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On Jordan Higher Bi-Derivations On Prime Gamma Rings

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Abstract; *In this study , we define the concepts of a higher bi-derivation , Jordan higher bi-derivation and Jordan triple higher bi-derivation on Γ-rings and show that a Jordan higher bi-derivation on 2-torsion free prime Γ-ring is a higher bi-derivation .*

I. Introduction

Let M and Γ be two additive abelian groups . If there exists a mapping(a, α ,b) $\to a\alpha b$ of $M\times\Gamma\times M \to M$ satisfying the following for all a,b,c ∈ M and α, β ∈ Γ

i) (a+b) α c =a α c + b α c
a(α + β)b = a α b + a β b
a α (b+c) = a α b + a α c
ii) (a α b) β c = a α (b β c)

Then M is called Γ -ring .

The notion of a Γ -ring was introduced by Nobusawa [4] and generalized by Barnes [1] as defined above . Many properties of were obtained by Barnes [1] , kyuno [2] , Luh [3] and others . let M be a Γ -ring . then M is called 2-torsion free if 2a= 0 implies a= 0 for all a ∈ m . Besides , M is called a prime if a ΓMΓ b=(0), for all a,b ∈ M , implies either a= 0 or b= 0 . and, M is called semiprime if a ΓMΓ a=(0) with implies a= 0 . Note that every prime is obviously semiprime.

M is said to be a commutative Γ -ring if a α b=b α a holds for all a,b ∈ M and α ∈ Γ and α ∈ Γ . Let M be a Γ -ring,then ,for a,b ∈ M and α ∈ Γ ,we define $[a,b]_\alpha$ =a α b-b α a known as the commutator of aand b with respect to α .The notion of derivation and Jordan derivation on a Γ-ring were defined by M. Sapanci and A. Nakajima in [5] as follows an additive mapping d:M → M is called a derivation of M if d(a α b)= d(a) α b +a α d(b) for all a,b ∈ M ,And , if d(a α a)= d(a) α a +a α d(a) for all a ∈ M and α ∈ Γ ,then d is called a Jordan derivation of M .

A mapping d:M → M is said to be symmetric if d(a,b)=d(b,a) ,for all a,b ∈ M .An bi-additive mapping d:M × M → M is called a symmetric bi-derivation on M×M into M if d(a α b,c) =d(a,c) α b+a α d(b,c) for all a,b,c ∈ M and α ∈ Γ .And , if d(a α a,c) =d(a,c) α a+a α d(a,c) for all a,c ∈ M and α ∈ Γ ,then d is called a Jordan bi-derivation on M×M into M. The notion of symmetric bi-derivation was introduced by G.Maksa [6]

A mapping F:M → M defined by F(a) =D(a,a),where D:M × M → M is a symmetric mapping is called the trace of D it is obvious that in the case is a symmetric mapping which is also bi-additive (i.e. additive in both arguments) .the trace F of D satisfies the relation F(a+b) = F(a)+F(b),for all a,b ∈ M .

In our work we need the following lemma.

Lemma 1.1. [7] let M be a 2-torsion free semi prime Γ-ring and suppose that a,b ∈ M if a Γ m Γ b+b Γ m Γ a =(0) for all a,b ∈ M , thenn a Γ m Γ b=b Γ m Γ a =(0) .

II. Higher bi-derivation on Γ-ring :

In this section we present the concepts of higher bi-derivation , Jordan higher bi-derivation and Jordan triple higher bi-derivation on Γ-rings and we study the properties of them.

Definition 2.1: Let M be a Γ-ring and D=$(d_i)_{i\in I}$ be a family of bi-additive mappings on M×M into M , such that d_0 (a,b)=a for all a,b ∈ M ,then D is called a higher bi-derivation on M×M into M if for every a,b,c,d ∈ M , and α ∈ Γ

$$d_n(a\,\alpha\,b,c\,\alpha\,d) = \sum_{i+j=n} d_i(a,c)\,\alpha\, d_j(b,d)$$

D is said to be a Jordan higher bi-derivation if

$$d_n(a\,\alpha\,a,c\,\alpha\,c) = \sum_{i+j=n} d_i(a,c)\,\alpha\, d_j(a,c)$$

D is called a Jordan triple higher bi-derivation

$$d_n(a\,\alpha\,b\,\beta\,a,c\,\alpha\,d\,\beta\,c) = \sum_{i+j+k=n} d_i(a,c)\,\alpha\, d_j(b,d)\,\beta\,d_k(a,c)$$

Note that $d_n(a+b,c+d)= d(a,c)+ d(b,d)$ for all $a,b,c,d \in M$ and $n \in N$..

Example 2.2.
Let M= { (a) : x,y ϵR }, R is real number .
M be a Γ-ring of 2×2 matrices and Γ= { : rϵR } we use the usual addition and multiplication on matrices of M×Γ×M , we define di : M×Γ×M→M , iϵN by
di (,) = for all, ϵ M
= Such that m =
Then d is a higher bi- derivation on Γ-ring

Lemma 2.3.
let M be a Γ-ring and D =(di)IϵN be a Jordan higher bi- derivation on M×M into M. then for all a,b,c,d,s,t ϵ M , α , βϵΓ and n ϵ N , the following statements hold :

(i) $d\,n\,(a\alpha\, b+b\,\alpha\,a\, ,\, c\,\alpha\,d + d\,\alpha\, c) = \sum_{i+j=n} d_i(a,c)\,\alpha\,d_j(b,d) + d_i(b,d)\,\alpha\,d_j(a,c)$

(ii) $dn\ (a\,\alpha\,b\beta a + a\beta b\,\alpha\,a\, ,\, c\,\alpha\,d\beta c) = \sum_{i+j+k=n} d_i(a,c)\,\alpha\,d_j(b,d)\,\beta\,d_k(a,c) + d_i(a,c)\,\beta\,d_j(b,d)\,\alpha\,d_k(a,c)$

Especially , if M is 2-torsion free ,then

(iii) $d_n\ (a\,\alpha\,b\,\alpha\,c + c\,\alpha\,b\,\alpha\,a\, ,\, s\,\alpha\,d\,\alpha\,t+t\,\alpha\, d\,\alpha\,s) = \sum d_i(a,s)\,\alpha\,d_j(b,d)\,\alpha\,d_k(c,t) + d_i(c,t)\,\alpha\,d_j(b,d)\,\alpha\,d_k(a,s)$

Proof. (i) is obtained by computing and (ii) is also obtained by replacing aβb+bβa for b and cβd+dβc for d in (i) , in (ii) . If we replace a+c for a and s+t for c in (iii) , we can get (iv).

Definition 2.4. let M be a Γ-ring and D =(d_i),iϵN be a Jordan higher bi- derivation on M×M into. then for all a,b,c,d ϵ M , α ϵΓ and n ϵ N , we define

$$\psi_n(a,b,c,d)_\alpha = dn\,(a\,\alpha\,b,c\ \alpha\,d) - \sum_{i+j=n} d_i(a,c)\,\alpha\,d_j(b,d)$$

Lemma 2.5 . let M be a Γ-ring and D=(d_i),iϵN be a Jordan higher bi- derivation on M×M into M. then for all a,b,c,d,s,t ϵ M , α , βϵΓ and n ϵ N.
i) $\psi_n(a,b,c,d)_\alpha = -\ \psi_n(b,a,d,c)$
ii) $\psi_n(a+s,b,c,d)_\alpha = \psi_n(a,b,c,d)_\alpha + \psi_n(s,b,c,d)_\alpha$
iii) $\psi_n(a,b+s,c,d)_\alpha = \psi_n(a,b,c,d)_\alpha + \psi_n(a,s,c,d)_\alpha$
iv) $\psi_n(a,b,c+s,d)_\alpha = \psi_n(a,b,c,d)_\alpha + \psi_n(a,b,s,d)_\alpha$
v) $\psi_n(a,b,c,d+s)_\alpha = \psi_n(a,b,c,d)_\alpha + \psi_n(a,b,c,s)_\alpha$
Proof . These results follow easily by lemma 1 and the definition of ψ_n.

Note that d is a higher bi-derivation iff $\psi_n(a,b,c,d)_\alpha = 0$ for all $a,b,c,d \in M$, $\alpha \in \Gamma$ and $n \in N$.

III. The Main Results

Throughout the following , we assume that M is an arbitrary Γ-ring and f a higher bi-derivation on M×M into M . clearly , every higher bi- derivation on M×M in to M is a Jordan bi-derivation . the converse in general is not true . in the present paper ,it is shown that every Jordan higher bi- derivation on certain Γ-rings is a higher bi-derivation .

Lemma 3.1. let M be a 2-torsion free and D =(d_i),iϵN be a Jordan higher bi- derivation on M×M into M . then for all a,b,c,d,s,t ϵ M , α , βϵΓ and n ϵ N , if $\psi_n(a,b,c,d)_\alpha = 0$ for every $t \leq n$ then:

$\psi_n(a,b,c,d)_\alpha\ \beta m \beta [a,b]_\alpha + [a,b]_\alpha \beta m \beta\ \psi_n(a,b,c,d)_\alpha = 0$

Proof . let s∈M , since d_n is bi- additive mapping then by Lemma 2.1. (iv) we obtain :

$d_n(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b,\ c\alpha d\beta s\beta d\alpha c + d\alpha c\beta s\beta c\alpha d)$

$= d_n((a\alpha b)\beta m\beta (b\alpha a) + (b\alpha a)\beta m\beta (a\alpha b),\ (c\alpha d)\beta s\beta (d\alpha c) + (d\alpha c)\beta s\beta (c\alpha d))$

$= \sum_{i+j+k=n} d_i(a\alpha b, c\alpha d)\ \beta d_j(m,s)\ \beta d_k(b\alpha a, d\alpha c) + d_i(b\alpha a, d\alpha c)\beta d_j(m,s)\ \beta d_k(a\alpha b, c\alpha d)$

$= d_n(a\alpha b, c\alpha d)\beta m\beta b\alpha a + a\alpha b\beta m\beta d_n(b\alpha a, c\alpha d) + \sum_{i+j+k=n}^{i,k<n} d_i(a\alpha b, c\alpha d)\beta d_j(m,s)\beta d_k(b\alpha a, d\alpha c) + d_i(b\alpha a, d\alpha c)\beta d_j(m,s)\beta d_k(a\alpha b, c\alpha d)$

$= d_n(a\alpha b, c\alpha d)\beta m\beta b\alpha a + a\alpha b\beta m\beta d_n(b\alpha a, d\alpha c) + d_n(b\alpha a, d\alpha c)\beta m\beta a\alpha b + b\alpha a\beta m\beta d_n(a\alpha b, c\alpha d) + \sum_{q+t+j+h+g=n}^{q+t,h+g<n} d_q(a,c)\alpha d_t(b,d)\beta d_j(m,s)\beta d_h(b,d)\alpha d_g(a,c) + d_q(b,d)\alpha d_t(a,c)\beta d_j(m,s)\beta d_h(a,c)\alpha d_g(b,d) \quad \ldots(1)$

On the other hand : by lemma 2.3. (iii)

$d_n(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b,\ c\alpha d\beta s\beta d\alpha c + d\alpha c\beta s\beta c\alpha d)$

$= d_n(a\alpha(b\beta m\beta b)\alpha a + b\alpha(a\beta m\beta a)\alpha b,\ c\alpha(d\beta s\beta d)\alpha c + d\alpha(c\beta s\beta c)\alpha d)$

$= \sum_{q+k+g=n} d_q(a,c)\alpha d_k(b\beta m\beta b, d\beta s\beta d)\alpha d_g(a,c) + d_q(b,d)\alpha d_k(a\beta m\beta a)\alpha d_g(b,d)$

$= \sum_{q+t+j+h+g=n} d_q(a,c)\alpha d_t(b,d)\beta d_j(m,s)\beta d_h(b,d)\alpha d_g(a,c) + d_q(b,d)\alpha d_t(a,c)\beta d_j(m,s)\beta d_h(a,c)\alpha d_g(b,d)$

$= \sum_{q+t=n} d_q(a,c)\alpha d_t(b,d)\beta m\beta b\alpha a + a\alpha b\beta m\beta \sum_{h+g=n} d_h(b,d)\alpha d_g(a,c) + \sum_{q+t=n} d_q(b,d)\alpha d_t(a,c)\beta m\beta a\alpha b + b\alpha a\beta m\beta \sum_{h+g} d_h(a,c)\alpha d_g(b,d) + \sum_{q+t+j+h+g=n}^{q+t,h+g<n} d_q(b,d)\alpha d_t(b,d)\beta d_j(m,s)\beta d_h(b,d)\alpha d_g(a,c) + d_q(a,c)\alpha d_t(a,c)\beta d_j(m,s)\beta d_h(a,c)\alpha d_g(b,d) \quad \ldots(2)$

Compare (1) and (2) we get :

$d_n(a\alpha b, c\alpha d)\beta m\beta b\alpha a - \sum_{q+t=n} d_q(a,c)\alpha d_t(b,d)\beta m\beta b\alpha a + a\alpha b\beta m\beta d_n(b\alpha a, d\alpha c) - a\alpha b\beta m\beta \sum_{h+g=n} d_h(b,d)\alpha d_g(a,c) + d_n(b\alpha a, d\alpha c)\beta m\beta a\alpha b - \sum_{q+t=n} d_q(b,d)\alpha d_t(a,c)\beta m\beta a\alpha b + b\alpha a\beta m\beta d_n(a\alpha b, c\alpha d) - b\alpha a\beta m\beta \sum_{g+h=n} d_h(a,c)\alpha d_g(b,d) = 0$

$\psi_n(a,b,c,d)_\alpha m\beta\ \beta b\alpha a + a\alpha b\beta m\beta\ \psi_n(a,b,c,d)_\alpha + \psi_n(b,a,d,c)_\alpha\ \beta m\beta a\alpha b + b\alpha a\beta m\beta\ \psi_n(a,b,c,d)_\alpha = 0$

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [b,a] + [b,a]\beta m\beta \psi n(a,b,c,d)\alpha = 0$

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [a,b] + [a,b]\beta m\beta \psi n(a,b,c,d)\alpha = 0.$

Lemma 3.2. let M be 2-torsion free prime Γ -ring and $D=(d_i)$, i ∈ N be a Jordan higher bi-derivation on M×M into M . then for all a,b,c,d,m ∈ M, α, β ∈ Γ and n ∈ N

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [a,b] = [a,b]\beta m\beta \psi n(a,b,c,d)\alpha = 0.$

Proof : By lemma 3.1. and lemma 1.1. , we obtain the proof .

Theorem 3.3. let M be 2-torsion free prime Γ -ring and $D=(d_i)$, i ∈ N be a Jordan higher bi-derivation on M×M into M , then for all a,b,c,d,m ∈ M α, β ∈ Γ and n ∈ N

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [s,t] = 0.$

Proof . Replacing a+s for a in lemma 3.2. we get

$\psi_n(a+s,b,c,d)_\alpha\ \beta m\beta [a+s,b] = 0$

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [a,b] + \psi_n(a,b,c,d)_\alpha\ \beta m\beta [s,b] + \psi_n(s,b,c,d)_\alpha\ \beta m\beta [a,b] + \psi_n(s,b,c,d)_\alpha\ \beta m\beta [s,b] = 0$

By lemma 3.2. we get

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [s,b] + \psi_n(s,b,c,d)_\alpha\ \beta m\beta [a,b] = 0$

There fore

$\psi_n(a,b,c,d)_\alpha\ \beta m\beta [s,b]\beta m\beta\ \psi_n(a,b,c,d)_\alpha\ \beta m\beta [s,b]$

$=-\psi_n(a,b,c,d)_\alpha \beta m \beta [s,b] \beta m \beta \psi_n(s,b,c,d)_\alpha \beta m \beta [a,b] = 0$

Hence , by the primmess on M :
$\psi_n(a,b,c,d)_\alpha \beta m \beta [s,b] = 0$ …(1)
Similarly , by replacing b+t for b in this equality we get :
$\psi_n(a,b,c,d)_\alpha \beta m \beta [a,t] = 0$ …(2)
Thus : $\psi_n(a,b,c,d)_\alpha \beta m \beta [a+s,b+t] = 0$
$\psi_n(a,b,c,d)_\alpha \beta m \beta [a,b] + \psi_n(a,b,c,d)_\alpha \beta m \beta [a,t] + \psi_n(a,b,c,d)_\alpha \beta m \beta [s,b] + \psi_n(a,b,c,d)_\alpha \beta m \beta [s,t] = 0$
By using (1) , (2) and lemma 3.2. we get
$\psi_n(a,b,c,d)_\alpha \beta m \beta [s,t] = 0$

Theorem 3.4. Let M be 2-torsion free prime Γ-ring .Then every Jordan higher bi-derivation on M×M into M is a higher bi-derivation on M×M into M.
Proof. Let M be 2-torsion free prime Γ-ring and D=(d_i) i ∈ N be a Jordan higher bi-derivation on M×M into M By Theorem 3.3.
$\psi_n(a,b,c,d)_\alpha \beta m \beta [s,t] = 0$ for all a,b,c,d,m,s,t ∈ M , α, β ∈ Γ .and n ∈ N since M is prime , we get either $\psi_n(a,b,c,d)_\alpha = 0$ or [s,t]=0,for alla,b,c,d,s,t ∈ M , α ∈ Γ .and n ∈ N
if [s,t] ≠ 0 for all s,t ∈ *M* Then $\psi_n(a,b,c,d)_\alpha = 0$ for all a,b,c,d ∈ M, α ∈ Γ and n ∈ *N* hence we get , D is a higher bi-derivation on M×M into M.
But, if [s,t] = 0 for all s,t ∈ *M* and α ∈ Γ , then M is commutative and therefore , we have from lemma 2.2.(i)

$$2d_n(a\alpha b, c\alpha d) = 2\sum_{i+j=n} d_i(a,c)\alpha d_j(b,d)$$

Since M is 2-torsion free , we obtain that D is a higher bi-derivation on M×M into M.

Proposition 3.5. Let M be 2-torsion free Γ-ring then every Jordan higher be-derivation on M×M into M such that aαbβc=aβbαc for all a,b,c ∈ M and α, β ∈ Γ is a Jordan triple higher bi-derivation on M × M into M .
Proof. Let M be 2-torsion free Γ-ring and D=(d_i) i ∈ N be a Jordan higher bi-derivation on M×M into M By lemma 2.3(ii).
$d_n(a\alpha b\beta a + a\beta b\alpha a, c\alpha d\beta c + c\beta d\alpha c)$

$$= \sum_{i+j+k=n} d_i(a,c)\alpha d_j(b,d)\beta d_k(a,c) + d_i(a,c)\beta d_j(b,d)\alpha d_k(a,c)$$

for all a,b,c,d ∈ M , α, β ∈ Γ . and n ∈ N
$d_n(a\alpha b\beta a, c\alpha d\beta c) + d_n(a\beta b\alpha a, c\beta d\alpha c)$

$$= \sum_{i+j+k=n} d_i(a,c)\alpha d_j(b,d)\beta d_k(a,c) + \sum_{i+j+k=n} d_i(a,c)\beta d_j(b,d)\alpha d_k(a,c)$$

Since aαbβc=aβbαc for all a,b,c ∈ M and α, β ∈ Γ , we get

$$2d_n(a\alpha b\beta a, c\alpha d\beta c) = 2\sum_{i+j+k=n} d_i(a,c)\alpha d_j(b,d)\beta d_k(a,c)$$

Since M is a 2-torsion free we have :

$$d_n(a\alpha b\beta a, c\alpha d\beta c) = \sum_{i+j+k=n} d_i(a,c)\alpha d_j(b,d)\beta d_k(a,c)$$

i.e D is Jordan triple higher bi-derivation on M × M into M .

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