IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 3 Ver. VII (May. - Jun. 2016), PP 22-26
www.iosrjournals.org

A Review on Construction of Proximate Orders for Generalized Biaxisymmetric Potentials in Open Hyper Sphere

DP Singh

Abstract: *This paper deals with* an *approximation theorem on proximate orders and this result is followed by some additional asymptotic properties of these functions. Finally proximate order is constructed for a* class *of GBSP,* $F^{(\alpha,\beta)}$ *regular in open hyper sphere of radius R under certain conditions.*

Keywords: *GBSP, Proximate order, Analytic function,*

I. Introduction

An entire function is a function of a complex variable which is holomorphic in the finite complex plane. Since an entire function f (z) is holomorphic in the finite complex plane, it has derivatives of all orders at every point of the plane and thus f(z) can be represented by a MacLaurin Series, $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$, which converges for all values of z. It is natural to think of entire functions as being a generalization of polynomials, and indeed many of their properties are quite similar to those of polynomials.

We consider a positive function $\rho(r)$ in $0 < r < R, 0 < r < \infty$ and having the properties:

(i) $\rho(r) \to \rho$ as $r \to R, 0 \le \rho < \infty$,

(ii) $\frac{(R-r)ln\left(\frac{R-r}{R}\right)\rho^{'}(r)}{R\rho(r)} \to \alpha - R$ as $r \to R, 0 \le \alpha < \infty$,

where $\rho^{'}(r)$ denotes the derivative of $\rho(r)$ and $\alpha = R$ for $0 < \rho < \infty$ and $\alpha \ne R$ corresponds to $\rho = 0\ or\ \infty$. Such a function is called the proximate order.

The GBSP are natural extensions of harmonic or analytic functions. Hence we anticipate properties similar to those of the harmonic functions found from associated analytic function$f(z)$, by taking Ref the real part of $f(z)$. Let a real valued GBSP $F^{(\alpha,\beta)}$, regular in $\sum_R^{(\alpha,\beta)}$, having order $\rho(0 < \rho < \infty)$[4] and satisfying in addition to (i) and (ii). Then for a given $\sigma(0 < \sigma < \infty)$, $\rho(r)$ satisfies also:

(iii) $\rho(r)$ is continuous and piecewise differentiable for $r > r_0$; and

(iv) $\lim_{r\to R} Sup \frac{\ln M\left(r,F^{(\alpha,\beta)}\right)}{(R/R-r)^{\rho(r)}} = \sigma, M\left(r, F^{(\alpha,\beta)}\right) = \max_{|z|=r} \left|F^{(\alpha,\beta)}(z,0)\right|$, and $\lim_{r\to R} Sup \frac{\ln\ \ln \ln M\left(r,F^{(\alpha,\beta)}\right)}{\ln \ln (R/R-r)} = \sigma$

This comparison function $\rho(r)$ is called the proximate order of the given GBSP$F^{(\alpha,\beta)}$. The quantity σ is termed as type of GBSP with respect to $\rho(r)$. Obviously, the proximate of order a GBSP is not uniquely determined . If, we add, $c/\ln(R/R-r), 0 < c < \infty$, to the proximate order $\rho(r)$ we obtain a new proximate order for the same GBSP $F^{(\alpha,\beta)}$ and the corresponding value of σ is divided bye^c. The existence of such function $\rho(r)$ for analytic function has been established by Kasana[2].

By Hadamard three circle theorem, we know it $f(z)$ is analytic in first disc, $\ln M(r, f)$ is an increasing convex function of $\ln r$ in$0 < r < R$. Using above theorem for GBSP $F^{(\alpha,\beta)}(x, y)$, we have $F^{(\alpha,\beta)}(x, y)$ is regular in open hypersphere $\sum_R^{(\alpha,\beta)}$, $\ln M\left(r, F^{(\alpha,\beta)}\right)$ is an increasing convex function of $\ln r$ in $0 < r < R$. Thus we have

$$\ln M\left(r, F^{(\alpha,\beta)}\right) = \ln M\left(r_0, F^{(\alpha,\beta)}\right) + \int_{r_0}^{r} \frac{w\left(x,F^{(\alpha,\beta)}\right)}{x} dx,\ \ 0 < r_0 < r < R. \qquad (1.1)$$

Where $w\left(x, F^{(\alpha,\beta)}\right)$ is positive, continuous and piecewise differentiable function of *x*.

II. Basic Results

In this section we will proof some theorem and lemmas.

Asymptotic Properties:

Theorem 2.1: For every proximate order $\rho(r) \in C^{'}(0, R)$, there exists a proximate order $\rho_1(r) \in C^2(0, R)$ such that

$$\left|\ln \frac{\rho(r)}{\rho_1(r)}\right| = O[(\ln(R/R-r))^{-1}]\ \ as\ \ r \to R, \qquad (2.1)$$

and

$$\lim_{r\to R} \frac{(R-r)^2 \ln\left(\frac{R-r}{R}\right)\rho_1{}^{"}(r)}{R^2\rho_1(r)} = \vartheta,\ \ 0 \le \vartheta \le \frac{\sigma-R}{4\delta},\ \ \delta > 0, \qquad (2.2)$$

where $C^m(0, R), m = 1,2$ is the space of all functions defined on $(0, R)$, whose m^{th} derivatives are continuous.

Proof: Let us assume that $\rho_1(r)$ be a proximate order and coincide with $\rho(r)$ on the sequence $\{r_n\}$ as
$\rho(r) = \rho_1(r) r_n = 1 - \frac{1}{4^n}, \quad n = 0,1,2 \dots\dots\dots$ (2.3)
and
$\lim_{r\to R} \frac{(R-r)\ln\left(\frac{R-r}{R}\right)\rho_1{}'(r)}{R\rho_1(r)} \to \beta - R$ (2.4)
In this case, for r lying in the intervals $[r_n, r_{n+1}]$
$$\left|\ln\frac{\rho(r)}{\rho_1(r)}\right| = \left|\int_{r_n}^{r}\left\{\frac{\rho'(x)}{\rho_1(x)} - \frac{\rho_1{}'(x)}{\rho_1(x)}\right\}dx\right| = \left|\int_{r_n}^{r} o\left\{\frac{R}{(R-x)\ln\left(\frac{(R-x)}{R}\right)}\right\}dx\right|$$
$$= o\left[\ln\frac{\ln\left(\frac{R-r}{R}\right)}{\ln\left(\frac{R-r_n}{R}\right)}\right] o\left[\left(\ln\left(\frac{R}{R-r}\right)\right)^{-1}\right] \quad as\ r \to R.$$
Thus, it is suffices to construct a proximate order$\rho_1(r) \in C^2(0,R)$ satisfying conditions (2.2),(2.3),(2.4). On the interval [0,3/4], we define
$$\phi(t) = \begin{cases} t & 0 \le t \le \frac{1}{4} \\ -2t + \frac{3}{4} & \frac{1}{4} \le t \le \frac{1}{2} \\ t - \frac{3}{4} & \frac{1}{2} \le t \le \frac{3}{4} \end{cases}$$
and
$\phi(\alpha) = \int_0^{\alpha} \phi(t)dt.$
Since $\phi(t)$ in continuous on [0,3/4], it follows that $\phi(\alpha) > C'(0,R)$. We observe
(a) $0 = \phi(0) = \phi\left(\frac{3}{4}\right) = \phi'(0) = \phi'\left(\frac{3}{4}\right)$
(b) $0 \le \phi(\alpha) \le \frac{3}{16}$
(c) $|\phi'(\alpha)| \le \frac{1}{4}$
(d) $\int_0^{\frac{3}{4}} \phi(\alpha)d\alpha = \delta > 0.$
Consider a sequence $\{\varepsilon_n\}$ such that
$\varepsilon_n = \frac{\ln\rho(r_{n-1}) - \ln\rho(r_n)}{2\ln 2}\ln\left(\frac{R-r_n}{R}\right).$
Using property (ii) of $\rho(r)$ it can be shown that $\varepsilon_n \to \alpha - h\ as\ n \to \infty$. Finally, we define the function
$\ln\rho_1(r) = \ln\rho(r_n) - \frac{R\epsilon n\delta^{-1}}{(R-r_n)\ln\left(\frac{R-r_n}{R}\right)}\int_{r_n}^{r}\phi\left(\frac{t-r_n}{R-r_n}\right)dt.$ (2.5)
On the interval $[r_n, r_{n+1}]$, where $\beta \in \left[R, R + \frac{3(\alpha-R)}{16\delta}\right]$. The verification of properties (i) and (ii) and derivation of (2.2) for the positive function $\rho_1(r)$in quite easy so we omit the details.

Remarks2.1: This theorem remains valid if we understand proximate order to be an arbitrary continuous differentiable function $\rho(r)$ satisfying (ii).
A real valued function $L(r)$in $0 < r < R$ is said to be slowly changing at R, if for every k, $R < k < \infty$,
$\lim_{r\to R}\frac{L\left(r + \frac{R-r}{kR}\right)}{L(r)} = R.$

Theorem2.2: Let $\rho(r)$ be proximate order. For $\rho(0 \le \rho < \infty)$,we find
(a) $\left[\frac{R}{R-r}\right]^{\rho(r)-\rho}$ is a slowly changing function of r in $0 < r < R$.
(b) $\left[\frac{R}{R-r}\right]^{\rho(r)}$ is monotonic increasing function for $r > r_0\ and\ \alpha > 0$.
$L(r) = \left[\frac{R}{R-r}\right]^{\rho(r)-\rho}$ (2.6)
hence
$\frac{L'(r)}{L(r)} = \ln\left(\frac{R}{R-r}\right)\rho'(r) \to 0\ as\ r \to R$. Thus, for all values of r sufficiently close to R

$\frac{L'(r)}{L(r)} = o\left(\frac{R}{R-r}\right). \Rightarrow \lim_{r\to R}\frac{L\left(r+\frac{R-r}{kR}\right)}{L(r)} = 0.$ (2.7)
For $\rho = 0$, we have

$\frac{(R-r)L'(r)}{RL(r)} = \rho(r)\left\{\frac{(R-r)\left(\frac{R}{R-r}\right)\rho'(r)}{R\rho(r)} + R\right\}.$

Again in view of property (ii), $\frac{(R-r)L'(r)}{RL(r)} \to 0 \; as \; r \to R$; and hence (2.7) is obtained which means

$L\left(r + \frac{R-r}{kR}\right) \approx L(r) \; as \; r \to R.$ (2.8)

The proof of part (b) for $\rho > 0$is given as

$\frac{d}{dr}\left[\left(\frac{R}{R-r}\right)^{\rho(r)}\right] = \left(\frac{R}{R-r}\right)^{\rho(r)+1}\left\{\rho(r) + \left(\frac{R-r}{R}\right)\ln\left(\frac{R}{R-r}\right)\rho'(r)\right\} > (\rho-\varepsilon)\left(\frac{R}{R-r}\right)^{\rho(r)+1} > 0.$

Similarly, for $\rho = 0$,we have

$$\frac{d}{dr}\left[\left(\frac{R}{R-r}\right)^{\rho(r)}\right] = \left(\frac{R}{R-r}\right)^{\rho(r)+1}\rho(r)\left[\frac{(R-r)\ln\left(\frac{R}{R-r}\right)\rho'(r)}{R\rho(r)} + R\right]$$

$$> \left(\frac{R}{R-r}\right)^{\rho(r)+1}\rho(r)(\alpha-\varepsilon), 0 < \varepsilon < \alpha,$$

In this case $\rho(r)\ln\left(\frac{R}{R-r}\right) \to \infty \; as \; r \to R.$

Corollary2.1: For every $k > 1$,we have

$$\lim_{r\to R}\frac{\left[\left(\frac{k}{k-1}\right)/\left(\frac{R}{R-r}\right)\right]^{\rho\left(r+\frac{R-r}{kR}\right)}}{\left(\frac{R}{R-r}\right)^{\rho(r)}} = \frac{k}{k-1}\rho.$$

This corollary is the direct consequence of (2.6) and (2.8).

Remarks2.2: The proof of theorem 2.2 is simpler and generalized than the proof given by Juneja and Kapoor [1, thm.1.6.2. pp.58-59] based on Lagrange mean value theorem. Also, Levin [3. lemma, pp.32-33] used mean value theorem to prove a simpler result on slowly growing function in reference to proximate order of entire function having nonzero finite order.

Theorem2.3: For $\rho - 1 > \beta;\quad 0 \le \rho < \infty$ and $R > r > \vartheta > 0$,

$\int_\vartheta^r \left(\frac{R}{r-t}\right)^{\rho(t)-\beta} dt = \frac{\left(\frac{R}{R-r}\right)^{\rho(r)\beta}}{\rho-\beta-1} + o\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}.$ (2.9)

Proof: integrating by parts with $\rho > 0$ as

$$\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta} dt = \int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\rho}\left(\frac{R}{R-t}\right)^{\rho-\beta} dt$$

$= \frac{\left(\frac{R}{R-t}\right)^{\rho(t)-\beta-1}}{\rho-\beta-1}\Big|_\vartheta^r - \frac{1}{\beta+1-\rho}\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta}\left\{\left(\frac{R-t}{R}\right)ln\left(\frac{R}{R-t}\right)\rho'(t) + \rho(t) - \rho\right\}dt$ (2.10)

Using the definition of proximate order we have asymptotically.

$|\rho(t)-\rho| < \frac{\varepsilon}{2},\quad \left(\frac{R-t}{R}\right)ln\left(\frac{R}{R-t}\right)\rho'(t) < \frac{\varepsilon}{2}.$

Hence

$$\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta} dt = O(1) + \frac{\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}}{\rho-\beta-1} + O(1)\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta} dt$$

So,

$(1+O(1))\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta} dt = O(1) + \frac{\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}}{\rho-\beta-1}.$ (2.11)

For $\rho = 0$, (2.10) can be written as

$$\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta} dt$$

$$= \frac{\left(\frac{R}{R-t}\right)^{\rho(t)-\beta-1}}{\rho-\beta-1}\Big|_\vartheta^r - \frac{1}{\beta+1-\rho}\int_\vartheta^r \left(\frac{R}{R-t}\right)^{\rho(t)-\beta}\rho(t)\left\{\frac{(R-t)ln\left(\frac{R}{R-t}\right)\rho'(t)}{R\rho(t)} + R\right\}dt.$$

Again we have asymptotically

$$\left|\rho(t)\left\{\frac{(R-t)ln\left(\frac{R}{R-t}\right)\rho'(t)}{R\rho(t)}+R\right\}\right|<\varepsilon;$$

And (2.11) is sufficient to ensure the required result. For $\beta > \rho - 1$, we obtain a simpler formula

$\int_{\vartheta}^{r}\left(\frac{R}{R-t}\right)^{\rho(t)-\beta}dt=\frac{\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}}{\rho-\beta-1}+O\left(\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}\right).$ (2.12)

Let $\varphi(r)$ be a bounded function on (0,R) and $\rho(r)$ be a proximate order such that

$$\lim_{r\to R}{}^{\sup}_{\inf}\frac{\varphi(r)}{\left(R/(R-r)\right)^{\rho(r)}}=\frac{p}{q},$$

And for $\beta \geq 1$,

$$\lim_{r\to R}{}^{\sup}_{\inf}\left\{\frac{\varphi(r)}{\left(R/(R-r)\right)^{\rho(r)-\beta-1}}\int_{\vartheta}^{r}\frac{\varphi(r)}{\left(R/(R-r)\right)^{\beta}}=\frac{r}{s}\right\}.$$

Theorem 2.4: For the constants, p,q,r,s defined in theorem 2.3, we have
$\frac{q}{\rho-\beta-1}\leq s\leq r<\frac{p}{\rho-\beta-1}.$

Proof: For given $\varepsilon > 0$ and $r > r_0 > \vartheta > 0$.

$$\varphi(r)<(p+\varepsilon)\left(\frac{R}{R-r}\right)^{\rho(r)}$$

and

$$\int_{\vartheta}^{r}\frac{\varphi(t)}{\left(\frac{R}{R-t}\right)^{\beta}}dt\leq O(1)+(p+\varepsilon)\int_{r_0}^{t}\left(\frac{R}{R-t}\right)^{\rho(t)-\beta}dt$$

using (2.9), we get

$$\int_{\vartheta}^{r}\frac{\varphi(t)}{\left(\frac{R}{R-t}\right)^{\beta}}dt\leq O(1)+(p+\varepsilon)\frac{\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}}{\rho-\beta-1}+O\left(\left(\frac{R}{R-r}\right)^{\rho(r)-\beta-1}\right)$$

$\Rightarrow \lim_{r\to R} sup\left\{\left(\frac{R}{R-r}\right)^{-\rho(r)+\beta+1}\int_{\vartheta}^{r}\frac{\varphi(t)}{\left(\frac{R}{R-t}\right)^{\beta}}dt\leq\frac{p}{\rho-\beta-1}\right\}$

And thus the third part of the inequalities in (2.12) follows in a similar maner it can be shown that
$\lim_{r\to R} inf\left\{\left(\frac{R}{R-r}\right)^{-\rho(r)+\beta+1}\int_{\vartheta}^{r}\frac{\varphi(t)}{\left(\frac{R}{R-t}\right)^{\beta}}dt\geq\frac{q}{\rho-\beta-1}\right\}$, and hence the result.

III. Main Result

To prove the main result of this paper we need the following:

Lemma 3.1: For a real valued GBSP $F^{(\alpha,\beta)}$, regular in $\sum_R^{(\alpha,\beta)}$, having order ρ and lower order λ we have
$\lim_{r\to R} inf\frac{(R-r)w(r,F^{(\alpha,\beta)})}{rRlog\ M(r,F^{(\alpha,\beta)})}\leq\lambda\leq\rho\leq\lim_{r\to R} sup\frac{(R-r)w(r,F^{(\alpha,\beta)})}{rRlog\ M(r,F^{(\alpha,\beta)})}.$ (3.1)

Proof: For $R_+^* U\{0\}$, let

$\lim_{r\to R} sup\frac{(R-r)w(r,F^{(\alpha,\beta)})}{rRln\ M(r,F^{(\alpha,\beta)})}=A$ (3.2)

where R_+^* is the set of extended positive real's, for $A=0,\rho=0$ and hence assume $0<A<\infty$. On differentiation (1.1) gives

$\frac{M'(r,F^{(\alpha,\beta)})}{M(r,F^{(\alpha,\beta)})}=\frac{w(r,F^{(\alpha,\beta)})}{r}.$ (3.3)

(3.2) in conjunction with (3.3) is rewritten as

$$\lim_{r\to R} sup\frac{(R-r)M'(r,F^{(\alpha,\beta)})}{M(r,F^{(\alpha,\beta)})\ln M(r,F^{(\alpha,\beta)})}=A.$$

For given $\varepsilon > 0$ and r sufficiently close to R.

$$\frac{(R-r)M'(r,F^{(\alpha,\beta)})}{RM(r,F^{(\alpha,\beta)})\ln M(r,F^{(\alpha,\beta)})}<A+\varepsilon$$

integrating above inequality, we get

$\ln\ln M\left(r, F^{(\alpha,\beta)}\right) < O(1) + A + \varepsilon \ln\left(\frac{R}{R-r}\right).$

Passing to limits and taking the definition of order into account, we have

$\rho \leq \lim_{r\to R} sup \frac{(R-r)w(r,F^{(\alpha,\beta)})}{Rr \ln M(r,F^{(\alpha,\beta)})}$, which also holds for $A < \infty$.

By the parallel reasoning for lower order, it is easy to get

$\lambda \geq \lim_{r\to R} inf \frac{(R-r)w(r,F^{(\alpha,\beta)})}{Rr \ln M(r,F^{(\alpha,\beta)})},$

Combining above both inequalities (3.1) is immediate.

Lemma 3.2: For constants α_1 and β_1 defined by

$$\lim_{r\to R} \begin{matrix} sup \\ inf \end{matrix} \frac{\ln\ln\ln M(r,F^{(\alpha,\beta)})}{\ln\ln\left(\frac{R}{R-r}\right)} = \begin{matrix} \alpha_1 \\ \beta_1 \end{matrix}, (0 \leq \beta_1 \leq \alpha_1 < \infty),$$

we have

$\lim_{r\to R} \inf \frac{w\left(r,F^{(\alpha,\beta)}\right)M\left(r,F^{(\alpha,\beta)}\right)\log\left(\frac{R}{R-r}\right)}{r\Delta_2\left(M\left(r,F^{(\alpha,\beta)}\right)\right)\left(\frac{R}{R-r}\right)} \leq \beta_1 \leq \alpha_1 \leq \lim_{r\to R} \sup \frac{w\left(r,F^{(\alpha,\beta)}\right)M\left(r,F^{(\alpha,\beta)}\right)\log\left(\frac{R}{R-r}\right)}{r\Delta_2\left(M\left(r,F^{(\alpha,\beta)}\right)\right)\left(\frac{R}{R-r}\right)}$

(3.4)

where, for convenience, $\Delta_2 x = \ln\ln x . \ln x . x,$

Proof: The above lemma can be proved along lines similar to those lemma 3.1 and so we omit the details.

Definition: A real valued GBSP, $F^{(\alpha,\beta)}$ is said to be regular growth of $0 < \lambda = \rho < \infty$.

Theorem 3.1: Let a real valued GBSP, $F^{(\alpha,\beta)}$ regular in $\sum_R^{(\alpha,\beta)}$, having order ρ such that limit in (3.1) and (3.4) exist. Then, for every $\sigma(0 < \sigma < \infty)$, $\ln\left(\frac{\sigma^{-1}\ln M\left(r,F^{(\alpha,\beta)}\right)}{\ln\left(\frac{R}{R-r}\right)}\right)$ is a proximate order of $F^{(\alpha,\beta)}$.

Proof: For a positive number σ, let

$\rho(r) = \ln\left(\frac{\sigma^{-1}\ln M\left(r,F^{(\alpha,\beta)}\right)}{\ln\left(\frac{R}{R-r}\right)}\right),$ (3.5)

$\ln M\left(r, F^{(\alpha,\beta)}\right)$ is positive, continuous and increasing functions of r for $r > r_0 > 0$, which is differentiable in adjacent open intervals, it follows that $\rho(r)$ satisfies (iii). Since the existence of limits in (3.1) implies that $F^{(\alpha,\beta)}$ is of regular growth and moreover, $\rho(r) \to \rho$ as $r \to R$. Differentiating (3.5), we have

$$\frac{S'(r)}{S(r)} = \frac{M'(r,F^{(\alpha,\beta)})}{M(r,F^{(\alpha,\beta)})\ln M(r,F^{(\alpha,\beta)})\ln(\sigma^{-1}\ln M(r,F^{(\alpha,\beta)}))} - \frac{R}{(R-r)\ln\left(\frac{R}{R-r}\right)}$$

Or

$$\frac{(R-r)\ln\left(\frac{R}{R-r}\right)\ S'(r)}{S(r)} = \frac{(R-r)\ln\left(\frac{R}{R-r}\right)M'(r,F^{(\alpha,\beta)})}{M(r,F^{(\alpha,\beta)})\ln M(r,F^{(\alpha,\beta)})\ln(\sigma^{-1}\ln M(r,F^{(\alpha,\beta)}))} - R$$

using $\ln \alpha x \approx \ln x$ as $x \to \infty$, we get

$$\lim_{r\to R} sup \frac{\ln ln\left(\sigma^{-1}\ln M\left(r,F^{(\alpha,\beta)}\right)\right)}{\ln\ln\left(\frac{R}{R-r}\right)} = 0.$$

Since, limits in (3.4) exist by assumption, it follows that

$\frac{(R-r)\ln\left(\frac{R}{R-r}\right)\ S'(r)}{S(r)} \to \alpha - R$ as $r \to R$.

Thus $S(r)$ satisfies assertion (ii). From (2.5) assertion (iv) is readily obtained. In this way all the assertions for $S(r)$ to be a proximate order of $F^{(\alpha,\beta)}$ are satisfied. Hence theorem is completed.

References

[1]. Juneja,O.P and Kappor, G.P., Analytic functions-growth aspects, research notes in mathematics:104, Pitamn advanced publishing program, Boston, London and Melbourne,1985.

[2]. Kasana, H.S., On the existence of proximate order for analytic functions, Ukrain, Mat. Zh.(kiev), 40(1988)

[3]. Levin, B. Ja., Distribution of zeroes of entire function, monographs, Amer. Math. Soc. Traslations, Vol.5(1968), revised edition (1980).

[4]. McCoy, P.A., Approximation of generalized biaxisymmetric potentials, J.Approx. Theory (1979), 125-137.