*

I. Introduction

Finding solutions to a polynomial of order higher than two has been unavoidable in engineering works. Mostly only real roots were required. In these cases the graphical method could give good initial guesses for some efficient numerical methods such as the Newton-Raphson method. However the determination of all possible roots has been very challenging. There are general solutions for cubic- and quartic polynomials [1]. Beyond the quartic polynomials some special forms and sufficient conditions for solvable polynomials have been studied [1-10].

The purpose of this technical paper was to propose a mathematical tool for solving for all possible roots of a polynomial of degree higher than two. It included the decomposition technique and the Newton-Raphson method for a system of nonlinear equations. The decomposition technique was applied for rewriting the original polynomial into the form of product of two polynomials of lower degrees. Based on the concept of undetermined coefficients each coefficient of an x power in the original polynomial would be equated to the corresponding collected-expanded one of the product of the two decomposed polynomials. The unknown coefficients in decomposed polynomials of the lower degree would be eliminated. Based on this a system of nonlinear equations of unknown coefficients in the decomposed polynomial of the higher degree was obtained. Then the n-D Newton-Raphson method was used to solve for the unknown coefficients from the system of nonlinear equations. The eliminated coefficients were obtained by back substitutions. The formulations and the concepts would be discussed in Section 2. In Section 3 the applicability of the proposed mathematical tool would be demonstrated in several numerical examples. From which critical conclusion could be drawn in Section 4.

II. Decomposition of the Original Polynomial Equation

2.1 Decomposition of a Polynomial Equation of Order m

A polynomial equation of order m may be generally expressed in form of (1):

$$a_m x^m + a_{m-1} x^{m-1} + ... + a_2 x^2 + a_1 x + a_0 = 0 \tag{1}$$

Where $a_i, i = 0,1,..., m$ are the coefficients of x^i and $a_m \neq 0$. Without losing generality $a_m = 1$ may be used throughout this technical paper. Thus:

$$x^m + a_{m-1} x^{m-1} + ... + a_2 x^2 + a_1 x + a_0 = 0 \tag{2}$$

Given $r_i, i = 0,1,..., m$ are all possible roots of the polynomial equation. The polynomial equation of (2) can be rewritten as:

$$(x - r_1)\cdot(x - r_2)\cdot(x - r_3)\cdots(x - r_m) = 0 \tag{3}$$

In the case that a root x_r is found, $(x-x_r)$ will be one factor of the original polynomial equation. The deflated equation may be obtained by the direct long division of the original equation of (2) by $(x-x_r)$ as:

$$x^{m-1}+(a_{m-1}+x_r)\cdot x^{m-2}+\left(a_{m-2}+a_{m-1}\cdot x_r+x_r{}^2\right)\cdot x^3+...+\left(a_1+a_2\cdot x_r+a_3\cdot x_r{}^2+...+x_r{}^m\right)=0 \quad (4)$$

Further deflations can be done as soon as additional roots will be found. Real roots of polynomial equations could be obtained in Bernstein form by numerical analysis [11]. The graphical method can be served as a powerful tool for determining the number of real roots, approximate values of real roots and nature of the real roots e.g. double real roots or triple real roots etc. Usually any value of a real root from the graphical method can be used as an initial guess value of a numerical method e.g. the Newton-Raphson method.

The rest of this technical paper will be devoted for the decomposition of an original polynomial equation into two decomposed polynomial equations of lower degrees.

2.2 Decomposition of a Polynomial Equation
2.2.1 Proposed Decomposition

An original polynomial equation of order m as in the form of (2) can always be decomposed into two polynomial equations of lower degrees. For an original polynomial equation of an odd degree the equation will be decomposed into two decomposed equations of lower degrees i.e. one decomposed polynomial equation with an odd degree and the other decomposed polynomial equation with an even degree. For an original polynomial equation of even degree, however, the function will be decomposed to two decomposed equations of even degrees. The reason behind is that a polynomial equation of odd degree will always have at least one real root. However no real root is guaranteed for a polynomial equation of an even degree. Therefore for the case of an original polynomial equation of even degree it is conservative to assume both two decomposed polynomial equations of even degrees, since a quadratic equation can always be solved in a closed form formula. Our discussions will be focused on the polynomial equations of degree higher than two i.e. degree three onwards. The orders of the two decomposed equations are summarized in Table 1.

Table 1 Degrees of Two Decomposed Equations for Original Equations of Degree from 3-9

Degree of Equation		
Original Equation	1st Decomposed Equation	2nd Decomposed Equation
3	1	2
4	2	2
5	2	3
6	2	4
7	3	4
8	4	4
9	4	5

2.2.2 Bairstow's Decomposition

Bairstow [12] proposed decomposing a polynomial of order m in form of (5) into a product of two lower degrees, i.e. a quadratic function and a polynomial of degree $m-2$, plus a remainder term in form of (6).

$$P(x)=a_m x^m+a_{m-1}x^{m-1}+...+a_2x^2+a_1x+a_0 \quad (5)$$

$$P(x)=\left(x^2+u\cdot x+v\right)\cdot\left(b_{m-2}x^{m-2}+b_{m-3}x^{m-3}+...+b_2x^2+b_1x+b_0\right)+cx+d \quad (6)$$

where $b_i, i=0,1,...,m-2$ are the coefficients of x^i c,d are coefficients and $b_{m-2}\neq 0$. The latter polynomial can be obtained by long dividing $P(x)$ by $\left(x^2+u\cdot x+v\right)$ and the term $cx+d$ is the remainder.

Once the values of u and v are assumed, All b_i's as well as c and d can be determined. By iterative procedures, the actual u and v as well as all b_i's can be obtained as soon as $cx+d$ approaches zero. Thus, (6) is reduced to (7).

$$\left(x^2+u\cdot x+v\right)\cdot\left(b_{m-2}x^{m-2}+b_{m-3}x^{m-3}+...+b_2x^2+b_1x+b_0\right)=0 \quad (7)$$

For the sake of further discussions a_m and b_{m-2} can be set to 1 without losing any generality. Thus, (7) may be rewritten as:

$$\left(x^2+d_1\cdot x+d_0\right)\cdot\left(x^{m-2}+e_{m-3}x^{m-3}+...+e_2x^2+e_1x+e_0\right)=0 \quad (8)$$

Based on (8) a polynomial equation can be decomposed to the product of two polynomial equations, i.e. a quadratic equation and a polynomial equation of order $m-2$. There can be several pairs of the decomposed polynomials depending on the initial guess of the iterative procedures. Since any quadratic equation and linear equation is always solved, all possible roots of a polynomial equation can be determined by the Bairstow's method. The convergence of the method is quadratic, only if the zeros are complex conjugate pairs of multiplicity one, or are real of multiplicity at most two. For higher multiplicities it is impractically slow or subject to failure. The modifications of the Bairstow's method were proposed [13-14], but the details are out of the scope of this technical paper.

2.2.3 Complete Bairstow's Decomposition

The decomposed polynomial equation of (8) can be further decomposed until the degree of the last decomposed equation is two or one. For an original polynomial equation of an even degree or an odd degree, the complete Bairstow's solution can be rewritten in form of (9) or (10), respectively.

$$\left(x^2+d_1\cdot x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)..Q(x)=0 \tag{9}$$

$$\left(x+c_0\right)\cdot\left(x^2+d_1\cdot x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)..Q(x)=0 \tag{10}$$

where $Q(x)$ is a quadratic function.

2.3 Decomposition of a Cubic Function

Consider a general cubic equation:

$$x^3+a_2x^2+a_1x+a_0=0 \tag{11}$$

The cubic equation may be decomposed into a product of two equations i.e. one linear equation and one quadratic equation as shown below:

$$\left(x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)=0 \tag{12}$$

where d_0, e_1 and e_0 are the unknown coefficients of the equations. Expanding the product in (12) yields:

$$x^3+\left(d_0+e_1\right)x^2+\left(e_0+d_0e_1\right)x+d_0e_0=0 \tag{13}$$

Equating each coefficient in (13) to the corresponding term in the original equation in (11) leads to 3 equations:

$$d_0+e_1=a_2 \tag{14.a}$$

$$e_0+d_0e_1=a_1 \tag{14.b}$$

$$d_0e_0=a_0 \tag{14.c}$$

d_0 in (14.c) can be rewritten in term of a_0 and e_0 as:

$$d_0=\frac{a_0}{e_0} \tag{15}$$

Thus, d_0 can be eliminated from (14.a) and (14.b) such that a system of two nonlinear equations in two simultaneous unknowns e_1 and e_0 is formed.

$$\frac{a_0}{e_0}+e_1=a_2 \tag{16.a}$$

$$e_0+\frac{a_0\cdot e_1}{e_0}=a_1 \tag{16.b}$$

Once e_1 and e_0 are obtained from a numerical method i.e. the Newton-Raphson method in two dimensions, d_0 can be obtained by back substitution via (15).

The proposed decomposition for a cubic equation has exactly the same form as the Bairstow's method and it is already complete.

2.4 Decomposition of a Quartic Function

Consider a general quartic equation:

$$x^4+a_3x^3+a_2x^2+a_1x+a_0=0 \tag{17}$$

The quartic equation may be decomposed into a product of two equations i.e. two quadratic equations as shown below:

$$\left(x^2+d_1x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)=0 \tag{18}$$

where d_1, d_0, e_1 and e_0 are the unknown coefficients of the equations. Expanding the product in (18) yields:

$$x^4+\left(d_1+e_1\right)x^3+\left(d_0+e_0+d_1e_1\right)x^2+\left(d_0e_1+d_1e_0\right)x+\left(d_0e_0\right)=0 \tag{19}$$

Equating each coefficient in (19) to the corresponding term in the original equation in (17) leads to 4 equations:

$$d_1+e_1=a_3 \tag{20.a}$$

$$d_0 + e_0 + d_1 e_1 = a_2 \quad (20.b)$$
$$d_0 e_1 + d_1 e_0 = a_1 \quad (20.c)$$
$$d_0 e_0 = a_0 \quad (20.d)$$

d_1 in (20.a) and d_0 in (20.d), respectively, can be rewritten in term of the other terms as:

$$d_1 = a_3 - e_1 \quad (21.a)$$
$$d_0 = \frac{a_0}{e_0} \quad (21.b)$$

Thus, d_1 and d_0 can be eliminated from (20.b) and (20.c) such that a system of two simultaneous nonlinear equations in two unknowns e_1 and e_0 is formed.

$$\frac{a_0}{e_0} + e_0 + (a_3 - e_1) \cdot e_1 = a_2 \quad (22.a)$$
$$\frac{a_0 \cdot e_1}{e_0} + (a_3 - e_1) \cdot e_0 = a_1 \quad (22.b)$$

Once e_1 and e_0 are obtained by the Newton-Raphson method in two dimensions, d_1 and d_0 can be obtained by back substitution via (21.a) and (21.b), respectively.

The proposed decomposition for a quartic equation has exactly the same form as the Bairstow's method and it is already complete.

2.5 Decomposition of a Quintic Function

2.5.1 The Proposed Decomposition

Consider a general quintic equation:

$$x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \quad (23)$$

The quintic equation may be decomposed into a product of two equations i.e. one quadratic equation and one cubic equation as shown below:

$$\left(x^2 + d_1 x + d_0\right) \cdot \left(x^3 + e_2 x^2 + e_1 x + e_0\right) = 0 \quad (24)$$

where d_1, d_0, e_2, e_1 and e_0 are the unknown coefficients of the equations. Expanding the product in (24) yields:

$$x^5 + (d_1 + e_2)x^4 + (d_0 + e_1 + d_1 \cdot e_2)x^3 + (e_0 + d_0 \cdot e_2 + d_1 \cdot e_1)x^2 + (d_0 \cdot e_1 + d_1 \cdot e_0)x + (d_0 \cdot e_0) = 0 \quad (25)$$

Equating each coefficient in (25) to the corresponding term in the original equation in (23) leads to 5 equations:

$$d_1 + e_2 = a_4 \quad (26.a)$$
$$d_0 + e_1 + d_1 \cdot e_2 = a_3 \quad (26.b)$$
$$e_0 + d_0 \cdot e_2 + d_1 \cdot e_1 = a_2 \quad (26.c)$$
$$d_0 \cdot e_1 + d_1 \cdot e_0 = a_1 \quad (26.d)$$
$$d_0 \cdot e_0 = a_0 \quad (26.e)$$

d_1 in (26.a) and d_0 in (26.e), respectively, can be rewritten in term of the other terms as:

$$d_1 = a_4 - e_2 \quad (27.a)$$
$$d_0 = \frac{a_0}{e_0} \quad (27.b)$$

Thus, d_1 and d_0 can be eliminated from (26.b), (26.c) and (26.d) such that a system of three simultaneous nonlinear equations in three unknowns e_2, e_1 and e_0 is formed.

$$\frac{a_0}{e_0} + e_1 + (a_4 - e_2) \cdot e_2 = a_3 \quad (28.a)$$
$$e_0 + \frac{a_0 \cdot e_2}{e_0} + (a_4 - e_2) \cdot e_1 = a_2 \quad (28.b)$$
$$\frac{a_0 \cdot e_1}{e_0} + (a_4 - e_2) \cdot e_0 = a_1 \quad (28.c)$$

Once e_2, e_1 and e_0 are obtained by the Newton-Raphson method in three dimensions, d_1 and d_0 can be obtained by back substitution via (27.a) and (27.b), respectively.

2.5.2 Bairstow's Decomposition

The proposed decomposition for a quintic equation has exactly the same form as the Bairstow's method and the cubic equation obtained can be further decomposed by using the proposed decomposition as discussed in Section 2.3.

2.5.3 Complete Bairstow's Decomposition

The quintic equation of (23) can be rewritten in form of a complete Bairstow's decomposition as:

$$(x+c_0)\cdot\left(x^2+d_1x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)=0 \quad (29)$$

Expanding the product in (29) yields:

$$\begin{aligned}&x^5+(c_0+d_1+e_1)x^4+(d_0+e_0+c_0\cdot d_1+c_0\cdot e_1+d_1\cdot e_1)x^3\\&\quad+(c_0\cdot d_0+c_0\cdot e_0+d_0\cdot e_1+d_1\cdot e_0+c_0\cdot d_1\cdot e_1)x^2+(d_0\cdot e_0+c_0\cdot d_0\cdot e_1+c_0\cdot d_1\cdot e_0)x\\&\quad+(c_0\cdot d_0\cdot e_0)=0\end{aligned} \quad (30)$$

Equating each coefficient in (30) to the corresponding term in the original equation in (23) leads to 5 equations:

$$c_0+d_1+e_1=a_4 \quad (31.a)$$

$$d_0+e_0+c_0\cdot d_1+c_0\cdot e_1+d_1\cdot e_1=a_3 \quad (31.b)$$

$$c_0\cdot d_0+c_0\cdot e_0+d_0\cdot e_1+d_1\cdot e_0+c_0\cdot d_1\cdot e_1=a_2 \quad (31.c)$$

$$d_0\cdot e_0+c_0\cdot d_0\cdot e_1+c_0\cdot d_1\cdot e_0=a_1 \quad (31.d)$$

$$c_0\cdot d_0\cdot e_0=a0 \quad (31.e)$$

Thus, a system of five simultaneous nonlinear equations in five unknowns c_0, d_1, d_0, e_1 and e_0 is formed.

2.6 Decomposition of a Sextic Function

2.6.1 The Proposed Decomposition

Consider a general sextic equation: (32)

$$x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$$

The sextic equation may be decomposed into a product of two equations i.e. one quadratic equation and one quartic equation as shown below:

$$\left(x^2+d_1x+d_0\right)\cdot\left(x^4+e_3x^3+e_2x^2+e_1x+e_0\right)=0 \quad (33)$$

where d_1, d_0, e_3, e_2, e_1 and e_0 are the unknown coefficients of the equations. Expanding the product in (33) yields:

$$\begin{aligned}&x^6+(d_1+e_3)x^5+(d_0+e_2+d_1e_3)x^4+(e_1+d_0e_3+d_1e_2)x^3+({}_0+d_0e_2+d_1e_1)x^2+(d_0e_1+d_1e_0)x\\&\quad+(d_0e_0)=0\end{aligned} \quad (34)$$

Equating each coefficient in (34) to the corresponding term in the original equation in (32) leads to 6 equations:

$$d_1+e_3=a_5 \quad (35.a)$$

$$d_0+e_2+d_1e_3=a_4 \quad (35.b)$$

$$e_1+d_0e_3+d_1e_2=a_3 \quad (35.c)$$

$$e_0+d_0e_2+d_1e_1=a_2 \quad (35.d)$$

$$d_0e_1+d_1e_0=a_1 \quad (35.e)$$

$$d_0e_0=a_0 \quad (35.f)$$

d_1 in (35.a) and d_0 in (35.f), respectively, can be rewritten in term of the other terms as:

$$d_1=a_5-e_3 \quad (36.a)$$

$$d_0=\frac{a_0}{e_0} \quad (36.b)$$

Thus, d_1 and d_0 can be eliminated from (35.b), (35.c), (35.d) and (35.e) such that a system of four simultaneous nonlinear equations in three unknowns e_3, e_2, e_1 and e_0 is formed.

$$\frac{a_0}{e_0}+e_2+(a_5-e_3)\cdot e_3=a_4 \quad (37.a)$$

$$\left(a_5 - e_3\right)\cdot e_2 + e_1 + \frac{a_0 \cdot e_3}{e_0} = a_3 \tag{37.b}$$

$$e_0 + \frac{a_0 \cdot e_2}{e_0} + \left(a_5 - e_3\right)\cdot e_1 = a_2 \tag{37.c}$$

$$\frac{a_0 \cdot e_1}{e_0} + \left(a_5 - e_3\right)\cdot e_0 = a_1 \tag{37.d}$$

Once e_3, e_2, e_1 and e_0 are obtained by the Newton-Raphson method in four dimensions, d_1 and d_0 can be obtained by back substitution via (36.a) and (36.b), respectively.

2.6.2 Bairstow's Decomposition

The proposed decomposition for a sextic equation has exactly the same form as the Bairstow's method and the quartic equation obtained can be further decomposed by using the proposed decomposition as discussed in Section 2.4.

2.6.3 Complete Bairstow's Decomposition

The sextic equation of (32) can be rewritten in form of a complete Bairstow's decomposition as:

$$\left(x^2 + d_1 x + d_0\right)\cdot\left(x^2 + e_1 x + e_0\right)\cdot\left(x^2 + f_1 x + f_0\right) = 0 \tag{38}$$

Expanding the product in (38) yields:

$$\begin{aligned} &x^6 + \left(d_1 + e_1 + f_1\right)x^5 + \left(d_0 + e_0 + f_0 + d_1 e_1 + d_1 f_1 + e_1 f_1\right)x^4 \\ &+ \left(d_0 e_1 + d_1 e_0 + d_0 f_1 + d_1 f_0 + e_0 f_1 + e_1 f_0 + d_1 e_1 f_1\right)x^3 \\ &+ \left(d_0 e_0 + d_0 f_0 + e_0 f_0 + d_0 e_1 f_1 + d_1 e_0 f_1 + d_1 e_1 f_0\right)x^2 \\ &+ \left(d_0 e_0 f_1 + d_0 e_1 f_0 + d_1 e_0 f_0\right)x + \left(d_0 e_0 f_0\right) = 0 \end{aligned} \tag{39}$$

Equating each coefficient in (39) to the corresponding term in the original equation in (32) leads to 6 equations:

$$d_1 + e_1 + f_1 = a_5 \tag{40.a}$$

$$d_0 + e_0 + f_0 + d_1 e_1 + d_1 f_1 + e_1 f_1 = a_4 \tag{40.b}$$

$$d_0 e_1 + d_1 e_0 + d_0 f_1 + d_1 f_0 + e_0 f_1 + e_1 f_0 + d_1 e_1 f_1 = a_3 \tag{40.c}$$

$$d_0 e_0 + d_0 f_0 + e_0 f_0 + d_0 e_1 f_1 + d_1 e_0 f_1 + d_1 e_1 f_0 = a_2 \tag{40.d}$$

$$d_0 e_0 f_1 + d_0 e_1 f_0 + d_1 e_0 f_0 = a_1 \tag{40.e}$$

$$d_0 e_0 f_0 = a_0 \tag{40.f}$$

Thus, a system of six simultaneous nonlinear equations in six unknowns d_1, d_0, e_1, e_0, f_1 and f_0 is formed.

2.7 Decomposition of a Septic Function

2.7.1 The Proposed Decomposition

Consider a general septic equation:

$$x^7 + a_6 x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \tag{41}$$

The septic equation may be decomposed into a product of two equations i.e. one cubic equation and one quartic equation as shown below:

$$\left(x^3 + d_2 x^2 + d_1 x + d_0\right)\cdot\left(x^4 + e_3 x^3 + e_2 x^2 + e_1 x + e_0\right) = 0 \tag{42}$$

where $d_2, d_1, d_0, e_3, e_2, e_1$ and e_0 are the unknown coefficients of the equations. Expanding the product in (42) yields:

$$\begin{aligned} &x^7 + \left(d_2 + e_3\right)x^6 + \left(d_1 + e_2 + d_2 e_3\right)x^5 + \left(d_0 + e_1 + d_1 e_3 + d_2 e_2\right)x^4 + \left(e_0 + d_0 e_3 + d_1 e_2 + d_2 e_1\right)x^3 \\ &\quad + \left(d_0 e_2 + d_1 e_1 + d_2 e_0\right)x^2 + \left(d_0 e_1 + d_1 e_0\right)x + \left(d_0 e_0\right) = 0 \end{aligned} \tag{43}$$

Equating each coefficient in (43) to the corresponding term in the original equation in (41) leads to 7 equations:

$$d_2 + e_3 = a_6 \tag{44.a}$$

$$d_1 + e_2 + d_2 e_3 = a_5 \tag{44.b}$$

$$d_0 + e_1 + d_1 e_3 + d_2 e_2 = a_4 \tag{44.c}$$

$$e_0 + d_0 e_3 + d_1 e_2 + d_2 e_1 = a_3 \tag{44.d}$$

$$d_0e_2 + d_1e_1 + d_2e_0 = a_2 \tag{44.e}$$
$$d_0e_1 + d_1e_0 = a_1 \tag{44.f}$$
$$d_0e_0 = a_0 \tag{44.g}$$

d_2 in (44.a), d_1 in (44.b) and d_0 in (44.g), respectively, can be rewritten in term of the other terms as:

$$d_2 = a_6 - e_3 \tag{45.a}$$
$$d_1 = a_5 - e_2 - (a_6 - e_3)\cdot e_3 \tag{45.b}$$
$$d_0 = \frac{a_0}{e_0} \tag{45.c}$$

Thus, d_2, d_1 and d_0 can be eliminated from (44.c), (44.d), (44.e) and (44.f) such that a system of four simultaneous nonlinear equations in four unknowns e_3, e_2, e_1 and e_0 is formed.

$$\frac{a_0}{e_0} + e_1 + \{a_5 - e_2 - (a_6 - e_3)\cdot e_3\}\cdot e_3 + (a_6 - e_3)\cdot e_2 = a_4 \tag{46.a}$$
$$e_0 + \frac{a_0}{e_0}\cdot e_3 + \{a_5 - e_2 - (a_6 - e_3)\cdot e_3\}\cdot e_2 + (a_6 - e_3)\cdot e_1 = a_3 \tag{46.b}$$
$$\frac{a_0 \cdot e_2}{e_0} + \{a_5 - e_2 - (a_6 - e_3)\cdot e_3\}\cdot e_1 + (a_6 - e_3)\cdot e_0 = a_2 \tag{46.c}$$
$$\frac{a_0 \cdot e_1}{e_0} + \{a_5 - e_2 - (a_6 - e_3)\cdot e_3\}\cdot e_0 = a_1 \tag{46.d}$$

Once e_3, e_2, e_1 and e_0 are obtained by the Newton-Raphson method in four dimensions, d_2, d_1 and d_0 can be obtained by back substitution via (45.a), (45.b) and (45.c), respectively.

2.7.2 Bairstow's Decomposition

The septic equation of (41) can be rewritten in form of the Bairstow's decomposition as:

$$\left(x^2 + d_1x + d_0\right)\cdot\left(x^5 + e_4x^4 + e_3x^3 + e_2x^2 + e_1x + e_0\right) = 0 \tag{47}$$

Expanding the product in (47) yields:

$$\begin{aligned} x^7 &+ (d_1 + e_4)x^6 + (d_0 + e_3 + d_1e_4)x^5 + (e_2 + d_0e_4 + d_1e_3)x^4 + (e_1 + d_0e_3 + d_1e_2)x^3 \\ &+ (e_0 + d_0e_2 + d_1e_1)x^2 + (d_0e_1 + d_1e_0)x + (d_0e_0) = 0 \end{aligned} \tag{48}$$

Equating each coefficient in (48) to the corresponding term in the original equation in (41) leads to 7 equations:

$$d_1 + e_4 = a_6 \tag{49.a}$$
$$d_0 + e_3 + d_1e_4 = a_5 \tag{49.b}$$
$$e_2 + d_0e_4 + d_1e_3 = a_4 \tag{49.c}$$
$$e_1 + d_0e_3 + d_1e_2 = a_3 \tag{49.d}$$
$$e_0 + d_0e_2 + d_1e_1 = a_2 \tag{49.e}$$
$$d_0e_1 + d_1e_0 = a_1 \tag{49.f}$$
$$d_0e_0 = a_0 \tag{49.g}$$

Thus, a system of seven simultaneous nonlinear equations in seven unknowns $c_0, d_1, d_0, e_1, e_0, f_1$ and f_0 is formed.

2.7.3 Complete Bairstow's Decomposition

The septic equation of (41) can be rewritten in form of a complete Bairstow's decomposition as:

$$(x + c_0)\cdot\left(x^2 + d_1x + d_0\right)\cdot\left(x^2 + e_1x + e_0\right)\cdot\left(x^2 + f_1x + f_0\right) = 0 \tag{50}$$

Expanding the product in (50) and equating each coefficient to the corresponding term in the original equation in (41) leads to 7 equations:

$$c_0 + d_1 + e_1 + f_1 = a_6 \tag{51.a}$$
$$d_0 + e_0 + f_0 + c_0d_1 + c_0e_1 + c_0f_1 + d_1e_1 + d_1f_1 + e_1f_1 = a_5 \tag{51.b}$$
$$c_0d_0 + c_0e_0 + c_0f_0 + d_0e_1 + d_1e_0 + d_0f_1 + d_1f_0 + e_0f_1 + e_1f_0 + c_0d_1e_1 + c_0d_1f_1 + d_1e_1f_1 = a_4 \tag{51.c}$$

$$d_0e_0 + d_0f_0 + e_0f_0 + c_0d_0e_1 + c_0d_1e_0 + c_0d_0f_1 + c_0d_1f_0 + c_0e_0f_1 + d_0e_1f_1 + d_1e_0f_1 + d_1e_1f_0 + c_0d_1e_1f_1 = a_3 \quad (51.d)$$

$$c_0d_0e_0 + c_0d_0f_0 + c_0e_0f_0 + d_0e_0f_1 + d_0e_1f_0 + d_1e_0f_0 + c_0d_1e_0f_1 + c_0d_1e_1f_1 + c_0d_1e_1f_0 = a_2 \quad (51.e)$$

$$d_0e_0f_0 + c_0d_0e_0f_1 + c_0d_0e_1f_0 + c_0d_1e_0f_0 = a_1 \quad (51.f)$$

$$c_0d_0e_0f_0 = a_0 \quad (51.h)$$

Thus, a system of seven simultaneous nonlinear equations in seven unknowns $c_0, d_1, d_0, e_1, e_0, f_1$ and f_0 is formed.

2.8 Decomposition of an Octic Function

2.8.1 The Proposed Decomposition

Consider a general octic equation:

$$x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \quad (52)$$

The octic equation may be decomposed into a product of two equations i.e. two quartic equations as shown below:

$$\left(x^4 + d_3x^3 + d_2x^2 + d_1x + d_0\right)\cdot\left(x^4 + e_3x^3 + e_2x^2 + e_1x + e_0\right) = 0 \quad (53)$$

where $d_3, d_2, d_1, d_0, e_3, e_2, e_1$ and e_0 are the unknown coefficients of the equations. Expanding the product in (54) and equating each coefficient to the corresponding term in the original equation in (52) leads to 8 equations:

$$d_3 + e_3 = a_7 \quad (54.a)$$

$$d_2 + e_2 + d_3e_3 = a_6 \quad (54.b)$$

$$d_1 + e_1 + d_2e_3 + d_3e_2 = a_5 \quad (54.c)$$

$$d_0 + e_0 + d_1e_3 + d_2e_2 + d_3e_1 = a_4 \quad (54.d)$$

$$d_0e_3 + d_1e_2 + d_2e_1 + d_3e_0 = a_3 \quad (54.e)$$

$$d_0e_2 + d_1e_1 + d_2e_0 = a_2 \quad (54.f)$$

$$d_0e_1 + d_1e_0 = a_1 \quad (54.g)$$

$$d_0e_0 = a_0 \quad (54.h)$$

d_3 in (54.a), d_2 in (54.b), d_1 in (54.c) and d_0 in (54.h), respectively, can be rewritten in term of the other terms as:

$$d_3 = a_7 - e_3 \quad (55.a)$$

$$d_2 = a_6 - e_2 - (a_7 - e_3)\cdot e_3 \quad (55.b)$$

$$d_1 = a_5 - e_1 - (a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3))\cdot e_3 - (a_7 - e_3)\cdot e_2 \quad (55.c)$$

$$d_0 = \frac{a_0}{e_0} \quad (55.d)$$

Thus, d_3, d_2, d_1 and d_0 can be eliminated from (54.d), (54.e), (54.f) and (54.g) such that a system of four simultaneous nonlinear equations in four unknowns e_3, e_2, e_1 and e_0 is formed.

$$\frac{a_0}{e_0} + e_0 + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_3 + (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_2 + (a_7 - e_3)\cdot e_1 = a_4 \quad (56.a)$$

$$\frac{a_0\cdot e_3}{e_0} + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_2 + (a_7 - e_3)\cdot e_0 + (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_1 = a_3 \quad (56.b)$$

$$\frac{a_0\cdot e_2}{e_0} + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_1 + (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_0 = a_2 \quad (56.c)$$

$$\frac{a_0\cdot e_1}{e_0} + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_0 = a_1 \quad (56.d)$$

Once e_3, e_2, e_1 and e_0 are obtained by the Newton-Raphson method in four dimensions, d_3, d_2, d_1 and d_0 can be obtained by back substitution via (55.a), (55.b), (55.c) and (55.d), respectively.

2.8.2 Bairstow's Decomposition

The octic equation of (52) can be rewritten in form of the Bairstow's decomposition as:

$$\left(x^2+d_1x+d_0\right)\cdot\left(x^6+e_5x^5+e_4x^4+e_3x^3+e_2x^2+e_1x+e_0\right)=0 \quad (57)$$

Expanding the product in (57) and equating each coefficient to the corresponding term in the original equation in (52) leads to 8 equations:

$$d_1+e_5=a_7 \quad (58.a)$$
$$d_0+e_4+d_1e_5=a_6 \quad (58.b)$$
$$e_3+d_0e_5+d_1e_4=a_5 \quad (58.c)$$
$$e_2+d_0e_4+d_1e_3=a_4 \quad (58.d)$$
$$e_1+d_0e_3+d_1e_2=a_3 \quad (58.e)$$
$$e_0+d_0e_2+d_1e_1=a_2 \quad (58.f)$$
$$d_0e_1+d_1e_0=a_1 \quad (58.g)$$
$$d_0e_0=a_0 \quad (58.h)$$

Thus, a system of eight simultaneous nonlinear equations in eight unknowns $d_1,d_0,e_5,e_4,e_3,e_2,e_1$ and e_0 is formed.

2.8.3 Complete Bairstow's Decomposition

The octic equation of (52) can be rewritten in form of a complete Bairstow's decomposition as:

$$\left(x^2+d_1x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)\cdot\left(x^2+f_1x+f_0\right)\cdot\left(x^2+g_1x+g_0\right)=0 \quad (59)$$

Expanding the product in (59) and equating each coefficient to the corresponding term in the original equation in (52) leads to 8 equations:

$$d_1+e_1+f_1+g_1=a_7 \quad (60.a)$$
$$d_0+e_0+f_0+g_0+d_1e_1+d_1f_1+d_1g_1+e_1f_1+e_1g_1+f_1g_1=a_6 \quad (60.b)$$
$$\begin{aligned}&d_0e_1+d_1e_0+d_0f_1+d_1f_0+d_0g_1+d_1g_0+e_0f_1+e_1f_0+e_0g_1+e_1g_0+f_0g_1+f_1g_0\\&+d_1e_1f_1+d_1e_1g_1+d_1f_1g_1+e_1f_1g_1=a_5\end{aligned} \quad (60.c)$$
$$\begin{aligned}&d_0e_0+d_0f_0+d_0g_0+e_0f_0+e_0g_0+f_0g_0+d_0e_1f_1+d_1e_0f_1+d_1e_1f_0+d_0e_1g_1+d_1e_0g_1\\&+d_1e_1g_0+d_0f_1g_1+d_1f_0g_1+d_1f_1g_0+e_0f_1g_1+e_1f_0g_1+e_1f_1g_0+d_1e_1f_1g_1=a_4\end{aligned} \quad (60.d)$$
$$\begin{aligned}&d_0e_0f_1+d_0e_1f_0+d_1e_0f_0+d_0e_0g_1+d_0e_1g_0+d_1e_0g_0+d_0f_0g_1+d_0f_1g_0+d_1f_0g_0+d_0f_0g_1\\&+e_0f_1g_0+e_1f_0g_0+d_0e_1f_1g_1+d_1e_0f_1g_1+d_1e_1f_0g_1+d_1e_1f_1g_0=a_3\end{aligned} \quad (60.e)$$
$$\begin{aligned}&d_0e_0f_0+d_0e_0g_0+d_0f_0g_0+e_0f_0g_0+d_0e_0f_1g_1+d_0e_1f_0g_0+d_1e_0f_0g_1\\&+d_1e_0f_0g_1+d_1e_0f_1g_0+d_1e_1f_0g_0+d_0e_1f_0g_1=a_2\end{aligned} \quad (60.f)$$
$$d_0e_0f_0g_1+d_0e_0f_1g_0+d_0e_1f_0g_0+d_1e_0f_0g_0=a_1 \quad (60.g)$$
$$d_0e_0f_0g_0=a_0 \quad (60.h)$$

Thus, a system of eight simultaneous nonlinear equations in eight unknowns $d_1,d_0,e_1,e_0,f_1,f_0,g_1$ and g_0 is formed.

2.9 Decomposition of a Nonic Function

2.9.1 The Proposed Decomposition

Consider a general nonic equation:

$$x^9+a_8x^8+a_7x^7+a_6x^6+a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0 \quad (61)$$

The nonic equation may be decomposed into a product of two equations i.e. one quartic equations and one quintic equation as shown below:

$$\left(x^4+d_3x^3+d_2x^2+d_1x+d_0\right)\cdot\left(x^5+e_4x^4+e_3x^3+e_2x^2+e_1x+e_0\right)=0 \quad (62)$$

where $d_3,d_2,d_1,d_0,e_4,e_3,e_2,e_1$ and e_0 are the unknown coefficients of the equations. Expanding the product in (62) and equating each coefficient to the corresponding term in the original equation in (61) leads to 9 equations:

$$d_3+e_4=a_8 \quad (63.a)$$
$$d_2+e_3+d_3e_4=a_7 \quad (63.b)$$
$$d_1+e_2+d_2e_4+d_3e_3=a_6 \quad (63.c)$$

$d_0 + e_1 + d_1e_4 + d_2e_3 + d_3e_2 = a_5$ (63.d)

$e_0 + d_0e_4 + d_1e_3 + d_2e_2 + d_3e_1 = a_4$ (63.e)

$d_0e_3 + d_1e_2 + d_2e_1 + d_3e_0 = a_3$ (63.f)

$d_0e_2 + d_1e_1 + d_2e_0 = a_2$ (63.g)

$d_0e_1 + d_1e_0 = a_1$ (63.h)

$d_0e_0 = a_0$ (63.i)

d_3 in (63.a), d_2 in (63.b), d_1 in (63.c) and d_0 in (63.i), respectively, can be rewritten in term of the other terms as:

$d_3 = a_8 - e_4$ (64.a)

$d_2 = a_7 - e_3 - (a_8 - e_4)\cdot e_4$ (64.b)

$d_1 = a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3$ (64.c)

$d_0 = \dfrac{a_0}{e_0}$ (64.d)

Thus, d_3, d_2, d_1 and d_0 can be eliminated from (63.d), (63.e), (63.f), (63.g) and (63.h) such that a system of five simultaneous nonlinear equations in five unknowns e_4, e_3, e_2, e_1 and e_0 is formed.

$$\frac{a_0}{e_0} + e_1 + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_4 \\ + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_3 + (a_8 - e_4)\cdot e_2 = a_5 \quad (65.a)$$

$$e_0 + \frac{a_0 \cdot e_4}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_3 \\ + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_2 + (a_8 - e_4)\cdot e_1 = a_4 \quad (65.b)$$

$$\frac{a_0 \cdot e_3}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_2 \\ + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_1 + (a_8 - e_4)\cdot e_0 = a_3 \quad (65.c)$$

$$\frac{a_0 \cdot e_2}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_1 + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_0 = a_2 \quad (65.d)$$

$$\frac{a_0 \cdot e_1}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_0 = a_1 \quad (65.e)$$

Once e_4, e_3, e_2, e_1 and e_0 are obtained by the Newton-Raphson method in five dimensions, d_3, d_2, d_1 and d_0 can be obtained by back substitution via (64.a), (64.b), (64.c) and (64.d), respectively.

2.9.2 Bairstow's Decomposition

The nonic equation of (61) can be rewritten in form of the Bairstow's decomposition as:

$(x^2 + d_1x + d_0)\cdot(x^7 + e_6x^6 + e_5x^5 + e_4x^4 + e_3x^3 + e_2x^2 + e_1x + e_0) = 0$ (66)

Expanding the product in (66) and equating each coefficient to the corresponding term in the original equation in (61) leads to 9 equations:

$d_1 + e_6 = a_8$ (67.a)

$d_0 + e_5 + d_1e_6 = a_7$ (67.b)

$e_4 + d_0e_6 + d_1e_5 = a_6$ (67.c)

$e_3 + d_0e_5 + d_1e_4 = a_5$ (67.d)

$e_2 + d_0e_4 + d_1e_3 = a_4$ (67.e)

$e_1 + d_0e_3 + d_1e_2 = a_3$ (67.f)

$e_0 + d_0e_2 + d_1e_1 = a_2$ (67.g)

$d_0e_1 + d_1e_0 = a_1$ (67.h)

$d_0e_0 = a_0$ (67.i)

Thus, a system of nine simultaneous nonlinear equations in nine unknowns $d_1, d_0, e_6, e_5, e_4, e_3, e_2, e_1$ and e_0 is formed.

2.9.3 Complete Bairstow's Decomposition

The nonic equation of (61) can be rewritten in form of a complete Bairstow's decomposition as:

$$(x+c_0)\cdot\left(x^2+d_1x+d_0\right)\cdot\left(x^2+e_1x+e_0\right)\cdot\left(x^2+f_1x+f_0\right)\cdot\left(x^2+g_1x+g_0\right)=0 \qquad (68)$$

Expanding the product in (68) and equating each coefficient to the corresponding term in the original equation in (61) leads to 9 equations:

$$c_0+d_1+e_1+f_1+g_1=a_8 \qquad (69.a)$$

$$d_0+e_0+f_0+g_0+c_0d_1+c_0e_1+c_0f_1+c_0g_1+d_1e_1+d_1f_1+d_1g_1+e_1f_1+e_1g_1+f_1g_1=a_7 \qquad (69.b)$$

$$\begin{aligned}&c_0d_0+c_0e_0+c_0f_0+c_0g_0+d_0e_1+d_1e_0+d_0f_1+d_1f_0+d_0g_1+d_1g_0+e_0f_1+e_1f_0+e_0g_1+e_1g_0\\&\quad c_0d_1e_1+c_0d_1f_1+c_0d_1g_1+c_0e_1f_1+c_0e_1g_1+c_0f_1g_1+d_1e_1f_1+d_1e_1g_1+d_1f_1g_1+e_1f_1g_1=a_6\end{aligned} \qquad (69.c)$$

$$\begin{aligned}&d_0e_0+d_0f_0+d_0g_0+e_0f_0+e_0g_0+f_0g_0+c_0d_0e_1+c_0d_1e_0+c_0d_0f_1+c_0d_1f_0+c_0d_0g_1+c_0d_1g_0\\&\quad+c_0e_0f_1+c_0e_1f_0+c_0e_0g_1+c_0e_1g_0+c_0f_0g_1+c_0f_1g_0+d_0e_1f_1+d_1e_0f_1+d_1e_1f_0+d_0e_1g_1\\&\quad+d_1e_0g_1+d_1e_1g_0+d_0f_1g_1+d_1f_0g_1+d_1f_1g_0+e_0f_1g_1+e_1f_0g_1+e_1f_1g_0+c_0d_1e_1f_1+c_0d_1e_1f_1\\&\quad+c_0d_1e_1g_1+c_0d_1f_1g_1+d_1e_1f_1g_1=a_5\end{aligned} \qquad (69.d)$$

$$\begin{aligned}&c_0d_0e_0+c_0d_0f_0+c_0d_0g_0+c_0e_0f_0+c_0e_0g_0+c_0f_0d_{g_0}+d_0e_0f_1+d_0e_1f_0+d_1e_0f_0+d_0e_0g_1\\&\quad+d_1e_0f_1g_1+d_0e_1g_0+d_1e_0g_0+d_0f_0g_1+d_0f_1g_0+d_1f_0g_0+e_0f_0g_1+e_0f_1g_0+e_1f_0g_0\\&\quad+d_0e_1f_1g_1+d_1e_1f_0g_1+d_1e_1f_1g_0+c_0d_0e_1f_1+c_0d_1e_0f_1+c_0d_1e_1f_0+c_0d_1f_0g_1+c_0d_1e_0g_1\\&\quad+c_0d_1e_1g_0+c_0d_0f_1g_1+c_0d_1f_0g_1+c_0d_1f_1g_0+c_0e_0f_1g_1+c_0e_1f_0g_1+c_0e_1f_1g_0\\&\quad+c_0d_1e_0f_1g_1=a_4\end{aligned} \qquad (69.e)$$

$$\begin{aligned}&d_0e_0f_0+d_0e_0g_0+d_0f_0g_0+e_0f_0g_0+d_0e_0f_1g_1+d_0e_1f_1g_0+d_1e_0f_0g_1+d_1e_0f_1g_0+d_1e_1f_0g_0\\&\quad+d_0f_0e_1g_1+c_0d_0e_0f_1+c_0d_0e_1f_0+c_0d_1e_0f_0+c_0d_0e_0g_1+c_0d_0e_1g_0+c_0d_1e_0g_0+c_0d_0f_0g_1\\&\quad+c_0d_0f_1g_0+c_0d_1f_0g_0+c_0e_0f_0g_1+c_0e_0f_1g_0+c_0e_1f_0g_0+c_0d_0e_1f_1g_1+c_0d_1e_0f_1g_1\\&\quad+c_0d_1e_1f_0g_1+c_0d_1e_1f_1g_0=a_3\end{aligned} \qquad (69.f)$$

$$\begin{aligned}&c_0d_0e_0f_0+c_0d_0e_0g_0+c_0d_0f_0g_0+c_0e_0f_0g_0+d_0e_0f_0g_1+d_0e_0f_1g_0+d_0e_1f_0g_0+d_1e_0f_0g_0\\&\quad+c_0d_0e_0f_1g_1+c_0d_0e_1f_0g_1+c_0d_0e_1f_1g_0+c_0d_1e_0f_0g_1+c_0d_1e_0f_1g_0+c_0d_1e_1f_0g_0=a_2\end{aligned} \qquad (69.g)$$

$$d_0e_0f_0g_0+c_0d_0e_0f_0g_1+c_0d_0e_0f_1g_0+c_0d_0e_1f_0g_0+c_0d_1e_0f_0g_0=a_1 \qquad (69.h)$$

$$c_0d_0e_0f_0g_0=a_0 \qquad (69.i)$$

Thus, a system of nine simultaneous nonlinear equations in nine unknowns $c_0, d_1, d_0, e_1, e_0, f_1, f_0, g_1$ and g_0 is formed.

III. N-D Newton-Raphson Method and the *Jacobian* of the Functions

3.1 Newton-Raphson Method for a Nonlinear Function

The prediction of the Newton-Raphson method was based on a first order Taylor series expansion:

$$f(x_{i+1})=f(x_i)+(x_{i+1}-x_i)\cdot f'(x_i) \qquad (70)$$

where x_i is the initial guess at the root or the previous estimate of the root and x_{i+1} is the point at which the slope intercepts the x axis. At this intercept $f(x_{i+1})=0$ by definition and (70) can be rearranged to:

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)} \qquad (71)$$

which is the Newton-Raphson method for a nonlinear equation.

3.2 Newton-Raphson Method in more than one Dimension

3.2.1 Newton-Raphson Method in two Dimensions

The Newton-Raphson method for two simultaneous nonlinear equations can be derived in the similar fashion. However, a multivariate Taylor series has to be taken into account for the fact of more than one independent variables contributing to the determination of the root. For the two-variable case, a first order Taylor series can be written for each nonlinear equation as:

$$u_{i+1}=u_i+(x_{i+1}-x_i)\frac{\partial u_i}{\partial x}+(y_{i+1}-y_i)\frac{\partial u_i}{\partial y} \qquad (72.a)$$

$$v_{i+1}=v_i+(x_{i+1}-x_i)\frac{\partial v_i}{\partial x}+(y_{i+1}-y_i)\frac{\partial v_i}{\partial y} \qquad (72.b)$$

Just as for the single-equation case, the root estimate corresponds to the points which $u_{i+1}=0$ and $v_{i+1}=0$. (72.a) and (72.b) can be rearranged to:

$$\frac{\partial u_i}{\partial x}x_{i+1}+\frac{\partial u_i}{\partial y}y_{i+1}=-u_i+x_i\frac{\partial u_i}{\partial x}+y_i\frac{\partial u_i}{\partial y} \tag{73.a}$$

$$\frac{\partial v_i}{\partial x}x_{i+1}+\frac{\partial v_i}{\partial y}y_{i+1}=-v_i+x_i\frac{\partial v_i}{\partial x}+y_i\frac{\partial v_i}{\partial y} \tag{73.b}$$

In (73) only x_{i+1} and y_{i+1} are unknown. Thus (73) is a set of two simultaneous linear equations with two unknowns. Consequently, with simple algebraic manipulations, e.g. Cramer's rule, x_{i+1} and y_{i+1} can be solved as:

$$x_{i+1}=x_i-\frac{u_i\cdot\frac{\partial v_i}{\partial y}-v_i\cdot\frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x}\cdot\frac{\partial v_i}{\partial y}-\frac{\partial u_i}{\partial y}\cdot\frac{\partial v_i}{\partial x}} \tag{74.a}$$

$$y_{i+1}=y_i-\frac{u_i\cdot\frac{\partial v_i}{\partial x}-v_i\cdot\frac{\partial u_i}{\partial x}}{\frac{\partial u_i}{\partial x}\cdot\frac{\partial v_i}{\partial y}-\frac{\partial u_i}{\partial y}\cdot\frac{\partial v_i}{\partial x}} \tag{74.b}$$

The denominator of each of (74) is the determinant of the *Jacobian* of the system. (73) is the equation for the Newton-Raphson method in two dimensions.

For the benefits of further discussions on the method for more than two dimensions, (74) should be rewritten in term of the matrix notation i.e. the *Jacobian* of the function – $[Z]$.

$$\begin{Bmatrix}x_{i+1}\\ y_{i+1}\end{Bmatrix}=\begin{Bmatrix}x_i\\ y_i\end{Bmatrix}-[Z]^{-1}\cdot\begin{Bmatrix}u(x_i,y_i)\\ v(x_i,y_i)\end{Bmatrix} \tag{75}$$

$$[Z]=\begin{bmatrix}\frac{\partial u(x_i,y_i)}{\partial x} & \frac{\partial v(x_i,y_i)}{\partial x}\\ \frac{\partial u(x_i,y_i)}{\partial y} & \frac{\partial v(x_i,y_i)}{\partial y}\end{bmatrix} \tag{76}$$

where $[Z]$ is the *Jacobian* of the function and $[Z]^{-1}$ is the inverse of $[Z]$.

3.2.2 Newton-Raphson Method in More Than Two Dimensions

Consider a system of n simultaneous nonlinear equations:

$$\begin{aligned} &f1(x_1,x_2,\ldots,x_n)=0\\ &f2(x_1,x_2,\ldots,x_n)=0\\ &\quad\vdots\\ &fn(x_1,x_2,\ldots,x_n)=0\end{aligned} \tag{77}$$

The solution of this system consists of a set of x values that simultaneously result in all the equations equaling zero. Just for the case of two nonlinear equations a Taylor series expansion is written for each equation

$$f_{l,i+1}=f_{l,i}+\left(x_{1,i+1}-x_{1,i}\right)\frac{\partial f_{l,i}}{\partial x_1}+\left(x_{2,i+1}-x_{2,i}\right)\frac{\partial f_{l,i}}{\partial x_2}+\cdots+\left(x_{n,i+1}-x_{n,i}\right)\frac{\partial f_{l,i}}{\partial x_n} \tag{78}$$

where the subscript, l, represents the equation or unknown and the second subscript denotes whether the value of function under consideration is at the present value (i) or at the next vale ($i+1$).

Equations in the form of (78) are written for each of the original nonlinear equations. All $f_{l,i+1}$ terms are set to zero and (78) can be rewritten as:

$$-f_{l,i}+x_{1,i}\frac{\partial f_{l,i}}{\partial x_1}+x_{2,i}\frac{\partial f_{l,i}}{\partial x_2}+\cdots+x_{n,i}\frac{\partial f_{l,i}}{\partial x_n}=x_{1,i+1}\frac{\partial f_{l,i}}{\partial x_1}+x_{2,i+1}\frac{\partial f_{l,i}}{\partial x_2}+\cdots+x_{n,i+1}\frac{\partial f_{l,i}}{\partial x_n} \tag{79}$$

Notice that only the $x_{j,i+1}, j=1,2,\ldots,n$ terms on the right-hand side are unknowns. As a result, a set of n linear simultaneous equation is obtained.

The partial derivatives can be expressed in term of matrix notation as:

$$[Z]=\begin{bmatrix} \frac{\partial f_{1,i}}{\partial x_1} & \frac{\partial f_{1,i}}{\partial x_2} & \cdots & \frac{\partial f_{1,i}}{\partial xn} \\ \frac{\partial f_{2,i}}{\partial x_1} & \frac{\partial f_{2,i}}{\partial x_2} & \cdots & \frac{\partial f_{2,i}}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_{n,i}}{\partial x_1} & \frac{\partial f_{n,i}}{\partial x_2} & \cdots & \frac{\partial f_{n,i}}{\partial x_n} \end{bmatrix} \tag{80}$$

The present and the next values can be expressed in the vector form as

$$\{X_i\}^T=\{x_{1,i} \quad x_{2,i} \quad \cdots \quad x_{n,i}\} \tag{81}$$

$$\{X_{i+1}\}^T=\{x_{1,i+1} \quad x_{2,i+1} \quad \cdots \quad x_{n,i+1}\} \tag{82}$$

The function values at i can be expressed as

$$\{F_i\}^T=\{f_{1,i} \quad f_{2,i} \quad \cdots \quad f_{n,i}\} \tag{83}$$

Using these relationships, (79) can be rewritten as

$$[Z]\{X_{i+1}\}=\{W_i\} \tag{84}$$

$$\{W_i\}=-\{F_i\}+[Z]\{X_i\} \tag{85}$$

Assumed that the inverse of $[Z]$ can be obtained. Then $\{X_{i+1}\}$ in (84) can be solved.

$$\{X_{i+1}\}=[Z]^{-1}\cdot\{W_i\}=[Z]^{-1}\cdot(-\{F_i\}+[Z]\{X_i\})=-[Z]^{-1}\cdot\{F_i\}+[Z]^{-1}\cdot[Z]\cdot\{X_i\} \tag{86}$$

In (86) $[Z]^{-1}\cdot[Z]=[I]$ is a unit matrix. Thus, (86) can be rewritten as:

$$\{X_{i+1}\}=\{X_i\}-[Z]^{-1}\cdot\{F_i\} \tag{87}$$

3.3 Functions from the Decomposed Equations and the *Jacobian* of the Functions

For the benefits of applications the system of simultaneous nonlinear equations derived for the decomposed equations discussed in Section 2.3 – Section 2.9 are summarized in the following subsections.

3.3.1 Original Cubic Equations

$$\{F\}=\begin{Bmatrix} \left(\frac{a_0}{e_0}+e_1-a_2\right) \\ \left(e_0+\frac{a_0\cdot e_1}{e_0}-a_1\right) \end{Bmatrix} \tag{88}$$

$$[Z]=\begin{bmatrix} \left(\frac{a_0}{e_0{}^2}\right) & (1) \\ \left(1-\frac{a_0\cdot e_1}{e_0{}^2}\right) & \left(\frac{a_0}{e_0}\right) \end{bmatrix} \tag{89}$$

3.3.2 Original Quartic Equations

$$\{F\}=\begin{Bmatrix} \left(\frac{a_0}{e_0}+e_0+(a_3-e_1)\cdot e_1-a_2\right) \\ \left(\frac{a_0\cdot e_1}{e_0}+(a_3-e_1)\cdot e_0-a_1\right) \end{Bmatrix} \tag{90}$$

$$[Z]=\begin{bmatrix} \left(1-\frac{a_0}{e_0{}^2}\right) & (a_3-2e_1) \\ \left(a_3-e_1-\frac{a_0\cdot e_1}{e_0{}^2}\right) & \left(\frac{a_0}{e_0}-e_0\right) \end{bmatrix} \tag{91}$$

3.3.3 Original Quintic Equations
3.3.3.1 The Proposed Decomposition

$$\{F\}=\begin{Bmatrix} \left(\dfrac{a_0}{e_0}+e_1+(a_4-e_2)\cdot e_2-a_3\right) \\ \left(e_0+\dfrac{a_0\cdot e_2}{e_0}+(a_4-e_2)\cdot e_1-a_2\right) \\ \left(\dfrac{a_0\cdot e_1}{e_0}+(a_4-e_2)\cdot e_0-a_1\right) \end{Bmatrix} \tag{92}$$

$$[Z]=\begin{bmatrix} \left(-\dfrac{a_0}{{e_0}^2}\right) & (1) & (a_4-2e_2) \\ \left(1-\dfrac{a_0\cdot e_2}{{e_0}^2}\right) & (a_4-e_2) & \left(\dfrac{a_0}{e_0}-e_1\right) \\ \left(-\dfrac{a_0\cdot e_1}{{e_0}^2}+a_4-e_2\right) & \left(\dfrac{a_0}{e_0}\right) & (-e_0) \end{bmatrix} \tag{93}$$

3.3.3.2 Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} d_1+e_1-a_4 \\ d_0+e_1+d_1\cdot e_2-a_3 \\ e_0+d_0\cdot e_2+d_1\cdot e_1-a_2 \\ d_0\cdot e_1+d_0\cdot e_1-a_1 \\ d_0\cdot e_0-a0 \end{Bmatrix} \tag{94}$$

$$[Z]=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & e_2 & 0 & 1 & d_1 \\ e_2 & e_1 & 1 & d_1 & d_0 \\ e_1 & e_0 & d_1 & d_0 & 0 \\ e_0 & 0 & d_0 & 0 & 0 \end{bmatrix} \tag{95}$$

3.3.3.3 Complete Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} c_0+d_1+e_1-a_4 \\ d_0+e_0+c_0\cdot d_1+c_0\cdot e_1+d_1\cdot e_1-a_3 \\ c_0\cdot d_0+c_0\cdot e_0+d_0\cdot e_1+d_1\cdot e_0+c_0\cdot d_1\cdot e_1-a_2 \\ d_0\cdot e_0+c_0\cdot d_0\cdot e_1+c_0\cdot d_1\cdot e_0-a_1 \\ c_0\cdot d_0\cdot e_0-a0 \end{Bmatrix} \tag{96}$$

$$[Z]=\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ d_1+e_1 & 1 & c_0+e_1 & 1 & c_0+d_1 \\ d_0+e_0+d_1e_1 & c_0+e_1 & e_0+c_0e_1 & c_0+d_1 & d_0+c_0d_1 \\ d_0e_1+d_1e_0 & e_0+c_0e_1 & c_0e_0 & d_0+c_0d_1 & c_0d_0 \\ d_0e_0 & c_0e_0 & 0 & c_0d_0 & 0 \end{bmatrix} \tag{97}$$

3.3.4 Original Sextic Equations

3.3.4.1 The Proposed Decomposition and Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} \left(\frac{a_0}{e_0}+e_2+(a_5-e_3)\cdot e_3-a_4\right) \\ \left((a_5-e_3)\cdot e_2+e_1+\frac{a_0\cdot e_3}{e_0}-a_3\right) \\ \left(e_0+\frac{a_0\cdot e_2}{e_0}+(a_5-e_3)\cdot e_1-a_2\right) \\ \left(\frac{a_0\cdot e_1}{e_0}+(a_5-e_3)\cdot e_0-a_1\right) \end{Bmatrix} \tag{98}$$

$$[Z]=\begin{bmatrix} \left(-\frac{a_0}{e_0^{\ 2}}\right) & (0) & (1) & -2e_3 \\ \left(-\frac{a_0\cdot e_3}{e_0^{\ 2}}\right) & (1) & (a_5-e_3) & \left(\frac{a_0}{e_0}\right) \\ \left(1-\frac{a_0\cdot e_2}{e_0^{\ 2}}\right) & (a_5-e_3) & \left(\frac{a_0}{e_0}\right) & (-e_1) \\ \left(-\frac{a_0\cdot e_1}{e_0^{\ 2}}+a_5-e_3\right) & \left(\frac{a_0}{e_0}\right) & (0) & (-e_0) \end{bmatrix} \tag{99}$$

3.3.4.2 Complete Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} d_1+e_1+f_1-a_5 \\ d_0+e_0+f_0+d_1e_1+d_1f_1+e_1f_1-a_4 \\ d_0e_1+d_1e_0+d_0f_1+d_1f_0+e_0f_1+e_1f_0+d_1e_1f_1-a_3 \\ d_0e_0+d_0f_0+e_0f_0+d_0e_1f_1+d_1e_0f_1+d_1e_1f_0-a_2 \\ d_0e_0f_1+d_0e_1f_0+d_1e_0f_0-a_1 \\ d_0e_0f_0-a_0 \end{Bmatrix} \tag{100}$$

$$[Z]=\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & e_1+f_1 & 1 & d_1+f_1 & 1 & d_1+e_1 \\ e_1+f_1 & e_0+f_0+e_1f_1 & d_1+f_1 & d_0+f_0+d_1f_1 & d_1+e_1 & d_0+e_0+d_1e_1 \\ e_0+f_0+e_1f_1 & e_0f_1+e_1f_0 & d_0+f_0+d_1f_1 & d_0f_1+d_1f_0 & d_0+e_0+d_1e_1 & d_0e_1+d_1e_0 \\ e_0f_1+e_1f_0 & e_0f_0 & d_0f_1+d_1f_0 & d_0f_0 & d_0e_1+d_1e_0 & d_0e_0 \\ e_0f_0 & 0 & d_0f_0 & 0 & d_0e_0 & 0 \end{bmatrix} \tag{101}$$

3.3.5 Original Septic Equations

3.3.5.1 The Proposed Decomposition

$$\{F\}=\begin{Bmatrix} \left(\frac{a_0}{e_0}+e_1+\{a_5-e_2-(a_6-e_3)\cdot e_3\}\cdot e_3+(a_6-e_3)\cdot e_2-a_4\right) \\ \left(e_0+\frac{a_0\cdot e_3}{e_0}+\{a_5-e_2-(a_6-e_3)\cdot e_3\}\cdot e_2+(a_6-e_3)\cdot e_1-a_3\right) \\ \left(\frac{a_0\cdot e_2}{e_0}+\{a_5-e_2-(a_6-e_3)\cdot e_3\}\cdot e_1+(a_6-e_3)\cdot e_0-a_2\right) \\ \left(\frac{a_0\cdot e_1}{e_0}+\{a_5-e_2-(a_6-e_3)\cdot e_3\}\cdot e_0-a_1\right) \end{Bmatrix} \tag{102}$$

$$[Z]=\begin{bmatrix} \left(-\frac{a_0}{e_0^2}\right) & (1) & (a_6-2e_3) & (a_5-2e_2-2(a_6-e_3)\cdot e_3-e_1) \\ \left(1-\frac{a_0\cdot e_3}{e_0^2}\right) & (a_6-e_3) & (a_5-2e_2-(a_6-e_3)\cdot e_3) & \left(\frac{a_0}{e_0}-e_1-a_6\cdot e_2+2e_2\cdot e_3\right) \\ \left(-\frac{a_0\cdot e_2}{e_0^2}+a_6-e_3\right) & (a_5-e_2-(a_6-e_3)\cdot e_3) & \left(\frac{a_0}{e_0}-e_1\right) & (-e_0-a_6\cdot e_1+2e_1\cdot e_3) \\ \left(-\frac{a_0\cdot e_1}{e_0^2}+a_5-e_2-(a_6-e_3)\cdot e_3\right) & \left(\frac{a_0}{e_0}\right) & (-e_0) & (-a_6\cdot e_0+2e_0\cdot e_3) \end{bmatrix} \quad (103)$$

3.3.5.2 Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} d_1+e_4-a_6 \\ d_0+e_3+d_1e_4-a_5 \\ e_2+d_0e_4+d_1e_3-a_4 \\ e_1+d_0e_3+d_1e_2-a_3 \\ e_0+d_0e_2+d_1e_1-a_2 \\ d_0e_1+d_1e_0-a_1 \\ d_0e_0-a_0 \end{Bmatrix} \quad (104)$$

$$[Z]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & e_4 & 0 & 0 & 0 & 1 & d_1 \\ e_4 & e_3 & 0 & 0 & 1 & d_1 & d_0 \\ e_3 & e_2 & 0 & 1 & d_1 & d_0 & 0 \\ e_2 & e_1 & 1 & d_1 & d_0 & 0 & 0 \\ e_1 & e_0 & d_1 & d_0 & 0 & 0 & 0 \\ e_0 & 0 & d_0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (105)$$

3.3.5.3 Complete Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} (c_0+d_1+e_1+f_1-a_6) \\ (d_0+e_0+f_0+c_0d_1+c_0e_1+c_0f_1+d_1e_1+d_1f_1+e_1f_1-a_5) \\ (c_0d_0+c_0e_0+c_0f_0+d_0e_1+d_1e_0+d_0f_1+d_1f_0+e_0f_1+e_1f_0+c_0d_1e_1+c_0d_1f_1+c_0e_1f_1+d_1e_1f_1-a_4) \\ \left(\begin{matrix} d_0e_0+d_0f_0+e_0f_0+c_0d_0e_1+c_0d_1e_0+c_0d_0f_1+c_0d_1f_0+c_0e_0f_1+c_0e_1f_0+d_0e_1f_1+d_1e_0f_1+d_1e_1f_0 \\ +c_0d_1e_1f_1-a_3 \end{matrix}\right) \\ (c_0d_0e_0+c_0d_0f_0+c_0e_0f_0+d_0e_0f_1+d_0e_1f_0+d_1e_0f_0+c_0d_0e_1f_1+c_0d_1e_0f_1+c_0d_1e_1f_0-a_2) \\ (d_0e_0f_0+c_0d_0e_0f_1+c_0d_0e_1f_0+c_0d_1e_0f_0-a_1) \\ (c_0d_0e_0f_0-a_0) \end{Bmatrix} \quad (106)$$

$$[Z]=\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ d_1+e_1+f_1 & 1 & c_0+e_1+f_1 & 1 & c_0+d_1+e_1 & 1 & c_0+d_1+e_1 \\ z_{3,1} & c_0+e_1+f_1 & z_{3,3} & c_0+d_1+f_1 & z_{3,5} & c_0+d_1+e_1 & z_{3,7} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} & z_{4,7} \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} & z_{5,6} & z_{5,7} \\ z_{6,1} & z_{6,2} & c_0e_0f_0 & z_{6,4} & c_0d_0f_0 & z_{6,6} & c_0d_0e_0 \\ d_0e_0f_0 & c_0e_0f_0 & 0 & c_0d_0f_0 & 0 & c_0d_0e_0 & 0 \end{bmatrix} \quad (107)$$

$$z_{3,1}=d_0+e_0+f_0+d_1e_1+d_1f_1+e_1f_1 \quad (108.a)$$

$$z_{3,3}=e_0+f_0+c_0e_1+c_0f_1+e_1f_1 \quad (108.b)$$

$$z_{3,5}=d_0+f_0+c_0d_1+c_0f_1+d_1f_1 \quad (108.c)$$

$$z_{3,7}=e_0+f_0+d_0e_1+d_0f_1+e_1f_1 \quad (108.d)$$

$$z_{4,1}=d_0e_1+d_1e_0+d_0f_1+d_1f_0+e_0f_1+e_1f_0+d_1e_1f_1 \quad (108.e)$$

$$z_{4,2}=e_0+f_0+c_0e_1+c_0f_1+e_1f_1 \quad (108.f)$$

$$z_{4,3}=c_0e_0+c_0f_0+e_0f_1+e_1f_0+c_0e_1f_1 \quad (108.g)$$

$$z_{4,4} = d_0 + f_0 + c_0 d_1 + c_0 f_1 + d_1 f_1 \quad (108.h)$$
$$z_{4,5} = c_0 d_0 + c_0 f_0 + d_0 f_1 + d_1 f_0 + c_0 d_1 f_1 \quad (108.i)$$
$$z_{4,6} = e_0 + f_0 + d_0 e_1 + d_0 f_1 + e_1 f_1 \quad (108.j)$$
$$z_{4,7} = d_0 e_0 + d_0 f_0 + e_0 f_1 + e_1 f_0 + d_0 e_1 f_1 \quad (108.k)$$
$$z_{5,1} = d_0 e_0 + d_0 f_0 + e_0 f_0 + d_0 e_1 f_1 + d_1 e_0 f_1 + d_1 e_1 f_0 \quad (108.l)$$
$$z_{5,2} = c_0 e_0 + c_0 f_0 + e_0 f_1 + e_1 f_0 + c_0 e_1 f_1 \quad (108.m)$$
$$z_{5,3} = e_0 f_0 + c_0 e_0 f_1 + c_0 e_1 f_0 \quad (108.n)$$
$$z_{5,4} = c_0 d_0 + c_0 f_0 + d_0 f_1 + d_1 f_0 + c_0 d_1 f_1 \quad (108.o)$$
$$z_{5,6} = d_0 e_0 + d_0 f_0 + e_0 f_1 + e_1 f_0 + d_0 e_1 f_1 \quad (108.q)$$
$$z_{5,7} = e_0 f_0 + d_0 e_0 f_1 + d_0 e_1 f_0 \quad (108.r)$$
$$z_{6,1} = d_0 e_0 f_1 + d_0 e_1 f_0 + d_1 e_0 f_0 \quad (108.s)$$
$$z_{6,2} = e_0 f_0 + c_0 e_0 f_1 + c_0 e_1 f_0 \quad (108.t)$$
$$z_{6,4} = d_0 f_0 + c_0 d_0 f_1 + c_0 d_1 f_0 \quad (108.u)$$
$$z_{6,6} = e_0 f_0 + d_0 e_0 f_1 + d_0 e_1 f_0 \quad (108.v)$$

3.3.6 Original Octic Equations

3.3.6.1 The Proposed Decomposition

$$\{F\} = \left\{ \begin{matrix} \left(\begin{matrix} \frac{a_0}{e_0} + e_0 + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_3 + (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_2 \\ + (a_7 - e_3)\cdot e_1 - a_4 \end{matrix} \right) \\ \left(\begin{matrix} \frac{a_0 \cdot e_3}{e_0} + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_2 + (a_7 - e_3)\cdot e_0 \\ + (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_1 - a_3 \end{matrix} \right) \\ \left(\frac{a_0 \cdot e_2}{e_0} + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_1 + (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_0 - a_2 \right) \\ \left(\frac{a_0 \cdot e_1}{e_0} + \{a_5 - e_1 - (a_6 - e_2 - (a_7 - e_3)\cdot e_3)\cdot e_3 - (a_7 - e_3)\cdot e_2\}\cdot e_0 - a_1 \right) \end{matrix} \right\} \quad (109)$$

$$[Z] = \begin{bmatrix} \left(-\frac{a_0}{e_0^{\,2}} + 1 \right) & (a_7 - 2e_3) & \left(-2e_2 + 3e_3^{\,2} - 2a_7 \cdot e_3 + a_6 \right) & z_{1,4} \\ \left(a_7 - e_3 - \frac{a_0 \cdot e_3}{e_0^{\,2}} \right) & \left(+ a_6 - 2e_2 - a_7 \cdot e_3 + e_3^{\,2} \right) & z_{2,3} & z_{2,4} \\ \left(-\frac{a_0 \cdot e_2}{e_0^{\,2}} + a_6 - e_2 - a_7 \cdot e_3 + e_3^{\,2} \right) & z_{3,2} & \left(\frac{a_0}{e_0} - e_0 - a_7 \cdot e_1 + 2e_1 \cdot e_3 \right) & z_{3,4} \\ z_{4,1} & \left(\frac{a_0}{e_0} - e_0 \right) & \left(- a_7 \cdot e_0 + 2e_0 \cdot e_3 \right) & z_{4,4} \end{bmatrix} \quad (110)$$

$$z_{1,4} = a_5 - 2e_1 - 2a_7 \cdot e_2 - 2a_6 \cdot e_3 + 3a_7 \cdot e_3^{\,2} - 4e_3^{\,3} + 6e_2 \cdot e_3 \quad (111.a)$$
$$z_{2,3} = a_5 - 2e_1 - 2a_7 \cdot e_2 - a_6 \cdot e_3 + a_7 \cdot e_3^{\,2} + 4e_2 \cdot e_3 - e_3^{\,3} \quad (111.b)$$
$$z_{2,4} = \frac{a_0}{e_0} - e_0 - a_7 \cdot e_1 - a_6 \cdot e_2 + 2e_2^{\,2} + 2e_1 \cdot e_3 + 2a_7 \cdot e_2 \cdot e_3 - 3e_2 \cdot e_3^{\,2} \quad (111.c)$$
$$z_{3,2} = a_5 - 2e_1 - a_7 \cdot e_2 - a_6 \cdot e_3 + a_7 \cdot e_3^{\,2} + 2e_2 \cdot e_3 - e_3^{\,3} \quad (111.d)$$
$$z_{3,4} = -a_7 \cdot e_0 - a_6 \cdot e_1 + 2e_0 \cdot e_3 + 2e_1 \cdot e_2 + 2a_7 \cdot e_1 \cdot e_3 - 3e_1 \cdot e_3^{\,2} \quad (111.e)$$
$$z_{4,1} = -\frac{a_0 \cdot e_1}{e_0^{\,2}} + a_5 - e_1 - a_7 \cdot e_2 - a_6 \cdot e_3 + a_7 \cdot e_3^{\,2} - e_3^{\,3} + 2e_2 \cdot e_3 \quad (111.f)$$
$$z_{4,4} = -a_6 \cdot e_0 + 2e_0 \cdot e_2 + 2a_7 \cdot e_0 \cdot e_3 - 3e_0 \cdot e_3^{\,2} \quad (111.g)$$

3.3.6.2 Bairstow's Decomposition

$$\{F\}=\left\{\begin{matrix} d_1+e_5-a_7 \\ d_0+e_4+d_1e_5-a_6 \\ e_3+d_0e_5+d_1e_4-a_5 \\ e_2+d_0e_4+d_1e_3-a_4 \\ e_1+d_0e_3+d_1e_2-a_3 \\ e_0+d_0e_2+d_1e_1-a_2 \\ d_0e_1+d_1e_0-a_1 \\ d_0e_0-a_0 \end{matrix}\right\} \tag{112}$$

$$[Z]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & e_5 & 0 & 0 & 0 & 0 & 1 & d_1 \\ e_5 & e_4 & 0 & 0 & 0 & 1 & d_1 & d_0 \\ e_4 & e_3 & 0 & 0 & 1 & d_1 & d_0 & 0 \\ e_3 & e_2 & 0 & 1 & d_1 & d_0 & 0 & 0 \\ e_2 & e_1 & 1 & d_1 & d_0 & 0 & 0 & 0 \\ e_1 & e_0 & d_1 & d_0 & 0 & 0 & 0 & 0 \\ e_0 & 0 & d_0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{113}$$

3.3.6.3 Complete Bairstow's Decomposition

$$\{F\}=\left\{\begin{matrix} \left(d_1+e_1+f_1+g_1-a_7\right) \\ \left(d_0+e_0+f_0+g_0+d_1e_1+d_1f_1+d_1g_1+e_1f_1+e_1g_1+f_1g_1-a_6\right) \\ \left(\begin{matrix} d_0e_1+d_1e_0+d_0f_1+d_1f_0+d_0g_1+d_1g_0+e_0f_1+e_1f_0+e_0g_1+e_1g_0+f_0g_1+f_1g_0 \\ +d_1e_1f_1+d_1e_1g_1+d_1f_1g_1+e_1f_1g_1-a_5 \end{matrix}\right) \\ \left(\begin{matrix} d_0e_0+d_0f_0+d_0g_0+e_0f_0+e_0g_0+f_0g_0+d_0e_1f_1+d_1e_0f_1+d_1e_1f_0+d_0e_1g_1+d_1e_0g_1 \\ +d_1e_1g_0+d_0f_1g_1+d_1f_0g_1+d_1f_1g_0+e_0f_1g_1+e_1f_0g_1+e_1f_1g_0+d_1e_1f_1g_1-a_4 \end{matrix}\right) \\ \left(\begin{matrix} d_0e_0f_1+d_0e_1f_0+d_1e_0f_0+d_0e_0g_1+d_0e_1g_0+d_1e_0g_0+d_0f_0g_1+d_0f_1g_0+d_1f_0g_0+d_0f_0g_1 \\ +e_0f_1g_0+e_1f_0g_0+d_0e_1f_1g_1+d_1e_0f_1g_1+d_1e_1f_0g_1+d_1e_1f_1g_0-a_3 \end{matrix}\right) \\ \left(\begin{matrix} d_0e_0f_0+d_0e_0g_0+d_0f_0g_0+e_0f_0g_0+d_0e_0f_1g_1+d_0e_1f_0g_0+d_1e_0f_0g_1 \\ +d_1e_0f_0g_1+d_1e_0f_1g_0+d_1e_1f_0g_0+d_0e_1f_0g_1-a_2 \end{matrix}\right) \\ \left(d_0e_0f_0g_1+d_0e_0f_1g_0+d_0e_1f_0g_0+d_1e_0f_0g_0-a_1\right) \\ \left(d_0e_0f_0g_0-a_0\right) \end{matrix}\right\} \tag{114}$$

$$[Z]=\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & e_1+f_1+g_1 & 1 & d_1+f_1+g_1 & 1 & d_1+e_1+g_1 & 1 & d_1+e_1+f_1 \\ e_1+f_1+g_1 & z_{3,2} & d_1+f_1+g_1 & z_{3,4} & d_1+e_1+g_1 & z_{3,6} & d_1+e_1+f_1 & z_{3,8} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} & z_{4,7} & z_{4,8} \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} & z_{5,6} & z_{5,7} & z_{5,8} \\ z_{6,1} & z_{6,2} & z_{6,3} & z_{6,4} & z_{6,5} & z_{6,6} & z_{6,7} & z_{6,8} \\ z_{7,1} & e_0f_0g_0 & z_{7,3} & d_0f_0g_0 & z_{7,5} & d_0e_0g_0 & z_{7,7} & d_0e_0f_0 \\ e_0f_0g_0 & 0 & d_0f_0g_0 & 0 & d_0e_0g_0 & 0 & d_0e_0f_0 & 0 \end{bmatrix} \tag{115}$$

$$z_{3,2}=e_0+f_0+g_0+e_1f_1+e_1g_1+f_1g_1 \tag{116.a}$$

$$z_{3,4}=d_0+f_0+g_0+d_1f_1+d_1g_1+f_1g_1 \tag{116.b}$$

$$z_{3,6}=d_0+e_0+g_0+d_1e_1+d_1g_1+e_1g_1 \tag{116.c}$$

$$z_{3,8}=d_0+e_0+f_0+d_1e_1+d_1f_1+e_1f_1 \tag{116.d}$$

$$z_{4,1}=e_0+f_0+g_0+e_1f_1+e_1g_1+f_1g_1 \tag{116.e}$$

$$z_{4,2}=e_0f_1+e_1f_0+e_0g_1+e_1g_0+f_0g_1+f_1g_0+e_1f_1g_1 \tag{116.f}$$

$$z_{4,3}=d_0+f_0+g_0+d_1f_1+d_1g_1+f_1g_1 \tag{116.g}$$

$$z_{4,4}=d_0f_1+d_1f_0+d_0g_1+d_1g_0+f_0g_1+f_1g_0+d_1f_1g_1 \tag{116.h}$$

$$z_{4,5}=d_0+e_0+g_0+d_1e_1+d_1g_1+e_1g_1 \tag{116.i}$$

$$z_{4,6} = d_0e_1 + d_1e_0 + d_0g_1 + d_1g_0 + e_0g_1 + e_1g_0 + d_1e_1g_1 \tag{116.j}$$
$$z_{4,7} = d_0 + e_0 + f_0 + d_1e_1 + d_1f_1 + e_1f_1 \tag{116.k}$$
$$z_{4,8} = d_0e_1 + d_1e_0 + d_0f_1 + d_1f_0 + e_0f_1 + e_1f_0 + d_1e_1f_1 \tag{116.l}$$
$$z_{5,1} = e_0f_1 + e_1f_0 + e_0g_1 + e_1g_0 + f_0g_1 + f_1g_0 + e_1f_1g_1 \tag{116.m}$$
$$z_{5,2} = e_0f_0 + e_0g_0 + f_0g_0 + e_0f_1g_1 + e_1f_0g_1 + e_1f_1g_0 \tag{116.n}$$
$$z_{5,3} = d_0f_1 + d_1f_0 + d_0g_1 + d_1g_0 + f_0g_1 + f_1g_0 + d_1f_1g_1 \tag{116.o}$$
$$z_{5,4} = d_0f_0 + d_0g_0 + f_0g_0 + d_0f_1g_1 + d_1f_0g_1 + d_1f_1g_0 \tag{116.p}$$
$$z_{5,5} = d_0e_1 + d_1e_0 + d_0g_1 + d_1g_0 + e_0g_1 + e_1g_0 + d_1e_1g_1 \tag{116.q}$$
$$z_{5,6} = d_0e_0 + d_0g_0 + e_0g_0 + d_0e_1g_1 + d_1e_0g_1 + d_1e_1g_0 \tag{116.r}$$
$$z_{5,7} = d_0e_1 + d_1e_0 + d_0f_1 + d_1f_0 + e_0f_1 + e_1f_0 + d_1e_1f_1 \tag{116.s}$$
$$z_{5,8} = d_0e_0 + d_0f_0 + e_0f_0 + d_0e_1f_1 + d_1e_0f_1 + d_1e_1f_0 \tag{116.u}$$
$$z_{6,1} = e_0f_0 + e_0g_0 + f_0g_0 + e_0f_1g_1 + e_1f_1g_0 + e_1f_0g_1 \tag{116.v}$$
$$z_{6,2} = e_0f_0g_1 + e_0f_1g \quad + e_1f_0g_0 \tag{116.w}$$
$$z_{6,3} = d_0f_0 + d_0g_0 + f_0g_0 + d_0f_1g_1 + d_1f_0g_1 + d_1f_1g_0 \tag{116.x}$$
$$z_{6,4} = d_0f_1g_0 + d_1f_0g_0 + d_0f_0g_1 \tag{116.z}$$
$$z_{6,5} = d_0e_0 + d_0g_0 + e_0g_0 + d_1e_0g_1 + d_1e_1g_0 + d_0e_1g_1 \tag{116.aa}$$
$$z_{6,6} = d_0e_0g_1 + d_0e_1g_0 + d_1e_0g_0 \tag{116.ab}$$
$$z_{6,7} = d_0e_0 + d_0f_0 + e_0f_0 + d_0e_1f_1 + d_1e_0f_1 + d_1e_1f_0 \tag{116.ac}$$
$$z_{6,8} = d_0e_0f_1 + d_1e_0f_0 + d_0e_1f_0 \tag{116.ad}$$
$$z_{7,1} = e_0f_0g_1 + e_0f_1g_0 + e_1f_0g_0 \tag{116.ae}$$
$$z_{7,3} = d_0f_0g_1 + d_0f_1g_0 + d_1f_0g_0 \tag{116.af}$$
$$z_{7,5} = d_0e_0g_1 + d_0e_1g_0 + d_1e_0g_0 \tag{116.ag}$$
$$z_{7,7} = d_0e_0f_1 + d_0e_1f_0 + d_1e_0f_0 \tag{116.ah}$$

3.3.7 Original Nonic Equations

3.3.7.1 The Proposed Decomposition

$$\{F\} = \left\{ \begin{array}{c} \left(\begin{array}{l} \frac{a_0}{e_0} + e_1 + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_4 + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_3 \\ \quad + (a_8 - e_4)\cdot e_2 - a_5 \end{array} \right) \\ \left(\begin{array}{l} e_0 + \frac{a_0 \cdot e_4}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_3 + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_2 \\ \quad + (a_8 - e_4)\cdot e_1 - a_4 \end{array} \right) \\ \left(\begin{array}{l} \frac{a_0 \cdot e_3}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_2 + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_1 \\ \quad + (a_8 - e_4)\cdot e_0 - a_3 \end{array} \right) \\ \left(\frac{a_0 \cdot e_2}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_1 + (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_0 - a_2 \right) \\ \left(\frac{a_0 \cdot e_1}{e_0} + \{a_6 - e_2 - (a_7 - e_3 - (a_8 - e_4)\cdot e_4)\cdot e_4 - (a_8 - e_4)\cdot e_3\}\cdot e_0 - a_1 \right) \end{array} \right\} \tag{117}$$

$$[Z]=\begin{bmatrix} \left(-\dfrac{a_0}{{e_0}^2}\right) & (1) & (a_8-2e_4) & z_{1,4} & z_{1,5} \\ \left(1-\dfrac{a_0\cdot e_4}{{e_0}^2}\right) & (a_8-e_4) & \left(a_7-2e_3-a_8\cdot e_4+{e_4}^2\right) & z_{2,4} & z_{2,5} \\ \left(-\dfrac{a_0\cdot e_3}{{e_0}^2}-a_8-e_4\right) & \left(a_7-e_3-a_8\cdot e_4+{e_4}^2\right) & z_{3,3} & z_{3,4} & z_{3,5} \\ z_{4,1} & z_{4,2} & \left(\dfrac{a_0}{e_0}-e_1\right) & z_{4,4} & z_{4,5} \\ z_{5,1} & \left(\dfrac{a_0}{e_0}\right) & (-e_0) & (a_8\cdot e_0+2e_0\cdot e_4) & z_{5,5} \end{bmatrix} \tag{118}$$

$$z_{1,4}=a_7-2e_3-2a_8\cdot e_4+3{e_4}^2 \tag{119.a}$$

$$z_{1,5}=a_6-2e_2-2a_8\cdot e_3-2a_7\cdot e_4+3a_8\cdot {e_4}^2+6e_3\cdot e_4-4{e_4}^3 \tag{119.b}$$

$$z_{2,4}=a_6-2e_2-2a_8\cdot e_3-7e_4+a_8\cdot {e_4}^2+4e_3\cdot e_4-{e_4}^3 \tag{119.c}$$

$$z_{2,5}=\frac{a_0}{e_0}-e_1-a_8\cdot e_2-a_7\cdot e_3+2e_2\cdot e_4+2a_8\cdot e_3\cdot e_4-3e_3\cdot {e_4}^2+2{e_3}^2 \tag{119.d}$$

$$z_{3,3}=a_6-2e_2-a_8\cdot e_3-a_7\cdot e_4+a_8\cdot {e_4}^2+2e_3\cdot e_4-{e_4}^3 \tag{119.e}$$

$$z_{3,4}=\frac{a_0}{e_0}-e_1-a_8\cdot e_2+2e_2\cdot e_4 \tag{119.f}$$

$$z_{3,5}=-e_0-a_8\cdot e_1-a_7\cdot e_2+2e_1\cdot e_4+2e_2\cdot e_3+2a_8\cdot e_2\cdot e_4-3e_2\cdot {e_4}^2 \tag{119.g}$$

$$z_{4,1}=-\frac{a_0\cdot e_2}{{e_0}^2}+a_7-a_8\cdot e_4+{e_4}^2 \tag{119.h}$$

$$z_{4,2}=a_6-e_2-a_8\cdot e_3-a_7\cdot e_4+a_8\cdot {e_4}^2+2e_3\cdot e_4-{e_4}^3 \tag{119.i}$$

$$z_{4,4}=-e_0-a_8\cdot e_1+2e_1\cdot e_4 \tag{119.j}$$

$$z_{4,5}=-a_8\cdot e_0-a_7\cdot e_1+2e_0\cdot e_4+2e_1\cdot e_3+2a_8\cdot e_1\cdot e_4-3e_1\cdot {e_4}^2 \tag{119.k}$$

$$z_{5,1}=-\frac{a_0\cdot e_1}{{e_0}^2}+a_6-e_2-a_8\cdot e_3-a_7\cdot e_4+a_8\cdot {e_4}^2+2e_3\cdot e_4-{e_4}^3 \tag{119.l}$$

$$z_{5,5}=-a_7\cdot e_0-2e_0\cdot e_3+2a_8\cdot e_0\cdot e_4-3e_0\cdot {e_4}^2 \tag{119.m}$$

3.3.7.2 Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} d_1+e_6-a_8 \\ d_0+e_5+d_1e_6-a_7 \\ e_4+d_0e_6+d_1e_5-a_6 \\ e_3+d_0e_5+d_1e_4-a_5 \\ e_2+d_0e_4+d_1e_3-a_4 \\ e_1+d_0e_3+d_1e_2-a_3 \\ e_0+d_0e_2+d_1e_1-a_2 \\ d_0e_1+d_1e_0-a_1 \\ d_0e_0-a_0 \end{Bmatrix} \tag{120}$$

$$[Z]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & e_6 & 0 & 0 & 0 & 0 & 0 & 1 & d_1 \\ e_6 & e_5 & 0 & 0 & 0 & 0 & 1 & d_1 & d_0 \\ e_5 & e_4 & 0 & 0 & 0 & 1 & d_1 & d_0 & 0 \\ e_4 & e_3 & 0 & 0 & 1 & d_1 & d_0 & 0 & 0 \\ e_3 & e_2 & 0 & 1 & d_1 & d_0 & 0 & 0 & 0 \\ e_2 & e_1 & 1 & d_1 & d_0 & 0 & 0 & 0 & 0 \\ e_1 & e_0 & d_1 & d_0 & 0 & 0 & 0 & 0 & 0 \\ e_0 & 0 & d_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{121}$$

3.3.7.3 Complete Bairstow's Decomposition

$$\{F\}=\begin{Bmatrix} (c_0+d_1+e_1+f_1+g_1-a_8) \\ (d_0+e_0+f_0+g_0+c_0d_1+c_0e_1+c_0f_1+c_0g_1+d_1e_1+d_1f_1+d_1g_1+e_1f_1+e_1g_1+f_1g_1-a_7) \\ \left(\begin{array}{l} c_0d_0+c_0e_0+c_0f_0+c_0g_0+d_0e_1+d_1e_0+d_0f_1+d_1f_0+d_0g_1+d_1g_0+e_0f_1+e_1f_0+e_0g_1+e_1g_0 \\ c_0d_1e_1+c_0d_1f_1+c_0d_1g_1+c_0e_1f_1+c_0e_1g_1+c_0f_1g_1+d_1e_1f_1+d_1e_1g_1+d_1f_1g_1+e_1f_1g_1-a_6 \end{array}\right) \\ \left(\begin{array}{l} d_0e_0+d_0f_0+d_0g_0+e_0f_0+e_0g_0+f_0g_0+c_0d_0e_1+c_0d_1e_0+c_0d_0f_1+c_0d_1f_0+c_0d_0g_1+c_0d_1g_0 \\ +c_0e_0f_1+c_0e_1f_0+c_0e_0g_1+c_0e_1g_0+c_0f_0g_1+c_0f_1g_0+d_0e_1f_1+d_1e_0f_1+d_1e_1f_0+d_0e_1g_1 \\ +d_1e_0g_1+d_1e_1g_0+d_0f_1g_1+d_1f_0g_1+d_1f_1g_0+e_0f_1g_1+e_1f_0g_1+e_1f_1g_0+c_0d_1e_1f_1+c_0d_1e_1f_1 \\ +c_0d_1e_1g_1+c_0d_1f_1g_1+d_1e_1f_1g_1-a_5 \end{array}\right) \\ \left(\begin{array}{l} c_0d_0e_0+c_0d_0f_0+c_0d_0g_0+c_0e_0f_0+c_0e_0g_0+c_0f_0d_{g_0}+d_0e_0f_1+d_0e_1f_0+d_1e_0f_0+d_0e_0g_1 \\ +d_1e_0f_1g_1+d_0e_1g_0+d_1e_0g_0+d_0f_0g_1+d_0f_1g_0+d_1f_0g_0+e_0f_0g_1+e_0f_1g_0+e_1f_0g_0 \\ +d_0e_1f_1g_1+d_1e_1f_0g_1+d_1e_1f_1g_0+c_0d_0e_1f_1+c_0d_1e_0f_1+c_0d_1e_1f_0+c_0d_1f_0g_1+c_0d_1e_0g_1 \\ +c_0d_1e_1g_0+c_0d_0f_1g_1+c_0d_1f_0g_1+c_0d_1f_1g_0+c_0e_0f_1g_1+c_0e_1f_0g_1+c_0e_1f_1g_0 \\ +c_0d_1e_0f_1g_1-a_4 \end{array}\right) \\ \left(\begin{array}{l} d_0e_0f_0+d_0e_0g_0+d_0f_0g_0+e_0f_0g_0+d_0e_0f_1g_1+d_0e_1f_1g_0+d_1e_0f_0g_1+d_1e_0f_1g_0+d_1e_1f_0g_0 \\ +d_0f_0e_1g_1+c_0d_0e_0f_1+c_0d_0e_1f_0+c_0d_1e_0f_0+c_0d_0e_0g_1+c_0d_0e_1g_0+c_0d_1e_0g_0+c_0d_0f_0g_1 \\ +c_0d_0f_1g_0+c_0d_1f_0g_0+c_0e_0f_0g_1+c_0e_0f_1g_0+c_0e_1f_0g_0+c_0d_0e_1f_1g_1+c_0d_1e_0f_1g_1 \\ +c_0d_1e_1f_0g_1+c_0d_1e_1f_1g_0-a_3 \end{array}\right) \\ \left(\begin{array}{l} c_0d_0e_0f_0+c_0d_0e_0g_0+c_0d_0f_0g_0+c_0e_0f_0g_0+d_0e_0f_0g_1+d_0e_0f_1g_0+d_0e_1f_0g_0+d_1e_0f_0g_0 \\ +c_0d_0e_0f_1g_1+c_0d_0e_1f_0g_1+c_0d_0e_1f_1g_0+c_0d_1e_0f_0g_1+c_0d_1e_0f_1g_0+c_0d_1e_1f_0g_0-a_2 \end{array}\right) \\ (d_0e_0f_0g_0+c_0d_0e_0f_0g_1+c_0d_0e_0f_1g_0+c_0d_0e_1f_0g_0+c_0d_1e_0f_0g_0-a_1) \\ (c_0d_0e_0f_0g_0-a_0) \end{Bmatrix} \tag{122}$$

$$[Z]=\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ z_{2,1} & 1 & z_{2,3} & 1 & z_{2,5} & 1 & z_{2,7} & 1 & z_{2,9} \\ z_{3,1} & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} & z_{3,7} & z_{3,8} & z_{3,9} \\ z_{4,1} & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} & z_{4,7} & z_{4,8} & z_{4,9} \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & z_{5,5} & z_{5,6} & z_{5,7} & z_{5,8} & z_{5,9} \\ z_{6,1} & z_{6,2} & z_{6,3} & z_{6,4} & z_{6,5} & z_{6,6} & z_{6,7} & z_{6,8} & z_{6,9} \\ z_{7,1} & z_{7,2} & z_{7,3} & z_{7,4} & z_{7,5} & z_{7,6} & z_{7,7} & z_{7,8} & z_{7,9} \\ z_{8,1} & z_{8,2} & z_{8,3} & z_{8,4} & z_{8,5} & z_{8,6} & z_{8,7} & z_{8,8} & z_{8,9} \\ z_{9,1} & z_{9,2} & 0 & z_{9,4} & 0 & z_{9,6} & 0 & 0 & z_{9,9} \end{bmatrix} \tag{123}$$

$$z_{2,1}=d_1+e_1+f_1+g_1 \tag{124.a}$$

$$z_{2,3}=c_0+e_1+f_1+g_1 \tag{124.b}$$

$$z_{2,5}=c_0+d_1+f_1+g_1 \tag{124.c}$$

$$z_{2,7}=c_0+d_1+e_1+g_1 \tag{124.d}$$

$$z_{2,9}=c_0+d_1+e_1+f_1 \tag{124.e}$$

$$z_{3,1}=d_0+e_0+f_0+g_0+d_1e_1+d_1f_1+d_1g_1+e_1f_1+e_1g_1+f_1g_1 \tag{124.f}$$

$$z_{3,2} = c_0 + e_1 + f_1 + g_1 \tag{124.g}$$

$$z_{3,3} = e_0 + f_0 + g_0 + c_0e_1 + c_0 f_1 + c_0g_1 + e_1 f_1 + e_1g_1 + f_1g_1 \tag{124.h}$$

$$z_{3,4} = c_0 + d_1 + f_1 + g_1 \tag{124.i}$$

$$z_{3,5} = d_0 + f_0 + g_0 + c_0d_1 + c_0 f_1 + c_0g_1 + d_1 f_1 + d_1g_1 + f_1g_1 \tag{124.j}$$

$$z_{3,6} = c_0 + d_1 + e_1 + g_1 \tag{124.k}$$

$$z_{3,7} = d_0 + e_0 + g_0 + c_0d_1 + c_0e_1 + c_0g_1 + d_1e_1 + d_1g_1 + e_1g_1 \tag{124.l}$$

$$z_{3,8} = c_0 + d_1 + e_1 + f_1 \tag{124.m}$$

$$z_{3,9} = d_0 + e_0 + f_0 + c_0d_1 + c_0e_1 + c_0 f_1 + d_1e_1 + d_1 f_1 + e_1 f_1 \tag{124.n}$$

$$\begin{aligned} z_{4,1} = {} & d_0e_1 + d_1e_0 + d_0 f_1 + d_1 f_0 + d_0g_1 + d_1g_0 + e_0 f_1 + e_1 f_0 + e_0g_1 + e_1g_0 + f_0g_1 + f_1g_0 \\ & + d_1e_1 f_1 + d_1e_1g_1 + d_1 f_1g_1 + e_1 f_1g_1 \end{aligned} \tag{124.o}$$

$$z_{4,2} = e_0 + f_0 + g_0 + c_0e_1 + c_0 f_1 + c_0g_1 + e_1 f_1 + e_1g_1 + f_1g_1 \tag{124.p}$$

$$\begin{aligned} z_{4,3} = {} & c_0e_0 + c_0 f_0 + c_0g_0 + e_0 f_1 + e_1 f_0 + e_0g_1 + e_1g_0 + f_0g_1 + f_1g_0 + c_0e_1 f_1 + c_0e_1g_1 + c_0 f_1g_1 \\ & + e_1 f_1g_1 \end{aligned} \tag{124.q}$$

$$z_{4,4} = d_0 + f_0 + g_0 + c_0d_1 + c_0 f_1 + c_0g_1 + d_1 f_1 + d_1g_1 + f_1g_1 \tag{124.r}$$

$$\begin{aligned} z_{4,5} = {} & c_0d_0 + c_0 f_0 + c_0g_0 + d_0 f_1 + d_1 f_0 + d_0g_1 + d_1g_0 + f_0g_1 + f_1g_0 + c_0d_1 f_1 + c_0d_1g_1 + c_0 f_1g_1 \\ & + d_1 f_1g_1 \end{aligned} \tag{124.s}$$

$$z_{4,6} = d_0 + e_0 + g_0 + c_0d_1 + c_0e_1 + c_0g_1 + d_1e_1 + d_1g_1 + e_1g_1 \tag{124.t}$$

$$\begin{aligned} z_{4,7} = {} & c_0d_0 + c_0e_0 + c_0g_0 + d_0e_1 + d_1e_0 + d_0g_1 + d_1g_0 + e_0g_1 + e_1g_0 + c_0d_1e_1 + c_0d_1g_1 + c_0e_1g_1 \\ & + d_1e_1g_1 \end{aligned} \tag{124.u}$$

$$z_{4,8} = d_0 + e_0 + f_0 + c_0d_1 + c_0e_1 + c_0 f_1 + d_1e_1 + d_1 f_1 + e_1 f_1 \tag{124.v}$$

$$\begin{aligned} z_{4,9} = {} & c_0d_0 + c_0e_0 + c_0 f_0 + d_0e_1 + d_1e_0 + d_0 f_1 + d_1 f_0 + e_0 f_1 + e_1 f_0 + c_0d_1e_1 + c_0d_1 f_1 + c_0e_1 f_1 \\ & + d_1e_1 f_1 \end{aligned} \tag{124.w}$$

$$\begin{aligned} z_{5,1} = {} & d_0e_0 + d_0 f_0 + d_0g_0 + e_0g_0 + f_0g_0 + d_0e_1 f_1 + d_1e_0 f_1 + d_1e_1 f_0 + d_0e_1g_1 + d_1e_0g_1 + d_1e_1g_0 \\ & + d_0 f_1g_1 + d_1 f_0g_1 + d_1 f_1g_0 + e_0 f_1g_1 + e_1 f_0g_1 + e_1 f_1g_0 + d_1e_1 f_1g_1 \end{aligned} \tag{124.x}$$

$$\begin{aligned} z_{5,2} = {} & c_0e_0 + c_0 f_0 + c_0g_0 + e_0 f_1 + e_1 f_0 + e_0g_1 + e_1g_0 + f_0g_1 + f_1g_0 + c_0e_1 f_1 + c_0e_1g_1 \\ & + c_0 f_1g_1 + e_1 f_1g_1 \end{aligned} \tag{124.y}$$

$$\begin{aligned} z_{5,3} = {} & e_0 f_0 + e_0g_0 + f_0g_0 + c_0e_0 f_1 + c_0e_1 f_0 + c_0e_0g_1 + c_0e_1g_0 + c_0 f_0g_1 + c_0 f_1g_0 + e_0 f_1g_1 \\ & + e_1 f + e_1 f_0g_{11}g_0 + c_0e_1 f_1g_1 \end{aligned} \tag{124.z}$$

$$\begin{aligned} z_{5,4} = {} & c_0d_0 + c_0 f_0 + c_0g_0 + d_0 f_1 + d_1 f_0 + d_0g_1 + d_1g_0 + f_0g_1 + f_1g_0 + c_0d_1 f_1 + c_0d_1g_1 + c_0 f_1g_1 \\ & + d_1 f_1g_1 \end{aligned} \tag{124.aa}$$

$$\begin{aligned} z_{5,5} = {} & d_0 f_0 + d_0g_0 + f_0g_0 + c_0d_0 f_1 + c_0d_1 f_0 + c_0d_0g_1 + c_0d_1g_0 + c_0 f_0g_1 + c_0 f_1g_0 + d_0 f_1g_1 \\ & + d_1 f_0g_1 + d_1 f_1g_0 + c_0d_1 f_1g_1 \end{aligned} \tag{124.ab}$$

$$\begin{aligned} z_{5,6} = {} & c_0d_0 + c_0e_0 + c_0g_0 + d_0e_1 + d_1e_0 + d_0g_1 + d_1g_0 + e_0g_1 + e_1g_0 + c_0d_1e_1 + c_0d_1g_1 + c_0e_1g_1 \\ & + d_1e_1g_1 \end{aligned} \tag{124.ac}$$

$$\begin{aligned} z_{5,7} = {} & d_0e_0 + d_0g_0 + e_0g_0 + c_0d_0e_1 + c_0d_1e_0 + c_0d_0g_1 + c_0d_1g_0 + c_0e_0g_1 + c_0e_1g_0 + d_0e_1g_1 \\ & + d_1e_0g_1 + d_1e_1g_0 + c_0d_1e_1g_1 \end{aligned} \tag{124.ad}$$

$$\begin{aligned} z_{5,8} = {} & c_0d_0 + c_0e_0 + c_0 f_0 + d_0e_1 + d_1e_0 + d_0 f_1 + d_1 f_0 + e_0 f_1 + e_1 f_0 + c_0d_1e_1 + c_0d_1 f_1 + c_0e_1 f_1 \\ & + d_1e_1 f_1 \end{aligned} \tag{124.ae}$$

$$\begin{aligned} z_{5,9} = {} & d_0e_0 + d_0 f_0 + e_0 f_0 + c_0d_0e_1 + c_0d_1e_0 + c_0d_0 f_1 + c_0d_1 f_0 + c_0e_0 f_1 + c_0e_1 f_0 + d_0e_1 f_1 + d_1e_0 f_1 \\ & + d_1e_1 f_0 + c_0d_1e_1 f_1 \end{aligned} \tag{124.af}$$

$$\begin{aligned} z_{6,1} = {} & d_0e_0 f_1 + d_0e_1 f_0 + d_1e_0 f_0 + d_0e_0g_1 + d_0e_1g_0 + d_1e_0g_0 + d_0 f_0g_1 + d_0 f_1g_0 + d_1 f_0g_0 \\ & + e_0 f_0g_1 + e_0 f_1g_0 + e_1 f_0g_0 + d_0e_1 f_1g_1 + d_1e_0 f_1g_1 + d_1e_1 f_0g_1 + d_1e_1 f_1g_0 \end{aligned} \tag{124.ag}$$

$$z_{6,2} = e_0 f_0 + e_0 g_0 + f_0 g_0 + c_0 e_0 f_1 + c_0 e_1 f_0 + c_0 e_0 g_1 + c_0 e_1 g_0 + c_0 f_0 g_1 + c_0 f_1 g_0 + e_0 f_1 g_1 + e_1 f_0 g_1 + e_1 f_1 g_0 + c_0 e_1 f_1 g_1 \quad (124.ah)$$

$$z_{6,3} = c_0 e_0 f_0 + c_0 e_0 g_0 + c_0 f_0 g_0 + e_0 f_0 g_1 + e_0 f_1 g_0 + e_1 f_0 g_0 + c_0 e_0 f_1 g_1 + c_0 e_1 f_0 g_1 + c_0 e_1 f_1 g_0 \quad (124.ai)$$

$$z_{6,4} = d_0 f_0 + d_0 g_0 + f_0 g_0 + c_0 d_0 f_1 + c_0 d_1 f_0 + c_0 d_0 g_1 + c_0 d_1 g_0 + c_0 f_0 g_1 + c_0 f_1 g_0 + d_0 f_1 g_1 + d_1 f_0 g_1 + d_1 f_1 g_0 + c_0 d_1 f_1 g_1 \quad (124.aj)$$

$$z_{6,5} = c_0 d_0 f_0 + c_0 d_0 g_0 + c_0 f_0 g_0 + d_0 f_0 g_1 + d_0 f_1 g_0 + d_1 f_0 g_0 + c_0 d_0 f_1 g_1 + c_0 d_1 f_0 g_1 + c_0 d_1 f_1 g_0 \quad (124.ak)$$

$$z_{6,6} = d_0 e_0 + d_0 g_0 + e_0 g_0 + c_0 d_0 e_1 + c_0 d_1 e_0 + c_0 d_0 g_1 + c_0 d_1 g_0 + c_0 e_0 g_1 + c_0 e_1 g_0 + d_0 e_1 g_1 + d_1 e_0 g_1 + d_1 e_1 g_0 + c_0 d_1 e_1 g_1 \quad (124.al)$$

$$z_{6,7} = c_0 d_0 e_0 + c_0 d_0 g_0 + c_0 e_0 g_0 + d_0 e_0 g_1 + d_0 e_1 g_0 + d_1 e_0 g_0 + c_0 d_0 e_1 g_1 + c_0 d_1 e_0 g_1 + c_0 d_1 e_1 g_0 \quad (124.am)$$

$$z_{6,8} = d_0 e_0 + d_0 f_0 + e_0 f_0 + c_0 d_0 e_1 + c_0 d_1 e_0 + c_0 d_0 f_1 + c_0 d_1 f_0 + c_0 e_0 f_1 + c_0 e_1 f_0 + d_0 e_1 f_1 + d_1 e_0 g_1 + d_1 e_1 f_0 + c_0 d_1 e_1 f_1 \quad (124.an)$$

$$z_{6,9} = c_0 d_0 e_0 + c_0 d_0 f_0 + c_0 e_0 f_0 + d_0 e_0 f_1 + d_0 e_1 f_0 + d_1 e_0 f_0 + c_0 d_0 e_1 f_1 + c_0 d_1 e_0 f_1 + c_0 d_1 e_1 f_0 \quad (124.ao)$$

$$z_{7,1} = d_0 e_0 f_0 + d_0 e_0 g_0 + d_0 f_0 g_0 + e_0 f_0 g_0 + d_0 e_0 f_1 g_1 + d_0 e_1 f_0 g_1 + d_0 e_1 f_1 g_0 + d_1 e_0 f_0 g_1 + d_1 e_0 f_1 g_0 + d_1 e_1 f_0 g_0 \quad (124.ap)$$

$$z_{7,2} = c_0 e_0 f_0 + c_0 e_0 g_0 + c_0 f_0 g_0 + e_0 f_0 g_1 + e_0 f_1 g_0 + e_1 f_0 g_0 + c_0 e_0 f_1 g_1 + c_0 e_1 f_0 g_1 + c_0 e_1 f_1 g_0 \quad (124.aq)$$

$$z_{7,3} = e_0 f_0 g_0 + c_0 e_0 f_0 g_1 + c_0 e_0 f_1 g_0 + c_0 e_1 f_0 g_0 \quad (124.ar)$$

$$z_{7,4} = c_0 d_0 f_0 + c_0 d_0 g_0 + c_0 f_0 g_0 + d_0 f_0 g_1 + d_0 f_1 g_0 + d_1 f_0 g_0 + c_0 d_0 f_1 g_1 + c_0 d_1 f_0 g_1 + c_0 d_1 f_1 g_0 \quad (124.as)$$

$$z_{7,5} = d_0 f_0 g_0 + c_0 d_0 f_0 g_1 + c_0 d_0 f_1 g_0 + c_0 d_1 f_0 g_0 \quad (124.at)$$

$$z_{7,6} = c_0 d_0 e_0 + c_0 d_0 g_0 + c_0 e_0 g_0 + d_0 e_0 g_1 + d_0 e_1 g_0 + d_1 e_0 g_0 + c_0 d_0 e_1 g_1 + c_0 d_1 e_0 g_1 + c_0 d_1 e_1 g_0 \quad (124.au)$$

$$z_{7,7} = d_0 e_0 g_0 + c_0 d_0 e_0 g_1 + c_0 d_0 e_1 g_0 + c_0 d_1 e_0 g_0 \quad (124.av)$$

$$z_{7,8} = c_0 d_0 e_0 + c_0 d_0 f_0 + c_0 e_0 f_0 + d_0 e_0 f_1 + d_0 e_1 f_0 + d_1 e_0 f_0 + c_0 d_0 e_1 f_1 + c_0 d_1 e_0 f_1 + c_0 d_1 e_1 f_0 \quad (124.aw)$$

$$z_{7,9} = d_0 e_0 f_0 + c_0 d_0 e_0 f_1 + c_0 d_0 e_1 f_0 + c_0 d_1 e_0 f_0 \quad (124.ax)$$

$$z_{8,1} = d_0 e_0 f_0 g_1 + d_0 e_0 f_1 g_0 + d_0 e_1 f_0 g_0 + d_1 e_0 f_0 g_0 \quad (124.ay)$$

$$z_{8,2} = e_0 f_0 g_0 + c_0 e_0 f_0 g_1 + c_0 e_0 f_1 g_0 + d_1 e_0 f_0 g_0 \quad (124.az)$$

$$z_{8,3} = c_0 e_0 f_0 g_0 \quad (124.ba)$$

$$z_{8,4} = d_0 f_0 g_0 + c_0 d_0 f_0 g_1 + c_0 d_0 f_1 g_0 + c_0 d_1 f_0 g_0 \quad (124.bb)$$

$$z_{8,5} = c_0 d_0 f_0 g_0 \quad (124.bc)$$

$$z_{8,6} = d_0 e_0 g_0 + c_0 d_0 e_0 g_1 + c_0 d_0 e_1 g_0 + c_0 d_1 e_0 g_0 \quad (124.bd)$$

$$z_{8,7} = c_0 d_0 e_0 g_0 \quad (124.be)$$

$$z_{8,8} = d_0 e_0 f_0 + c_0 d_0 e_0 f_1 + c_0 d_0 e_1 f_0 + c_0 d_1 e_0 f_0 \quad (124.bf)$$

$$z_{8,9} = c_0 d_0 e_0 f_0 \quad (124.bg)$$

$$z_{9,1} = d_0 e_0 f_0 g_0 \quad (124.bh)$$

$$z_{9,2} = c_0 e_0 f_0 g_0 \quad (124.bi)$$

$$z_{9,4} = c_0 d_0 f_0 g_0 \quad (124.bj)$$

$$z_{9,6} = c_0 d_0 e_0 g_0 \quad (124.bk)$$

$$z_{9,9} = c_0 d_0 e_0 f_0 \quad (124.bl)$$

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IV. Numerical Examples

Seven numerical examples will be demonstrated in this section to verify the applicability of the proposed procedures. The first four examples are selected from real applications in civil engineering. They are the cubic-, quartic-, quintic- and sextic equations, respectively. The last three examples are beyond the sextic equation demonstrated in technical papers of the other author [7-9]. The intention is just to serve the researchers to exercise some challenging problems and the formulas given herein should be useful for some existing applications, if any. All calculations in this technical paper were done in the environments of Software Mathcad Prime 3.0. Only results from the proposed decomposition will be shown rather in details. The results from the other two alternative decompositions were also calculated to verify the correctness of the equations and to compare the efficiency with the proposed method, but the results from the alternative methods will be excluded because of the space limitation.

Example 1: Roots of a Cubic Equation

The required depth of a square timber section could be determined by solving the a cubic equation, $x^3 - 1.40368 \times 10^{-2} x - 3.55872 \times 10^{-3} = 0$. This equation can be decomposed to the product of a linear equation and a quadratic equation as shown in (12). Firstly the two unknown e_1 and e_0 may be obtained by the Newton-Raphson method in 2 dimensions via (75). In this case $\{F\}$ and $[Z]$ are calculated via (88) and (89), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 0.01 \\ 0.2 \end{Bmatrix}$

Iteration 1: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 0.01 \\ 0.2 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -1.55872 \times 10^{-1} \\ -4.71376 \times 10^{-2} \end{Bmatrix}$, $[Z] = \begin{bmatrix} 3.55872 \times 10 & 1.00000 \\ 8.11744 & -3.55872 \times 10^{-1} \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 1.71241 \times 10^{-2} & 4.81188 \times 10^{-2} \\ 3.90601 \times 10^{-1} & -1.71241 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 0.01 \\ 0.2 \end{Bmatrix} + \begin{bmatrix} 1.71241 \times 10^{-2} & 4.81188 \times 10^{-2} \\ 3.90601 \times 10^{-1} & -1.71241 \end{bmatrix} \cdot \begin{Bmatrix} -1.55872 \times 10^{-1} \\ -4.71376 \times 10^{-2} \end{Bmatrix} = \begin{Bmatrix} 1.49374 \times 10^{-2} \\ 1.80165 \times 10^{-1} \end{Bmatrix}$$

Iteration 2: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.49374 \times 10^{-2} \\ 1.80165 \times 10^{-1} \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -5.80779 \times 10^{-2} \\ -1.39488 \times 10^{-2} \end{Bmatrix}$, $[Z] = \begin{bmatrix} 1.59494 \times 10 & 1.00000 \\ 3.87353 & -2.38243 \times 10^{-1} \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 3.10480 \times 10^{-2} & 1.30321 \times 10^{-1} \\ 5.04802 \times 10^{-1} & -2.07855 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.49374 \times 10^{-2} \\ 1.80165 \times 10^{-1} \end{Bmatrix} + \begin{bmatrix} 3.10480 \times 10^{-2} & 1.30321 \times 10^{-1} \\ 5.04802 \times 10^{-1} & -2.07855 \end{bmatrix} \cdot \begin{Bmatrix} -5.80779 \times 10^{-2} \\ -1.39488 \times 10^{-2} \end{Bmatrix} = \begin{Bmatrix} 1.85584 \times 10^{-2} \\ 1.80489 \times 10^{-1} \end{Bmatrix}$$

Iteration 3: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.85584 \times 10^{-2} \\ 1.80489 \times 10^{-1} \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -1.12685 \times 10^{-2} \\ -2.01510 \times 10^{-3} \end{Bmatrix}$, $[Z] = \begin{bmatrix} 1.03327 \times 10 & 1.00000 \\ 2.86494 & -1.91758 \times 10^{-1} \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 3.95678 \times 10^{-2} & 2.06342 \times 10^{-2} \\ 5.91159 \times 10^{-1} & -2.13207 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.85584 \times 10^{-2} \\ 1.80489 \times 10^{-1} \end{Bmatrix} + \begin{bmatrix} 3.95678 \times 10^{-2} & 2.06342 \times 10^{-2} \\ 5.91159 \times 10^{-1} & -2.13207 \end{bmatrix} \cdot \begin{Bmatrix} -1.12685 \times 10^{-2} \\ -2.01510 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} 1.94201 \times 10^{-2} \\ 1.82855 \times 10^{-1} \end{Bmatrix}$$

Iteration 4: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.94201 \times 10^{-2} \\ 1.82855 \times 10^{-1} \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -3.95045 \times 10^{-4} \\ -5.11779 \times 10^{-5} \end{Bmatrix}$, $[Z] = \begin{bmatrix} 9.43609 & 1.00000 \\ 2.72543 & -1.83250 \times 10^{-1} \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 4.11372 \times 10^{-2} & 2.24487 \times 10^{-1} \\ 6.11825 \times 10^{-1} & -2.11828 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.94201 \times 10^{-2} \\ 1.82855 \times 10^{-1} \end{Bmatrix} + \begin{bmatrix} 4.11372 \times 10^{-2} & 2.24487 \times 10^{-1} \\ 6.11825 \times 10^{-1} & -2.11828 \end{bmatrix} \cdot \begin{Bmatrix} -3.95045 \times 10^{-4} \\ -5.11779 \times 10^{-5} \end{Bmatrix} = \begin{Bmatrix} 1.94478 \times 10^{-2} \\ 1.82988 \times 10^{-1} \end{Bmatrix}$$

Iteration 5: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.94478\times10^{-2} \\ 1.82988\times10^{-1} \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -3.73362\times10^{-7} \\ -3.34315\times10^{-8} \end{Bmatrix}$, $[Z] = \begin{bmatrix} 9.40920 & 1.00000 \\ 2.72177 & -1.82988\times10^{-1} \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 4.11807\times10^{-2} & 2.25046\times10^{-1} \\ 6.12523\times10^{-1} & -2.11750 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 1.94478\times10^{-2} \\ 1.82988\times10^{-1} \end{Bmatrix} + \begin{bmatrix} 4.11807\times10^{-2} & 2.25046\times10^{-1} \\ 6.12523\times10^{-1} & -2.11750 \end{bmatrix} \cdot \begin{Bmatrix} -1.55872\times10^{-1} \\ -4.71376\times10^{-2} \end{Bmatrix} = \begin{Bmatrix} 1.94478\times10^{-2} \\ 1.82988\times10^{-1} \end{Bmatrix}$$

Then the estimates of e_1 and e_0 are 0.019447 and 0.182998, respectively. Then d_0 can be obtained from (9).

$d_0 = \dfrac{a_0}{e_0} = \dfrac{-3.55872\times10^{-3}}{1.94478\times10^{-2}} = -0.18299$. The decomposed equation becomes,

$$(x-0.18299)\cdot\left(x^2+1.82988\times10^{-1}x+1.94478\times10^{-2}\right)=0$$

$\therefore x = 0.183, -0.091\pm0.105i$. However, for the design purpose only the positive real root $x = 0.183$ m is taken.

Example 2: Roots of a Quartic Equation

The minimum anchored length of a sheet pile can be obtained from the equilibrium of the lateral earth pressure acting on a sheet pile in form of a quartic equation $x^4+5.971x^3-12.132x^2-87.925x-109.496=0$. This equation can be decomposed into the product of two quadratic equations as shown in (18). Firstly the two unknown e_1 and e_0 may be obtained by the Newton-Raphson method in 2 dimensions via (75). In this case $\{F\}$ and $[Z]$ are calculated via (90) and (91), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$

Iteration 1: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -3.35750 \\ -13.68500 \end{Bmatrix}$, $[Z] = \begin{bmatrix} 7.84338 & -2.02900 \\ 29.34450 & -31.37350 \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 0.16819 & -0.01088 \\ 0.15731 & -0.04205 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix} + \begin{bmatrix} 0.16819 & -0.01088 \\ 0.15731 & -0.04205 \end{bmatrix} \cdot \begin{Bmatrix} -3.35750 \\ -13.68500 \end{Bmatrix} = \begin{Bmatrix} 4.41585 \\ 3.95276 \end{Bmatrix}$$

Iteration 2: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4.41585 \\ 3.95276 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -0.27022 \\ -1.17411 \end{Bmatrix}$, $[Z] = \begin{bmatrix} 6.61516 & -1.93451 \\ 24.21361 & -29.21154 \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 0.19954 & -0.01321 \\ 0.16540 & -0.04519 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4.41585 \\ 3.95276 \end{Bmatrix} + \begin{bmatrix} 0.19954 & -0.01321 \\ 0.16540 & -0.04519 \end{bmatrix} \cdot \begin{Bmatrix} -0.27022 \\ -1.17411 \end{Bmatrix} = \begin{Bmatrix} 4.45425 \\ 3.94440 \end{Bmatrix}$$

Iteration 3: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4.45425 \\ 3.94440 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -0.00193 \\ -0.00882 \end{Bmatrix}$, $[Z] = \begin{bmatrix} 6.51875 & -1.91780 \\ 23.79475 & -29.03616 \end{bmatrix}$

$$[Z]^{-1} = \begin{bmatrix} 0.20214 & -0.01335 \\ 0.16565 & -0.04538 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4.45425 \\ 3.94440 \end{Bmatrix} + \begin{bmatrix} 0.20214 & -0.01335 \\ 0.16565 & -0.04538 \end{bmatrix} \cdot \begin{Bmatrix} -0.00193 \\ -0.00882 \end{Bmatrix} = \begin{Bmatrix} 4.45452 \\ 3.94432 \end{Bmatrix}$$

Iteration 4: $\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 4.45452 \\ 3.94432 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -9.83190\times10^{-8} \\ -4.61274\times10^{-7} \end{Bmatrix}$, $[Z] = \begin{bmatrix} 6.51808 & -1.91763 \\ 23.79173 & -29.03493 \end{bmatrix}$

$$[Z]^{-1}=\begin{bmatrix}0.20215 & -0.01335\\ 0.16565 & -0.04538\end{bmatrix}$$

$$\begin{Bmatrix}e_0\\ e_1\end{Bmatrix}=\begin{Bmatrix}4.45452\\ 3.94432\end{Bmatrix}+\begin{bmatrix}0.20215 & -0.01335\\ 0.16565 & -0.04538\end{bmatrix}\cdot\begin{Bmatrix}-9.83190\times10^{-8}\\ -4.61274\times10^{-7}\end{Bmatrix}=\begin{Bmatrix}4.45452\\ 3.94432\end{Bmatrix}$$

Thus the estimates of e_1 and e_0 are 3.94432 and 4.45452, respectively. Then d_1 and d_0 from can be obtained from (15.a) and (15.b), respectively.

$d_1=2.02668$ and $d_0=-24.58043$. The decomposed equation becomes,

$\left(x^2+2.02668\,x-24.58043\right)\cdot\left(x^2+3.94432\,x+4.45452\right)=0$

$\therefore x=-6.074, 4.047, 1.972\pm0.752i$. However, for the design purpose only the positive real root $x=4.497$ m is taken.

Example 3: Roots of a Quintic Equation

The diameter of a circular steel column subjected to an axial force may be determined by solving a quintic equation $x^5-5.866\times10^{-3}x^3+4.951\times10^{-5}x^2+9.850\times10^{-8}=0$. This equation can be decomposed into the product of two equations i.e. one quadratic equation and one cubic equation as shown in (24). Firstly the three unknown e_2, e_1 and e_0 may be obtained by the Newton-Raphson method in 3 dimensions via (87). In this case $\{F\}$ and $[Z]$ are calculated via (92) and (93), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix}e_0\\ e_1\\ e_2\end{Bmatrix}=\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}$

Iteration 1:

$$\begin{Bmatrix}e_0\\ e_1\\ e_2\end{Bmatrix}=\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix},\ \{F\}=\begin{Bmatrix}5.86581\times10^{-3}\\ -4.94163\times10^{-5}\\ -10.00000\times10^{-1}\end{Bmatrix},\ [Z]=\begin{bmatrix}-9.84984\times10^{-8} & 1.00000 & -2.00000\\ 10.00000\times10^{-1} & -1.00000 & -10.00000\times10^{-1}\\ -1.00000 & 9.84984\times10^{-8} & -1.00000\end{bmatrix}$$

$$[Z]^{-1}=\begin{bmatrix}2.50000\times10^{-1} & 2.50000\times10^{-1} & -7.50000\times10^{-1}\\ 5.00000\times10^{-1} & -5.00000\times10^{-1} & -5.00000\times10^{-1}\\ -2.50000\times10^{-1} & -2.50000\times10^{-1} & -2.50000\times10^{-1}\end{bmatrix}$$

$$\begin{Bmatrix}e_0\\ e_1\\ e_2\end{Bmatrix}=\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}+\begin{bmatrix}2.50000\times10^{-1} & 2.50000\times10^{-1} & -7.50000\times10^{-1}\\ 5.00000\times10^{-1} & -5.00000\times10^{-1} & -5.00000\times10^{-1}\\ -2.50000\times10^{-1} & -2.50000\times10^{-1} & -2.50000\times10^{-1}\end{bmatrix}\cdot\begin{Bmatrix}5.86581\times10^{-3}\\ -4.94163\times10^{-5}\\ -10.00000\times10^{-1}\end{Bmatrix}=\begin{Bmatrix}2.48546\times10^{-1}\\ 4.97042\times10^{-1}\\ 7.51454\times10^{-1}\end{Bmatrix}$$

Iteration 2:

$$\begin{Bmatrix}e_0\\ e_1\\ e_2\end{Bmatrix}=\begin{Bmatrix}2.48546\times10^{-1}\\ 4.97042\times10^{-1}\\ 7.51454\times10^{-1}\end{Bmatrix},\ \{F\}=\begin{Bmatrix}-6.17748\times10^{-2}\\ -1.25008\times10^{-1}\\ -1.86771\times10^{-1}\end{Bmatrix},\ [Z]=\begin{bmatrix}-1.59447\times10^{-6} & 1.00000 & -1.50291\\ 9.99999\times10^{-1} & -7.51454\times10^{-1} & -4.97042\times10^{-1}\\ -7.51455\times10^{-1} & 3.96299\times10^{-7} & -2.48546\times10^{-1}\end{bmatrix}$$

$$[Z]^{-1}=\begin{bmatrix}1.26993\times10^{-1} & 1.68996\times10^{-1} & -1.10586\\ 4.22957\times10^{-1} & -7.67902\times10^{-1} & -1.02189\\ -3.83951\times10^{-1} & -5.10944\times10^{-1} & -6.79939\times10^{-1}\end{bmatrix}$$

$$\begin{Bmatrix}e_0\\ e_1\\ e_2\end{Bmatrix}=\begin{Bmatrix}2.48546\times10^{-1}\\ 4.97042\times10^{-1}\\ 7.51454\times10^{-1}\end{Bmatrix}+\begin{bmatrix}1.26993\times10^{-1} & 1.68996\times10^{-1} & -1.10586\\ 4.22957\times10^{-1} & -7.67902\times10^{-1} & -1.02189\\ -3.83951\times10^{-1} & -5.10944\times10^{-1} & -6.79939\times10^{-1}\end{bmatrix}\cdot\begin{Bmatrix}-6.17748\times10^{-2}\\ -1.25008\times10^{-1}\\ -1.86771\times10^{-1}\end{Bmatrix}=\begin{Bmatrix}7.09744\times10^{-2}\\ 2.36318\times10^{-1}\\ 5.36871\times10^{-1}\end{Bmatrix}$$

Iteration 3:

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \end{Bmatrix} = \begin{Bmatrix} 7.09744 \times 10^{-2} \\ 2.36318 \times 10^{-1} \\ 5.36871 \times 10^{-1} \end{Bmatrix}, \{F\} = \begin{Bmatrix} -4.60452 \times 10^{-2} \\ -5.59467 \times 10^{-2} \\ -3.81038 \times 10^{-2} \end{Bmatrix}, [Z] = \begin{bmatrix} -1.95536 \times 10^{-5} & 1.00000 & -1.07374 \\ 9.99990 \times 10^{-1} & -5.36871 \times 10^{-1} & -2.36317 \times 10^{-1} \\ -5.36876 \times 10^{-1} & 1.38780 \times 10^{-6} & -7.09744 \times 10^{-2} \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 7.51075 \times 10^{-2} & 1.39894 \times 10^{-1} & -1.60206 \\ 3.89974 \times 10^{-1} & -1.13627 & -2.11644 \\ -5.68132 \times 10^{-1} & -1.05823 & -1.97105 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \end{Bmatrix} = \begin{Bmatrix} 7.09744 \times 10^{-2} \\ 2.36318 \times 10^{-1} \\ 5.36871 \times 10^{-1} \end{Bmatrix} + \begin{bmatrix} 7.51075 \times 10^{-2} & 1.39894 \times 10^{-1} & -1.60206 \\ 3.89974 \times 10^{-1} & -1.13627 & -2.11644 \\ -5.68132 \times 10^{-1} & -1.05823 & -1.97105 \end{bmatrix} \cdot \begin{Bmatrix} -4.60452 \times 10^{-2} \\ -5.59467 \times 10^{-2} \\ -3.81038 \times 10^{-2} \end{Bmatrix} = \begin{Bmatrix} 2.12147 \times 10^{-2} \\ 1.10060 \times 10^{-1} \\ 3.76402 \times 10^{-1} \end{Bmatrix}$$

After 13 cycles of iteration the estimates of e_2, e_1 and e_0 are 1.00751×10^{-1}, 2.12283×10^{-3} and 4.55564×10^{-5}, respectively. Then $d_1 = -1.00751 \times 10^{-1}$ and $d_0 = 2.16212 \times 10^{-3}$ can be obtained from (27.a) and (27.b), respectively.

The decomposed equation becomes,

$$\left(x^2 - 1.00751 \times 10^{-1} x + 2.16212 \times 10^{3}\right) \cdot \left(x^3 + 1.00751 \times 10^{-1} x^2 + 2.12283 \times 10^{-3} x + 4.55564 \times 10^{-5}\right) = 0$$

$\therefore x = -8.157 \times 10^{-2}, 3.100 \times 10^{-2}, 6.975 \times 10^{-2}, -9.59 \times 10^{-3} \pm 2.160 \times 10^{-2} i$. However, for the design purpose the larger value of the positive real roots $x = 0.06975$ m is taken.

Example 4: Roots of a Sextic Equation

The critical water height (unit in m) in an open channel of a trapezoidal section may be considered by solving a sextic equation $x^6 + 30x^5 + 300x^4 + 1000x^3 - 7.964 \times 10^4 x - 3.982 \times 10^5 = 0$. This equation can be decomposed into the product of one quadratic equation and one quartic equation as shown in (33). Firstly the four unknown e_3, e_2, e_1 and e_0 may be obtained by the Newton-Raphson method in 4 dimensions via (87). In this case $\{F\}$ and $[Z]$ are calculated via (98) and (99), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} -2000 \\ -100 \\ 20 \\ 1 \end{Bmatrix}$

Iteration 1: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} -2000 \\ -100 \\ 20 \\ 1 \end{Bmatrix}, \{F\} = \begin{Bmatrix} -51.90000 \\ -320.90000 \\ -918.00000 \\ 1.73000 \times 10^3 \end{Bmatrix},$

$$[Z] = \begin{bmatrix} 0.09955 & 0 & 1 & 1 \\ 0.09955 & 1 & 29.00000 & 179.10000 \\ 2.99100 & 29.00000 & 199.10000 & 100.00000 \\ 19.04500 & 199.10000 & 0 & 2.00000 \times 10^3 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 10.12801 & -0.10085 & -0.03618 & 0.00578 \\ -0.98312 & -0.04122 & 0.01094 & 0.00364 \\ -0.00967 & 0.00498 & 0.00435 & -0.00066 \\ 0.00143 & 0.00506 & -0.00074 & 0.00008 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} -2000 \\ -100 \\ 20 \\ 1 \end{Bmatrix} + \begin{bmatrix} 10.12801 & -0.10085 & -0.03618 & 0.00578 \\ -0.98312 & -0.04122 & 0.01094 & 0.00364 \\ -0.00967 & 0.00498 & 0.00435 & -0.00066 \\ 0.00143 & 0.00506 & -0.00074 & 0.00008 \end{bmatrix} \cdot \begin{Bmatrix} -51.90000 \\ -320.90000 \\ -918.00000 \\ 1.73000 \times 10^3 \end{Bmatrix} = \begin{Bmatrix} -1.54993 \times 10^3 \\ -160.49717 \\ 26.22340 \\ 1.87166 \end{Bmatrix}$$

Iteration 2: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} -1.54993 \times 10^3 \\ -160.49717 \\ 26.22340 \\ 1.87166 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} 35.78562 \\ 57.98137 \\ 672.75595 \\ -5.19106 \times 10^3 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} 0.16576 & 0 & 1 & 1 \\ 0.31025 & 1 & 28.12834 & 230.69222 \\ 5.34679 & 28.12834 & 256.91562 & 160.49717 \\ 1.52433 & 256.91562 & 0 & 1.54993 \times 10^3 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 6.81178 & -0.03127 & -0.02309 & 0.00265 \\ -0.08827 & -0.02633 & 0.00323 & 0.00364 \\ -0.13705 & 0.00079 & 0.00434 & -0.00048 \\ 0.00793 & 0.00439 & -0.00051 & 0.00004 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} -1.54993 \times 10^3 \\ -160.49717 \\ 26.22340 \\ 1.87166 \end{Bmatrix} + \begin{bmatrix} 6.81178 & -0.03127 & -0.02309 & 0.00265 \\ -0.08827 & -0.02633 & 0.00323 & 0.00364 \\ -0.13705 & 0.00079 & 0.00434 & -0.00048 \\ 0.00793 & 0.00439 & -0.00051 & 0.00004 \end{bmatrix} \cdot \begin{Bmatrix} 35.78562 \\ 57.98137 \\ 672.75595 \\ -5.19106 \times 10^3 \end{Bmatrix} = \begin{Bmatrix} -1.76259 \times 10^3 \\ -139.07837 \\ 25.68058 \\ 1.87966 \end{Bmatrix}$$

After 8 cycles of iteration the estimates of e_3, e_2, e_1 and e_0 are 1.96104, 24.95821, -131.34117 and -1.80954×10^3, respectively. Then $d_1 = 28.03896$ and $d_0 = 220.05627$ can be obtained from (36.a) and (36.b), respectively.

The decomposed equation becomes,

$$\left(x^2 + 28.03896\,x + 220.05627\right) \cdot \left(x^4 + 1.87966\,x^3 + 125.68058\,x^2 - 139.07837\,x - 1.76259 \times 10^3\right) = 0$$

Again the quartic equation can be decomposed further to two quadratic equations as discussed earlier in Example 2.

$\therefore x = -5.198, 5.983, -14.019 \pm 4.949i, -1.375 \pm 7.505i$. However, for the design purpose the larger value of the positive real roots $x = 5.983$ m is taken.

Example 5: Roots of a Septic Equation

Let's consider a septic equation $x^7 - x^6 + 14x^5 - 28x^4 + 14x^3 - 35x^2 + 28x - 35 = 0$. This equation can be decomposed into the product of one cubic equation and one quartic equation as shown in (42). Firstly the four unknown e_3, e_2, e_1 and e_0 may be obtained by the Newton-Raphson method in 4 dimensions via (87). In this case $\{F\}$ and $[Z]$ are calculated via (102) and (103), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$

Iteration 1: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} 7.00000 \\ -35.00000 \\ 13.00000 \\ -48.00000 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} 35.00000 & 1 & -3.00000 & 17.00000 \\ 36.00000 & -2.00000 & 14.00000 & -33.00000 \\ 33.00000 & 15.00000 & -36.00000 & 2.00000 \\ 50.00000 & -35.00000 & -1.00000 & 3.00000 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 0.01655 & 0.00872 & 0.00199 & 0.00083 \\ 0.02527 & 0.01071 & 0.00282 & -0.02725 \\ 0.02725 & 0.01154 & -0.02527 & -0.01071 \\ 0.02808 & -0.01655 & -0.00872 & -0.00199 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} + \begin{bmatrix} 0.01655 & 0.00872 & 0.00199 & 0.00083 \\ 0.02527 & 0.01071 & 0.00282 & -0.02725 \\ 0.02725 & 0.01154 & -0.02527 & -0.01071 \\ 0.02808 & -0.01655 & -0.00872 & -0.00199 \end{bmatrix} \cdot \begin{Bmatrix} 7.00000 \\ -35.00000 \\ 13.00000 \\ -48.00000 \end{Bmatrix} = \begin{Bmatrix} 1.20319 \\ -0.14689 \\ 1.0274 \\ 0.24219 \end{Bmatrix}$$

Iteration 2: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1.20319 \\ -0.14689 \\ 1.0274 \\ 0.24219 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} 0.70239 \\ -6.02241 \\ 1.66935 \\ -7.75658 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} 24.17667 & 1 & -1.48439 & 12.60556 \\ 6.85544 & -1.24219 & 12.24605 & -27.41728 \\ 23.59696 & 13.27345 & -28.94235 & -1.42123 \\ 9.72217 & -29.08923 & -1.20319 & 1.78601 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 0.03189 & 0.01459 & 0.00444 & 0.00250 \\ 0.01070 & 0.00265 & 0.00196 & -0.03323 \\ 0.02988 & 0.01442 & -0.02944 & -0.01302 \\ 0.02084 & -0.02650 & -0.01213 & -0.00369 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1.20319 \\ -0.14689 \\ 1.0274 \\ 0.24219 \end{Bmatrix} + \begin{bmatrix} 0.03189 & 0.01459 & 0.00444 & 0.00250 \\ 0.01070 & 0.00265 & 0.00196 & -0.03323 \\ 0.02988 & 0.01442 & -0.02944 & -0.01302 \\ 0.02084 & -0.02650 & -0.01213 & -0.00369 \end{bmatrix} \cdot \begin{Bmatrix} 0.70239 \\ -6.02241 \\ 1.66935 \\ -7.75658 \end{Bmatrix} = \begin{Bmatrix} 1.28065 \\ -0.39944 \\ 1.04138 \\ 0.05959 \end{Bmatrix}$$

After 5 cycles of iteration the estimates of e_3, e_2, e_1 and e_0 are 0.04747, 1.04274, -0.41342 and 1.29095, respectively. Then $d2 = -1.04747$, $d_1 = 13.00698$ and $d_0 = -27.11172$ can be obtained from (45.a), (45.b) and (45.c), respectively.

The decomposed equation becomes,

$$\left(x^3 - 1.04747\, x^2 + 13.00698\, x - 27.11172\right) \cdot \left(x^4 + 0.04747\, x^3 + 1.04274\, x^2 - 0.41342\, x + 1.29095\right) = 0$$

Again the cubic equation can be decomposed further to one linear equation and one quadratic equation as discussed earlier in Example1. Whereas the quartic equation can be decomposed further to two quadratic equations as discussed earlier in Example 2.

$\therefore x = 1.86552, -0.40902 \pm 3.79022\, i, -0.57489 \pm 1.01612\, i, 0.55115 \pm 0.80211\, i$.

Example 6: Roots of an Octic Equation

Let's consider an octic equation $x^8 - x^7 + 10x^6 - 25x^5 + 14x^4 - 30x^3 + 28x^2 - 35x - 20 = 0$. This equation can be decomposed into the product of two quartic equations as shown in (52). Firstly the four unknown e_3, e_2, e_1 and e_0 may be obtained by the Newton-Raphson method in 4 dimensions via (87). In this case $\{F\}$ and $[Z]$ are calculated via (109) and (110), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

Iteration 1: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -33.00000 \\ 29.00000 \\ -18.00000 \\ 10.00000 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} 21.00000 & -1.00000 & 10.00000 & -25.00000 \\ -1.00000 & 10.00000 & -25.00000 & -21.00000 \\ 10.00000 & -25.00000 & -21.00000 & 1.00000 \\ -25.00000 & -21.00000 & 1.00000 & -10.00000 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 0.01501 & -0.00703 & 0.01449 & -0.02132 \\ -0.00703 & 0.01449 & -0.02132 & -0.01501 \\ 0.01449 & -0.02132 & -0.01501 & 0.00703 \\ -0.02132 & -0.01501 & 0.00703 & -0.01449 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0.01501 & -0.00703 & 0.01449 & -0.02132 \\ -0.00703 & 0.01449 & -0.02132 & -0.01501 \\ 0.01449 & -0.02132 & -0.01501 & 0.00703 \\ -0.02132 & -0.01501 & 0.00703 & -0.01449 \end{bmatrix} \cdot \begin{Bmatrix} -33.00000 \\ 29.00000 \\ -18.00000 \\ 10.00000 \end{Bmatrix} = \begin{Bmatrix} 2.17302 \\ -0.88578 \\ 0.75608 \\ 0.00320 \end{Bmatrix}$$

Iteration 2: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} 2.17302 \\ -0.88578 \\ 0.75608 \\ 0.00320 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} -13.22536 \\ 1.91850 \\ 5.84943 \\ -7.66411 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} 5.23550 & -1.00640 & 8.49426 & -21.76576 \\ -0.98965 & 8.49104 & -21.73860 & -18.69062 \\ 12.44951 & -22.49952 & -12.26825 & 12.26825 \\ -27.13699 & -11.37681 & 2.18691 & -18.45816 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} 0.02682 & 0.00091 & 0.01232 & -0.02606 \\ -0.01003 & 0.01195 & -0.03108 & -0.01663 \\ 0.02128 & -0.03076 & -0.01234 & -0.00043 \\ -0.03072 & -0.01234 & -0.00042 & -0.00567 \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \begin{Bmatrix} -1.54993 \times 10^3 \\ -160.49717 \\ 26.22340 \\ 1.87166 \end{Bmatrix} + \begin{bmatrix} 0.02682 & 0.00091 & 0.01232 & -0.02606 \\ -0.01003 & 0.01195 & -0.03108 & -0.01663 \\ 0.02128 & -0.03076 & -0.01234 & -0.00043 \\ -0.03072 & -0.01234 & -0.00042 & -0.00567 \end{bmatrix} \cdot \begin{Bmatrix} -13.22536 \\ 1.91850 \\ 5.84943 \\ -7.66411 \end{Bmatrix} = \begin{Bmatrix} 2.25417 \\ -0.98694 \\ 1.16540 \\ -0.42048 \end{Bmatrix}$$

After 5 cycles of iteration the estimates of e_3, e_2, e_1 and e_0 are -0.47893, 1.26342, -1.02269 and 2.28307, respectively. Then $d_3 = -0.52107$, $d_2 = 8.48702$, $d_1 = -19.25428$ and $d_0 = -8.76012$ can be obtained from (55.a), (55.b), (55.c) and (55.d), respectively.

The decomposed equation becomes,

$$\left(x^4 - 0.52107x^3 + 8.48702x^2 - 19.25428x - 8.76012\right) \cdot \left(x^4 - 0.47893x^3 + 1.26342x^2 - 1.02269x + 2.28307\right) = 0$$

Again each of the quartic equations can be decomposed further to two quadratic equations as discussed earlier in Example 2.

$$\therefore x = -0.38643, 2.03588, -0.56419 \pm 3.28887i, -0.55552 \pm 1.14734i, 0.79499 \pm 0.87919i.$$

Example 7: Roots of a Nonic Equation

Let's consider a nonic equation $x^9 + x^8 - x^7 + 10x^6 - 25x^5 + 14x^4 - 30x^3 + 28x^2 - 35x - 20 = 0$. This equation can be decomposed into the product of one quartic equation and one quintic equation as shown in (62). Firstly the five unknown e_4, e_3, e_2, e_1 and e_0 may be obtained by the Newton-Raphson method in 5 dimensions via (87). In this case $\{F\}$ and $[Z]$ are calculated via (117) and (118), respectively. The results of calculation are summarized as shown below.

Initial guess $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

Iteration 1: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} 1.46880 \times 10^4 \\ -1.88700 \times 10^3 \\ -254.00000 \times 10^0 \\ 1.00561 \times 10^4 \\ -1.88607 \times 10^4 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} -1.34400 \times 10^4 & 1 & 2.00000 & -56.00000 & -172.70100 \\ 1.00000 & 2.00000 & -56.00000 & -172.70100 & 1.34400 \times 10^4 \\ 2.00000 & -56.00000 & -172.70100 & 1.34400 \times 10^4 & -1.00000 \\ -56.00000 & -172.70100 & 1.34400 \times 10^4 & -1.00000 & -2.00000 \\ -172.70100 & 1.34400 \times 10^4 & -1.00000 & -2.00000 & 56.00000 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} -0.00007 & -9.56129 \times 10^{-7} & -3.22306 \times 10^{-7} & 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} \\ -9.56129 \times 10^{-7} & -3.22306 \times 10^{-7} & 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} & 0.00007 \\ -3.22306 \times 10^{-7} & 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} & 0.00007 & 9.56129 \times 10^{-7} \\ 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} & 0.00007 & 9.56129 \times 10^{-7} & 3.22306 \times 10^{-7} \\ 4.37329 \times 10^{-9} & 0.00007 & 9.56129 \times 10^{-7} & 3.22306 \times 10^{-7} & -2.94706 \times 10^{-9} \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} -0.00007 & -9.56129 \times 10^{-7} & -3.22306 \times 10^{-7} & 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} \\ -9.56129 \times 10^{-7} & -3.22306 \times 10^{-7} & 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} & 0.00007 \\ -3.22306 \times 10^{-7} & 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} & 0.00007 & 9.56129 \times 10^{-7} \\ 2.94706 \times 10^{-9} & 4.37329 \times 10^{-9} & 0.00007 & 9.56129 \times 10^{-7} & 3.22306 \times 10^{-7} \\ 4.37329 \times 10^{-9} & 0.00007 & 9.56129 \times 10^{-7} & 3.22306 \times 10^{-7} & -2.94706 \times 10^{-9} \end{bmatrix} \cdot \begin{Bmatrix} 1.46880 \times 10^4 \\ -1.88700 \times 10^3 \\ -254.00000 \times 10^0 \\ 1.00561 \times 10^4 \\ -1.88607 \times 10^4 \end{Bmatrix} = \begin{Bmatrix} 2.09103 \\ 1.41672 \\ -0.72545 \\ 0.01533 \\ 0.13728 \end{Bmatrix}$$

Iteration 2: $\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 2.09103 \\ 1.41672 \\ -0.72545 \\ 0.01533 \\ 0.13728 \end{Bmatrix}$, $\{F\} = \begin{Bmatrix} 7.65213 \times 10^3 \\ -962.57836 \\ -114.13033 \\ 5.09890 \times 10^3 \\ -9.92559 \times 10^3 \end{Bmatrix}$,

$$[Z] = \begin{bmatrix} -3.07383 \times 10^3 & 1 & 1.72543 & -56.52325 & -155.82024 \\ -420.98764 & 1.86272 & -56.28638 & -163.57998 & 6.42816 \times 10^3 \\ -45.25232 & -56.27105 & -163.55353 & 6.42730 \times 10^3 & -45.54030 \\ 2.17364 \times 10^3 & -164.27898 & 6.42604 \times 10^3 & -4.53548 & 76.46979 \\ -4.51904 \times 10^3 & 6.42746 \times 10^3 & -2.09103 & -3.60793 & 118.19168 \end{bmatrix}$$

$$[Z]^{-1} = \begin{bmatrix} -0.00032 & -7.88201 \times 10^{-6} & -3.05244 \times 10^{-6} & -5.96497 \times 10^{-8} & 2.44886 \times 10^{-8} \\ -0.00023 & -8.39269 \times 10^{-6} & -2.12775 \times 10^{-6} & -1.59255 \times 10^{-8} & 0.00016 \\ 0.00010 & 6.06794 \times 10^{-7} & 1.04307 \times 10^{-6} & 0.00016 & 3.97029 \times 10^{-6} \\ -1.77027 \times 10^{-6} & 9.85295 \times 10^{-7} & 0.00016 & 3.96985 \times 10^{-6} & 1.46370 \times 10^{-6} \\ -0.00002 & 0.00016 & 3.76944 \times 10^{-6} & 1.45978 \times 10^{-6} & 2.85265 \times 10^{-8} \end{bmatrix}$$

$$\begin{Bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{Bmatrix} = \begin{Bmatrix} 2.09103 \\ 1.41672 \\ -0.72545 \\ 0.01533 \\ 0.13728 \end{Bmatrix} + \begin{bmatrix} -0.00032 & -7.88201\times10^{-6} & -3.05244\times10^{-6} & -5.96497\times10^{-8} & 2.44886\times10^{-8} \\ -0.00023 & -8.39269\times10^{-6} & -2.12775\times10^{-6} & -1.59255\times10^{-8} & 0.00016 \\ 0.00010 & 6.06794\times10^{-7} & 1.04307\times10^{-6} & 0.00016 & 3.97029\times10^{-6} \\ -1.77027\times10^{-6} & 9.85295\times10^{-7} & 0.00016 & 3.96985\times10^{-6} & 1.46370\times10^{-6} \\ -0.00002 & 0.00016 & 3.76944\times10^{-6} & 1.45978\times10^{-6} & 2.85265\times10^{-8} \end{bmatrix} \cdot \begin{Bmatrix} 7.65213\times10^{3} \\ -962.57836 \\ -114.13033 \\ 5.09890\times10^{3} \\ -9.92559\times10^{3} \end{Bmatrix} = \begin{Bmatrix} 4.56508 \\ 4.69446 \\ -2.27552 \\ 0.04187 \\ 0.43521 \end{Bmatrix}$$

After 10 cycles of iteration the estimates of e_4, e_3, e_2, e_1 and e_0 are 1.87305, 20.41547, -20.31628, 71.49813 and 49.90072, respectively. Then $d_3 = 0.12695$, $d_2 = -76.65326$ $d_1 = -11.40134$ and $d_0 = 269.3348$ can be obtained from (64.a), (64.a), (64.c) and (64.d), respectively.

The decomposed equation becomes,

$$\left(x^4 + 0.12695\,x^3 - 76.65326\,x^2 - 11.40134\,x + 269.3348\right)\cdot$$
$$\left(x^5 + 1.87305\,x^4 + 20.41547\,x^3 - 20.31628\,x^2 + 71.49813\,x + 49.90072\right) = 0$$

Again the quartic equation $\left(x^4 + 0.12695\,x^3 - 76.65326\,x^2 - 11.40134\,x + 269.3348\right) = 0$ can be decomposed further to two quadratic equations as discussed earlier in Example 2. Whereas the quintic equation $\left(x^5 + 1.87305\,x^4 + 20.41547\,x^3 - 20.31628\,x^2 + 71.49813\,x + 49.90072 = 0\right)$ can be decomposed further to one quadratic equation and one cubic equation as discussed earlier in Example 3. Finally the cubic equation can be decomposed to one linear equation and one quadratic equation as discussed earlier in Example1.

$$\therefore x = -8.52604, -2.00192, -0.56031, 1.84408, 8.55692, -1.65546 \pm 4.42289\,i, 0.99909 \pm 1.73063\,i\,.$$

V. Conclusion

1) An approach for solving polynomial equations of degree higher than two was proposed.
2) The main concepts were decomposition of a polynomial of higher degrees to the product of two polynomials of lower degrees and the n-D Newton-Raphson method for a system of nonlinear equations.
3) The coefficient of each term in an original polynomial of order m will be equated to the corresponding term from the collected-expanded product of the two polynomials of the lower degrees based on the concept of undetermined coefficients. Consequently a system of m nonlinear equations was formed. Then the unknown coefficients of the decomposed polynomial of the lower degree of the two decomposed polynomials would be eliminated from the system of nonlinear equations. Therefore the number of nonlinear equations would be reduced to the number of unknown coefficients in the decomposed polynomial equation of higher degree.
4) The unknown coefficients in the decomposed polynomial of the higher degree would be obtained by the Newton-Raphson method for simultaneous nonlinear equations. Then the unknown coefficients for the decomposed polynomial of the lower degree would be obtained by back substitutions.
5) The formulations for the decomposed polynomials would be derived for the original polynomials of degree from three to nine.
6) The system of nonlinear equations and supplementary equations for determining the unknown coefficients of the decomposed polynomials were also summarized for the original polynomial for degree from three to nine.
7) For the case of an original polynomial equation of an odd degree the original polynomial equation will be decomposed to two polynomial equations i.e. one equation of an odd degree and the other equation of an even degree. Whereas for the case of an original polynomial equation of an even degree, two decomposed polynomial equations of even degrees were proposed to guarantee obtaining all possible roots i.e. complex conjugates, distinct real roots, double real root, triple real root etc.
8) Two alternative forms of decomposed polynomial equations were also given i.e. Bairstow's decomposition and complete Bairstow's decomposition. For the polynomial equation of degree five or higher the vector of nonlinear equations and the corresponding *Jacobian* matrix from the Bairstow's decomposition were simpler than the proposed decomposition. Whereas those from the complete Bairstow's decomposition were more complex than the proposed decomposition. Both alternative forms involved larger systems of simultaneous nonlinear equations.
9) Seven numerical examples were also given to verify the applicability of the proposed approach. Four numerical examples for polynomial equations of degree three, four, five and six were demonstrated. These problems were selected from the real applications in civil engineering. The other three problems for polynomial equations of degree seven, eight and nine were given to challenge to the researchers. Numerical results were given rather in details so that the readers can keep track for all steps of calculations.
10) For a given polynomial equation there exist several possible pairs of decomposed polynomial equations, but any pair of decomposed equations will always give the same final results.
11) The Newton-Raphson method in two and more dimensions were proved to be a very efficient tool for

solving a system of simultaneous nonlinear equations at least in the extent of numerical examples shown in this technical paper and author's experience on finding roots of a polynomial equation.

12) The decomposed polynomial equations can always decomposed further to the equations of lower degrees. Finally the original polynomial equations can be rewritten in form of a product of linear equations and quadratic equations. Therefore all possible roots can always be determined.
13) The method proposed can be extended for a polynomial equation of any degree beyond nine, but with longer equations and a larger system of simultaneous nonlinear equations. Further extension was not shown in this technical paper because of the limitation of the paper space, but it can be done systematically in form of matrix notations and computer programming.

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