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Connected Domination Polynomial of Some Graphs

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Abstract: *Let G be a simple connected graph. The connected domination polynomial of G is defined by* $C_d(G,x) = \sum_{i=\gamma_d(G)}^{|V(G)|} c_d(G,i)\, x^i$ *, where* γ_d *(G) is the minimum cardinality of connected dominating set of G. In this paper, we find the connected dominating polynomial and roots of some general graphs.*
Keywords: *Connected domination number, connected dominating set, connected domination polynomial, connected dominating roots, minimum connected dominating set.*

I. Introduction

A connected dominating set of a graph *G* is a set *D* of vertices with two properties: (i) Any node in *D* can reach any other node in *D* by a path that stays entirely within *D*. That is, D induces a connected subgraph of *G* (ii)Every vertex in *G* either belongs to *D* or is adjacent to a vertex in *D*. That is, *D* is a dominating set of *G*.

A minimum connected dominating set of a graph *G* is a connected dominating set with the smallest possible cardinality among all connected dominating sets of *G*. The connected domination number of *G* is the number of vertices in the minimum connected dominating set. By the definition of connected domination number, γ_d (G) is the minimum cardinality of a connected dominating set in G. For more details about domination number and its related parameters, we refer to [1] – [4].

For a detailed treatment of the domination polynomial of a graph, the reader is referred to [5], [6]. We introduce the connected domination polynomial of G, we obtain connected domination polynomial and compute its roots for some standard graphs.

II. Introduction to Connected Domination Polynomial

2.1 Definition

Let G be a simple connected graph. The connected domination polynomial of G is defined by $C_d(G,x) = \sum_{i=\gamma_d(G)}^{|V(G)|} c_d(G,i)\, x^i$, where γ_d (G) is the connected domination number of G.

2.2 Theorem

Let G be a graph with |V(G)| = n. Then

(i) If G is connected then $C_d(G,n) = 1$ and $C_d(G,n-1) = n$.

(ii) $C_d(G,i) = 0$ if and only if i < γ_d (G) and i > n.

(iii) $C_d(G,x)$ has no constant and first degree terms.

(iv) $C_d(G,x)$ is a strictly increasing function in $[0,\infty)$.

(v) Let G be a graph and H be any induced subgraph of G. Then $\deg(C_d(G,x)) \geq \deg(C_d(H,x))$.

(vi) Zero is a root of $C_d(G,x)$ with multiplicity γ_d (G).

Proof:

(i) Since G has n vertices, there is only one way to choose all these vertices and it connected and dominates all the vertices. Therefore, $c_d(G,n) = 1$. If we delete one vertex v, the remaining n-1 vertices are connected dominate all the vertices of G. (This is done in n ways). Therefore, $c_d(G,n-1) = n$.

(ii) Since $\mathcal{C}_d(G,i) = \Phi$ if $i < \gamma_d(G)$ or $\mathcal{C}_d(G,n+k) = \Phi$, k = 1,2,…. Therefore, we have $c_d(G,i) = 0$ if $i < \gamma_d(G)$ or i > n. Conversely, if $i < \gamma_d(G)$ or i > n, $c_d(G,i) = 0$. Hence the result.

(iii) Since γ_d (G) ≥ 2, the connected domination polynomial has no term of degree 0 and 1. Therefore, it has no constant and first degree terms.
(iv) The proof of (iv) follows from the definition of connected domination polynomial of a graph.
(v) We have, deg(C_d(H,x)) = number of vertices in H. Also, deg(C_d(G,x)) = number of vertices in G. Since the number of vertices in H ≤ the number of vertices in G, deg(C_d(G,x)) ≥ deg(C_d(H,x)).
(vi) As C_d(G,x) has no constant term, C_d(G,x) = 0 implies x = 0. Hence x = 0 is the root of polynomial C_d(G,x). Also since least power of x in expansion of C_d(G,x) is γ_d (G), multiplicity of root is γ_d (G).

III. Connected Domination Polynomial and Roots for Some Graphs

3.1Theorem

If F_m is a friendship graph with 2m+1 vertices, then the connected domination polynomial of F_m is $C_d(F_m,x) = x\,[(1+x)^{2m} - 1]$ and the connected dominating roots are 0 with multiplicity 2 and $e^{\frac{i\pi}{m}} - 1$, $e^{\frac{i2\pi}{m}} - 1$,…, $e^{\frac{i(m-1)\pi}{m}} - 1$ with multiplicity 1.

Proof:

Let G be a friendship graph of size 2m + 1 and m ≥ 2. By labeling the vertices of G as $v_1, v_2,\ldots, v_{2m+1}$ where v_1 is joined with all the vertices and $(v_2,v_3), (v_4,v_5),\ldots,(v_{2m},v_{2m+1})$ are joined itself. Clearly there are 2m connected dominating set of size two namely $\{v_1, v_2\}, \{v_1, v_3\},\ldots, \{v_1, v_{2m+1}\}$. Similarly for the connected dominating set of size three, we need to select the vertex v_1 and two vertices from the set of vertices $\{ v_2, v_3,\ldots, v_{2m+1}\}$. That means there are $\binom{2m}{2}$ connected dominating sets. In general, $c_d(G,i) = \binom{2m}{i-1}$, 2 ≤ i ≤ 2n+1.

Hence $C_d(F_m,x) = 2m\,x^2 + \binom{2m}{2}x^3 + \ldots + \binom{2m}{2m}x^{2m+1}$

$= x\,[(1+x)^{2m} - 1]$.

Consider, $x\,[(1+x)^{2m} - 1] = 0$. The roots of this polynomial are 0 with multiplicity 2 and $e^{\frac{i\pi}{m}} - 1$, $e^{\frac{i2\pi}{m}} - 1$,…, $e^{\frac{i(m-1)\pi}{m}} - 1$ with multiplicity 1.

3.2 Theorem

For any helm graph H_n with 2n+1 vertices, where n ≥ 3, $C_d(H_n,x) = x^n (1+x)^{n+1}$ and the connected dominating roots are 0 with multiplicity n and –1 with multiplicity n+1.

Proof:

Let G be a helm graph of size 2n+1vertices and n ≥ 3. By labeling the vertices of G as $v_1, v_2,\ldots, v_{2n+1}$ and v_1 is joined with $v_2, v_3,\ldots, v_{n+1}$. Also $(v_2,v_{n+2}), (v_3,v_{n+3}),\ldots,(v_{n+1},v_{2n+1})$ are joined itself. Clearly the only set $\{v_2, v_3,\ldots, v_{n+1}\}$ is connected dominating of size n. Clearly for the connected dominating set of size n+1 we need to select $\{v_1, v_2,\ldots, v_{n+1}\}, \{v_2, v_3,\ldots, v_{n+1}, v_{n+2}\}, \{v_2, v_3,\ldots, v_{n+1}, v_{n+3}\},\ldots, \{v_2, v_3,\ldots, v_{n+1}, v_{2n+1}\}$. That means there are $\binom{n+1}{1}$ connected dominating sets. In general, $c_d(G,i) = \binom{n+1}{i-n}$, n ≤ i ≤ 2n+1.

Hence $C_d(H_n,x) = x^n + \binom{n+1}{1}x^{n+1} + \ldots + \binom{n+1}{n+1}x^{2n+1}$

$= x^n (1+x)^{n+1}$

Consider, $x^n (1+x)^{n+1} = 0$. The roots of this polynomial are 0 with multiplicity n and –1 with multiplicity n+1.

3.3 Theorem

For any lollipop graph $L_{m,1}$ with m+1 vertices, where m ≥ 2, $C_d(L_{m,1},x) = x\,[(1+x)^m - 1]$ and the connected dominating roots are 0 with multiplicity 2 and $e^{\frac{i2\pi}{m}} - 1$, $e^{\frac{i4\pi}{m}} - 1$,…, $e^{\frac{i2(m-1)\pi}{m}} - 1$ with multiplicity 1.

Proof:

Let G be a lollipop graph of size m+1 and $m \geq 2$. By labeling the vertices of G as $v_1, v_2, \ldots, v_{m+1}$ where v_{m+1} is joined only with v_1 and the remaining vertices are joined with each other except v_{m+1}. Clearly there are m connected dominating set of size two namely $\{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_{m+1}\}$. Similarly for the connected dominating set of size three, we need to select the vertex v_1 and two vertices from the set of vertices $\{ v_2, v_3, \ldots, v_{m+1}\}$. That means there are $\binom{m}{2}$ connected dominating sets. In general, $c_d(G,i) = \binom{m}{i-1}$, $2 \leq i \leq m+1$.

Hence $C_d(L_{m,1},x) = m\,x^2 + \binom{m}{2} x^3 + \ldots + \binom{m}{m} x^{m+1}$

$= x\,[(1+x)^m - 1]$.

Consider, $x\,[(1+x)^m - 1] = 0$. The roots of this polynomial are 0 with multiplicity 2 and $e^{\frac{i2\pi}{m}} - 1$, $e^{\frac{i4\pi}{m}} - 1$, …, $e^{\frac{i2(m-1)\pi}{m}} - 1$ with multiplicity 1.

3.4 Theorem

For any barbell graph B_n with 2n vertices where $n \geq 3$, $C_d(B_n,x) = x^2\,(1+x)^{2(n-1)}$ and the connected dominating roots are 0 with multiplicity 2 and -1 with multiplicity 2(n-1).

Proof:

Let G be a barbell graph of size 2n and $n \geq 3$. By labeling the vertices of G as $v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}$, and v_n is joined with v_{n+1}. Also $v_1, v_2, \ldots, v_n$ are joined with each other and $v_{n+1}, v_{n+2}, \ldots, v_{2n}$ are joined with each other. Clearly the only set (v_n, v_{n+1}) is connected dominating of size 2. Clearly for the connected dominating sets of size 3, we need to select the vertices v_n, v_{n+1} and one vertex from the set $\{ v_1, v_2, \ldots, v_{n-1}, v_{n+2}, \ldots, v_{2n}\}$. That means there are $\binom{2n-2}{2}$ connected dominating sets. In general, $c_d(G,i) = \binom{2n-2}{i-2}$, $2 \leq i \leq 2n$.

Hence $C_d(B_n,x) = x^2 + \binom{2n-2}{1} x^3 + \binom{2n-2}{2} x^4 + \ldots + \binom{2n-2}{2n-2} x^{2n}$

$= x^2\,(1+x)^{2(n-1)}$.

Consider, $x^2\,(1+x)^{2(n-1)} = 0$. The roots of this polynomial are 0 with multiplicity 2 and -1 with multiplicity 2(n-1).

3.5 Theorem

For any tadpole graph $T_{n,1}$ with n+1 vertices, where $n \geq 4$, $C_d(T_{n,1},x) = (n-2)\,x^{n-2} + (2n-3)\,x^{n-1} + n\,x^n + x^{n+1}$ and the connected dominating roots are 0 with multiplicity n-2 and –1 with multiplicity 2 and – n+2 with multiplicity 1.

Proof:

Let G be a tadpole graph with n+1 vertices and $n \geq 4$. By labeling the vertices of G as $v_1, v_2, \ldots, v_n, v_{n+1}$. The first n vertices are a cycle C_n and v_{n+1} is connected with the vertex v_n. Clearly the set $\{v_1, v_2, \ldots, v_{n+1}\}$ is a connected dominating set of cardinality n+1. We remove one vertex from C_n we get the connected dominating number of cardinality n is n. That is $c_d(T_{n,1},i) = n$. Also connected dominating number with cardinality n – 1 is 2n – 3 and connected dominating number with cardinality n – 2 is n – 2. Therefore, $C_d(T_{n,1},x) = (n-2)\,x^{n-2} + (2n-3)\,x^{n-1} + n\,x^n + x^{n+1}$. Consider, $(n-2)\,x^{n-2} + (2n-3)\,x^{n-1} + n\,x^n + x^{n+1} = 0$

$\Rightarrow x^{n-2}\,[\,x^3 + n\,x^2 + (2n-3)\,x + n-2] = 0$

$\Rightarrow x^{n-2}\,(x+1)\,(x+1)\,(x + n-2) = 0$.

Hence the roots of this polynomial are 0 with multiplicity n-2 and -1 with multiplicity 2 and – n+2 with multiplicity 1.

IV. Conclusion

In this paper the connected domination polynomial for some standard graphs by identifying its connected dominating sets. It also helps us to find the roots of those polynomials.

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