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On Pseudo m – power Commutative near – rings

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Abstract: *A near ring N is called weak commutative if xyz = xzy for every x,y,z ϵ N[14]. N is called pseudo commutative if xyz = zyx for every x,y,z ϵ N[15]. N is called quasi weak commutative if xyz = yxz for every xyz = yxz forevery x,y,z ϵ N [11]. N is called pseudo m – power commutative if $x^m yz = zy^m x$ for every x,y,z ϵ N [10]. We obtain more results, generalising the results of [15].*

I. Introduction

S.Uma, R.Balakrishnan and T.Tammizhchelvam[15] called a near – ring N to be pseudo commutative if xyz =zyx forevery x,y,z ϵ N. G.GopalaKrishnamoorthy and S.Geetha[8] called a ring R to be m – power commutative if $x^m y = y^m x$ forall x,y ϵ R, where$m\geq 1$ is a fixed integer . They also called a ring R to be (m, n) power commutative if $x^m y^n = y^m x^n$ forallx,y ϵ R, where $m \geq 1$ and $n \geq 1$ are fixed integers. We [10] have defined a near – ring to be pseudo m-power commutative if $x^m yz = zy^m x$ forall x,y,z ϵ N, where $m \geq 1$ is a fixed integer. Also we have defined a near – ring N to be pseudo (m,n) power Commutative if $x^m y^n z = zy^m x^n$ for all x,y,z ϵ N, where $m\geq 1$ and $n\geq 1$ are fixed integers. In this paper we prove more general results on pseudo m – power commutative near – rings, this generalising the results of [15].

II. Preliminaries

Throughout this paper, N denotes a right near – ring with atleast two elements. For any non – empty subset A of N, we denote A – {0} as A^*. The following definitions and results are needed for the development of this paper.

2.1 Definition
Let N be a near – ring. An element a N is said to be Idempotent if $a^2 = a$
Nilpotent if there exists a positive integer k such that $a^k = 0$
2.2 Lemma (Pilz [14])
Each near – ring N is isomorphic to a sub direct product of subdirectly irreducible near – rings
2.3 Definition
A near – ring N is said to be zero – symmetric if ab = 0 implies ba = 0, where a,b ϵ N.
2.4 Lemma
If N is zero symmetric, then every left ideal Aof N is an N – subgroup of N every ideal I of N satisfies the condition NIN $\subseteq$ I
N*I*N* $\subseteq$ I*

2.5 Lemma
Let N be a near – ring. Then the following are true.
If A is an ideal of N and B is any subset of N, then (A : B) ={n ϵ N/ nB $\subseteq$ A} is always a left ideal.
If A is an ideal of N and B is any N – subgroup then (A :B) is an ideal . Inparticular if A and B are ideals of a zero symmetric near – ring, then (A: B) is an ideal .
2.6 Lemma
Let N be a regular near – ring, a ϵ N and a = axa, then
(i) ax and xa are idempotents and so the set of idempotent elements of N is non – empty.
(ii) axN = aN and Nxa = Na
(iii) N is S and S' near – rings.
2.7 Definition
A near – ring N is said to be reduced if N has no non – zero nilpotent elements
2.8 Lemma [3]
Let N be a zero – symmetric reduced near – ring. For any a,b ϵ N and for any idempotent element e ϵ N,
abe = aeb

2.9 Lemma [5, 6]
A near ring N is sub – directly irreducible if and only if the intersection of all non – zero ideals of N is not zero
2.10 Lemma [6]
Each simple near – ring is sub – directly irreducible
2.11 Lemma [13]
An N – subgroup A of N is essential if $A \cap B = \{0\}$ where B is any N subgroup of N implies $B = \{0\}$
2.12 Definition
A near – ring N is said to be an integral near – ring if N has no non – zero divisors.
2.13 Lemma
Let N be a near – ring such that for all $a \in N$, $a^2 = 0$ implies $a = 0$. Then N has no non – zero nilpotent elements. That is, N is reduced.
2.14 Definition
A near ring N is said to satisfy intersection of factors property (I F P) if $ab = 0$ implies $anb = 0$ forall $n \in N$, where $a, b \in N$
2.15 Lemma [14]
A non – zero symmetric near – ring n has intersection of factors property if and only if (O:S) is an ideal for any subset S of N
2.16 Definition
(i) Let N be a near – ring. An ideal I of N is called a prime ideal if for all ideals A, B of N, $AB \subseteq I$ $A \subseteq I$ or $B \subseteq I$
(ii) I is called a semi – prime ideal if for all ideals A of N, $A^2 \subseteq I$ implies $A \subseteq I$
(iii) I is called a completely semi – prime ideal if for any $x \in N$, $x^2 \in I$ implies $x \in I$
(iv) I is called a completely prime ideal if for any $x, y \in N$, $xy \in I$ implies $x \in I$ or $y \in I$
(v) N is said to have strong intersection of factors property if for all ideals I of N, $ab \in I$ implies $anb \in I$ for all $n \in N$

2.17 Lemma
Let N be a Pseudo Commutative near – ring. Then every idempotent element is central.

III. Main results

3.1 Lemma
Every pseudo m – power commutative (right) near – ring is zero symmetric
Proof
Let N be a pseudo m – power commutative near – ring. Then $x^m yz = zy^m x$ forall $x,y,z \in N$
Now for all $a \in N$,
$a.0 = a.0^{m+1}$
$= a.0^m.0$
$= 0^m.0a = 0a = 0$
This proves N is zero symmetric

3.2 Lemma
Every idempotent element in a pseudo m – power commutative near – ring is central
Proof
Let N be a pseudo m – power commutative near – ring and $e \in N$ be an idempotent element. Then it follows that $e^k = e$ for all $k \geq 2$
Now for any $a \in N$,
$e\,a = e^{m+1}\,a = e^m\,e\,a$
$= a\,e^m\,e = ae^{m+1}$
$= ae$
This proves e is central

3.3 Lemma
Homomorphic image of a pseudo m – power commutative near – ring is also a pseudo m – power commutative near – ring.
Proof
Let N be a pseudo m – power commutative near – ring. Let $f : N \to M$ be an endomorphism of near – rings. For all $x,y, z \in N$,
$f(x)^m\, f(y)\, f(z) = f(x^m yz)$

$= f(zy^m x)$
$= f(z)\ f(y)^m\ f(x)$
This proves M is pseudo m – power commutative .
3.4 Corrollary
Let N be a pseudo m – power commutative near – ring. If I is an ideal of N, then N / I is also pseudo m – power commutative
Proof
Since the canonical map η : N $\rightarrow$N/I is an endomorphism of near – rings, the corollary follows from the Lemma.
3.5 Theorem
Every pseudo m – power commutative near – ring N is isomorphic to a sub – direct product of sub – directly irreducible pseudo m – power commutative rings
Proof
By Lemma2.2, N is isomorphic to a subdirect product of sub- directly irreducible near – rings N_k and each N_k is a homomorphic image of N under the projection map π_k:N$\rightarrow$ N_k. The result follows from Lemma 3.4
3.6 Definition
Let N be a near- ring. N is said to be weak m – power commutative if $ab^mc = ac^mb$ for all a,b,c$\in$ N
3.7 Lemma
Any pseudo – m – power commutative near – ring with right identity is weak m – power commutative
Proof
Let N be a pseudo m – power commutative near – ring. Let a,b,c$\in$ N
Now, $ab^mc = (ab^mc)e$
$= a(b^mce)$
$= (ae)(c^mb)$ (N is pseudo m – power commutative)
$= (ae)(c^mb)$
$ab^mc = ac^mb$
This proves N is weak m – power commutative

3.8 Definition
Let N be a near – ring. N is said to be quasi – weak m – power commutative if $x^myz = y^mxz$ forall x,y,z$\in$ N
3.9 Lemma
Any weak m – power commutative near – ring with left identity is quasi – weak m – power commutative

Proof
Let N be a weak m – power commutative near – ring. Let a,b,c$\in$ N
Now $a^mbc = e(a^mbc)$
$= (ea^mb)c$
$= (eb^ma)c$
$= b^mac$
This proves N is quasi weak m – power commutative.

3.9 Definition
A near – ring N is said to be m – regular near – ring if for each a$\in$ N, where exists an element b$\in$N such that $a = ab^ma$ where m$\geq$ 1 is a fixed integer.
3.10 Lemma
Let N be a m – regular near – ring, a$\in$ N and $a = ab^ma$.
Then (i) ab^m,b^ma are idempotents
(ii) $ab^mN = aN$ and $Nb^ma = Na$
Proof
(i) Let a$\in$ N. Since N is m – regular, there exists b$\in$ N such that
$a = ab^ma$ ....(1)
Now $(ab^m)^2 = (ab^m)(ab^m)$
$= (ab^ma)b^m$
$= ab^m$
Similarly, $(b^ma)^2 = (b^ma)(b^ma) = b^m(ab^ma)$
$= b^ma$
Hence ab^m and b^ma are idempotents.
(ii) Let y$\in$ ab^mN
$\Rightarrow y = ab^mx$ for some x$\in$ N

$\in aN$
$\Rightarrow ab^mN \subseteq aN$
Let $y \in aN$
$y = az$ for some $z \in N$
$= (ab^ma)z$
$= ab^m(az)$
$\in ab^mN$
That is, $aN \subseteq ab^mN$
Hence $ab^mN = aN$
Similarly it can be proved $Nb^ma = Na$

3.11 Definition

Let N be a near – ring $A \subseteq N$ then $\sqrt{A} = \{x \in N / x^k \in A$ for some $k \geq 1.\}$

3.12 Theorem

Let N be a m – regular pseudo m power commutative near – ring.
Then $A = \sqrt{A}$ for every N – subgroup A of N.

Proof

Let A be an N – subgroup of N.
Since N is m – regular for every $a \in N$, there exists $b \in N$ such that
$a = ab^ma$.
By Lemma 3.10(i), ab^m, b^ma are idempotents
Since N is pseudo m – power commutative by Lemma 3.2, ab^m, b^ma are central.
Let $a \in \sqrt{A}$ .Then $a^k \in A$ for some positive integer k.
Now $a = a\, b^m\, a = a\, (b^m\, a)$
$a = (b^m\, a)\, a = b^m\, a^2$ ....(1)
$a = b^m\, a\, a = b^m\, (b^m\, a^2)\, a$
$= b^{2m}\, a^3$
$= b^{2m}\, a\, a^2$
$= b^{2m}\, (b^m\, a^2)\, a^2$
$= b^{3m}a^4$
. . .
. . .
$a = b^{(k-1)m}a^k \in NA \subseteq A$ for all $k \geq 1$ ....(2)
Hence $\sqrt{A} \subseteq A$
Obviously $A \subseteq \sqrt{A}$
Hence $A = \sqrt{A}$

3.13 Theorem

Let N be a m – regular pseudo m – power commutative near – ring. Then (i) N is reduced
(ii) N has IFP (A m – regular near – ring is said to have IFP if ab = 0 implies there exists $n \in N$ such that $an^mb = 0$)

Proof

Let $a \in N$ be such that $a^2 = 0$. By (i) of Theorem 3.12, $a = b^ma^2 = b^m \,.\, 0 = 0$
Hence N is reduced.
Let $x, y \in N$ such that $xy = 0$
Now $(yx)^2 = (yx)\,(yx) = y\,(xy)\,x$
$= y.0.x$
$= y.0$
$(yx)^2 = 0$
By (i) $yx = 0$
That is, N is zero commutative
Now for any $n \in N$, $(xn^my)^2 = xn^my.xn^my$
$= xn^m(yx)n^my$

$= xn^m 0 n^m y$
$= 0$

By (i) $xn^m y = 0$

3.14 Theorem

Let N be a m – regular pseudo m – power commutative near – ring . Then every N subgroup is an ideal.

Proof

Let $a \in N$. Since N is m – regular, there exists $b \in N$ such that $a = ab^m a$.
By Lemma 3.10(i) bma is idempotent
Let $b^m a = e$
Then $Ne = Nb^m a = Na$ (by Lemma 3.10 (ii))
Let $S = \{n-ne / n \in N\}$
Claim : $(O : S) = \{y \in N / sy = 0 \ \forall s \in S\} = Ne$
Now $(n-ne)e = ne - ne^2 = ne-ne = 0 \ \forall n \in N$
Since N has IFP, we have
$(n-ne)Ne = 0$
Hence $Ne \subseteq (O:S)$ ....(1)
Let y (O:S). Then $sy = 0 \ \forall s \in S$ ....(2)
Now N is m – regular. $y = yx^m y$ for some $x \in N$
Since $yx^m - (yx^m)e$ S, by (2) we get
$(yx^m - (yx^m)e) y = 0$
That is, $yx^m y - yx^m ey = 0$
$y - y(x^m ey) = 0$ ....(3)
Since N is zero symmetric reduced ring by Lemma 2.8, $x^m ey = x^m ye$
So, (3) becomes $y - y(x^m ye) = 0$
That is, $y - yx^m ye = 0$
$y - ye = 0$
Hence $y = ye \in Ne$
That is, $(O : S) \subseteq Ne$ ....(4)

and (4) gives $(O:S) = Ne = Nb^m a = Na$
Using Lemma 2.15, Na is an ideal of N.
Now if M is any subgroup of N, then $M = \sum_{a \in M} Na$
Thus M becomes an ideal of N.

3.15 Theorem

Let N be a m – regular pseudo m – power commutative near – ring. Then (i) N = Na = Na2 = aN = aNa forall $a \in N$
(ii)Any ideal of N is completely semi prime

Proof

Since N is m – regular, for every $a \in N$, there exists $b \in N$ such that
$a = ab^m a$
Then $a = ab^m a = (ab^m)a = a(ab^m) = a^2 b^m$ (by Lemma 3.10 (i))
Also $a = ab^m a = a(b^m a) = (b^m a)a = b^m a^2 \in Na^2$
Hence $N \subseteq Na^2$ ....(1)
Now $Na \subseteq N \subseteq Na^2 = (Na)a \subseteq Na \subseteq N$
So, $Na = Na^2 = N$ ....(2)
We shall now prove that $Na^2 = aN$
Let $x \in Na^2$.

Then $x = na^2$ for some $n \in N$
$= naa$
$= n(b^m a^2)a$
$= nb^m a^3$
$= (a^m bn)\, a^2$ (pseudo m – power commutative)
$= a(a^{m-1} bna^2)\, aN$

That is, $Na^2 \subseteq aN$ ....(3)

Let $y \in aN$.

Then $y = an$ for some $n \in N$

$= (b^m a^2)n$

$= b^m a(a^2 b^m)n$

$= b^m a(a^2 b^m n)$

$= b^m a(n^m ba^2)$ (pseudo m – power commutative)

$= (b^m a n^m b)a^2 \in Na^2$

So $aN \subseteq Na^2$ ....(4)

(3) and (4) gives $Na^2 = aN$ ....(5)

Next we shall prove that aN = aNa

Let $x \in aN$.

Then $x = an$ for some $n \in N$

$= (ab^m a)n$

$= a(b^m an) \in a(NaN) \subseteq aNa$

So, $aN \subseteq aNa$ ....(6)

Obviously $aNa \subseteq aN$

Hence aNa = Na ....(7)

From (2), (5) and (7) we get

$N = Na = Na^2 = aN = aNa$

Let I be any ideal of N and $a^2 \in I$

Now $a = a^2 b^m \in IN \subseteq I$

That is, $a^2 \in I$ implies $a \in I$

Hence I is Completely semi – prime.

3.16 Definition

A near – ring N is said to have the property P_4 if for all ideals I of N,
$xy \in I$ implies $yx \in I$, where $x,y \in N$

3.17 Theorem

Every m – regular pseudo m – power Commutative near – ring satisfies the property P_4

Proof

Let N be a m – regular pseudo m – power Commutative near – ring and I be an ideal of N. Let $a,b \in N$ such that $ab \in I$

Then $(ba)^2 = (ba)(ba)$

$= b(ab)a$

$\in N I N \subset I$

That is, $(ba)^2 \in I$

By Theorem 3.15 (ii), $ba \in I$

Thus N satisfies the property P_4

3.18 Theorem

Let N be a m – regular pseudo m – power Commutative near – ring. Then (i) For every ideal I of N, (I:S) is an ideal of N, where S is any subset of N

(ii) For every ideal I of N, $x_1,x_2,x_3,...,x_n \in N$ if $x_1,x_2,x_3,...,x_n \in I$, then $<x_1>, <x_2>, <x_3>,..., <x_n> \subset I$.

Proof

Let I be an ideal of N and S be any subset of N.

By Lemma 2.5, $(I:S) = \{n \in N / ns \subseteq I\}$ is a left ideal of N.

If $a \in (I:S)$, then $aS \subseteq I$. So, $as \in I$ forall $s \in S$.

Then by Theorem 3.16, $sa \in I$. Then for any $n \in N$, $(sa)n \in I$.

That is, $s(an) \in I$. By Theorem 3.17, $(an)s \in I$. So $an \in (I:S)$ forany $n \in I$.

Hence (I:S) is a right ideal. Consequently (I:S) is an ideal. This completes the proof 3.17 (i).

Let $x_1,x_2,x_3,...,x_n \in I$

$\Rightarrow x_1 \in (I : x_2.x_3....,x_n)$

$\Rightarrow < x_1 > \subseteq (I : x_2.x_3 .x_4....,x_n)$

$\Rightarrow < x_1 > x_2.x_3....,x_n \subseteq I$

$\Rightarrow x_2.x_3....,x_n < x_1 > \subseteq I$
$\Rightarrow x_2 \in (I : x_3.x_4....,x_n < x_1>)$

$\Rightarrow < x_2 > \subseteq (I : x_3.x_4....,x_n < x_1>)$
$\Rightarrow < x_2 > x_3.x_4....,x_n < x_1 > \subseteq I$
$\Rightarrow x_3.x_4....,x_n < x_1 >< x_2 > \subseteq I$
Continuing like this, we get $<x_1>, <x_2>, <x_3>,..., <x_n> \subset I$.

3.19 Theorem

Let N be a m – regular pseudo m – power Commutative near – ring. Then (i) N has strong IFP
(ii) N is a semi – prime near –ring

Proof

Let I be an ideal of N such that $ab \in I$, where $a,b \in N$. By Lemma 3.1, N is zero symmetric $NI \subseteq I$.
By Theorem 3.15 $aN = Na^2$.
Hence $an = ma^2$ for some $m,n \in N$
Then for any $n \in N$, $anb = ma^2b$
$= (ma)ab \in NI \subseteq I$
That is, N has strong IFP
Let M be an N – subgroup of N. Then by Theorem 3.14, M is an ideal of such that $I^2 \subseteq M$.
Since N is zero symmetric, $NI \subseteq I$.
If $a \in I$, then $a = ab^ma \in I(NI) \subseteq I^2 \subseteq M$.
So, any N – subgroup M of N is a semi – prime ideal. In particular {0} is semi – prime ideal and hence N is a semi – prime near – ring.

3.20 Note

When m = 1, all the results of [15] are obtained.

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