IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 3 Ver. V (May. - Jun. 2016), PP 104-112
www.iosrjournals.org

Application of Non Standard Finite Difference Method on Logistic Differential Equation and Comparison with Standard Difference methods

Bilelgn Dawit[1], ParchaKalyani[2]
[1,2]School of Mathematical & Statistical Sciences, Hawassa University, Hawassa, Ethiopia.

Abstract: *One of the most interesting themes of the mathematical ecology is the description of population biology and human epidemic models, which are almost nonlinear dynamic equations. In this communication we described the population biology by the logistic differential equation and applied an unconditionally stable nonstandard finite difference method through the methodology of Mickens. It has been proved that the Mickens scheme is dynamically consistent with the original differential equation regardless of the step sizes used in numerical simulations, in comparison with the standard finite difference methods.*

Keywords: *Logistic differential equation; Standard finite difference scheme; Non-standard finite difference method; Stability analysis;*

I. Introduction

The biological world is too complex and unpredictable. There are usually no laws associated with biological systems with a few exceptions such as for example the Hardy-Weinberg law in population genetics. In addition, it is very hard if not impossible to test for hypotheses by experiments. Nevertheless, it is very important to build relevant mathematical models so that some conclusions about the biology can be drawn.

Mathematical models play a significant role in understanding the dynamics of biological systems. In most cases, these models are described by autonomous systems of nonlinear ordinary differential equations, very often, such systems are so complex that their exact solutions are not obtainable and hence the need for numerical methods arises. Numerical methods have gained more performance since the advent of computers. Soft wares have been developed to simulate numerical experiment with expected real life situations.

Discrete time models become essential when one wants to describe experimental data that have been collected with certain interval of time. One of the critical aspects of the discretization methods is just the dependence from this interval, the time step. Since the time step should be selected only in relation to the characteristics of the problem under examination, it is necessary to choose a reliable discretization method that allows to make this transformation without any restriction on time step, as well as not to introduce artifacts, change the linear stability properties.

Since in general, these requirements are not satisfied in the application of most standard finite difference schemes, the concept of non-standard finite difference schemes was proposed for the first time by Mickens in 1988 as a solution to the numerical instability. It is a known and documented fact that a differential equation is said to have numerical instabilities if there exist solutions to the finite-difference equations that do not correspond qualitatively to any of the possible solutions of the differential equation Mickens [1].

Non-standard numerical method was introduced by Mickens [1] as a viable tool that provides approximate solution to differential equations and retain the qualitative properties of the equation. In Mickens [1, 2], valuable reasons for numerical instabilities were given in some particular investigated cases. The preservation of the qualitative properties of the considered differential equation with respect to these schemes is of great interest in finite difference methods of solving differential equations. The major consequence of this result is that such scheme does not allow numerical instabilities to occur.

Mickens proposed a new method of construction of discrete models whose solution have the same qualitative properties as that of the corresponding differential equations for all step-sizes and thus eliminate the elementary numerical instabilities that can arise. Numerical instabilities are indications that the discrete equations are not able to model the correct mathematical properties of the solutions to the differential equations of interest. In this work, the equation investigated is the autonomous first-order differential equation:

Researchers have carried out several studies describing various finite difference methods for the solution of continuous dynamical systems given by systems of ordinary differential equation. This review pertains to the study of first order nonlinear dynamic equation using both the standard and nonstandard. Ronald E. Mickens [3] has given a brief history of nonstandard finite difference (NSFD) methods along with the clarification to the remarks related to the views of some others on these techniques. Nonstandard finite difference (NSFD) methods for the numerical integration of differential equations had their genesis in a paper

[4] published in 1989. The basic rules to construct such schemes [2] and their application to specific nonlinear equations appear in a variety of publications [5]. In recent years, NSFD discrete models have been constructed and tested for a wide range of nonlinear dynamical systems. An essentially complete listing and summary of publications using NSFD methods, up to 2004, is presented in the paper by Patidar [6]. This paper [6] and other published works [1, 5, 7, 8, 9, 10, and 11] provided ample evidence that NSFD schemes are enjoying a growing applicability as the practical users of numerical techniques for differential equations. Arenas et al. [12] developed a nonstandard numerical scheme for a SIR (where S, I and R stand for susceptible, infected and removed individuals, respectively) seasonal epidemiological model for Respiratory Syncytial Virus (RSV). Ronald E.Mickens and Talitam. Washington [13] constructed a nonstandard finite difference (NSFD) scheme for an SIRS mathematical model of respiratory virus transmission. This discretization is in full compliance with the NSFD methodology as formulated by R. E. Mickens [3]. Gumel et al. [14] investigated a class of NSFD methods for solving system of differential equations arising in mathematical biology. Villanueva et al. [15] developed (and analyzed numerically) nonstandard finite difference schemes which are free of numerical instabilities, to obtain the numerical solution of a mathematical model of infant obesity with constant population size.

Some useful studies on dynamical systems are found in the work of Duleba [16],Rauh et al. [17], as well as Zhai and Michel [18]. Dimitrov and Kojouharov [19] formulated positive and elementary stable nonstandard finite-difference methods to solve a general class of Rosenzweig-MacArthur predator-prey systems which involve a logistic intrinsic growth of the prey population. Their methods preserve the positivity of solutions and the stability of the equilibria for arbitrary step-sizes.

In this research it is intended to investigate a class of finite difference methods, designed via the non-standard framework of Mickens, for solving systems of differential equations arising in population biology. It has been shown that this class of methods can often give numerical results that are asymptotically consistent with those of the corresponding continuous model. We tested their linear stability properties at different step-lengths. This fact is illustrated using a number of case studies arising from population biology (human epidemiology and ecology).

II. Logistic Differential Equation

Many ecological and epidemic models have been constructed and analyzed by researchers in diversified disciplines to helpus understand and interpret biological problems in ecologyand epidemics. Among these is theMalthus equation, whichcan be viewed as the simplest way tomodel population growth,

$$\frac{dx}{dT} = \lambda x \qquad (1)$$

Where$\lambda > 0$, independent of population size and time, is the growth rate of the population, and $x(T)$is the populationdensity or size at time T. Under the simple assumption that population growth rate is a constant, the population will alwaysgrow to unboundedly large over time as long as the initialpopulation size is positive. Therefore the equation doesnot capture the long time realistic population growth phenomenon.One way to modify the above biological assumption is to incorporate density dependence into the growth parameter. The well-known continuous-time logistic equation based onthe assumption that the population evolves in an environmentwith limited resources with no immigration or emigration phenomena. Let $x(T)$ the population at instant T, the lawthat regulates can be expressed from the following first orderautonomous ODEby replacing the growth parameter λin eq(1) with $r\left(1-\frac{x}{k}\right)$, which depends on both population density at time T and the carrying capacity k of the environment.

$$\frac{dx}{dT} = rx(1-\frac{x}{k}) \qquad (2)$$

Whereparameter$r > 0$ is the intrinsic growth rate of the population and $k > 0$ is the carrying capacity of the environment.

This equation possesses a very simple asymptoticdynamics: all solutions with positive initial conditionswill eventually approach the carrying capacity k. Therefore,population size will eventually be stabilized to k in the longrun even if population dynamics initially either overshoot orundershoot the carrying capacity.

2.1Non-dimensionalization(scaling)

We see that the model in equation (2) has four parameters. To make the analysis easy, we reduce the number of parameters by scaling as follows:

Let

$T = \frac{t}{r}$and$x = ky$

Substituting these in (2), we get

$$\frac{d(ky)}{d(\frac{t}{r})} = rky(1 - \frac{ky}{k}) \quad (3)$$

$$\frac{dy}{dt} = y(1 - y) \quad (4)$$

The differential equation (4), solved by variable separation method, admits the following solution:

$$y(t) = \frac{y_o}{y_o + (1 - y_o)e^{-t}} \quad (5)$$

where the initial condition is

$$y_o = y(0) \quad (6)$$

If $y_o > 0$, then all solution curves of our differential equation monotonically approach the stable equilibrium points.

2.2Stability analysis

Consider the differential equation

$$\frac{dx}{dT} = f(x) \quad (7)$$

Let x^*be the equilibrium point .i.e,

$f(x^*) = 0(8)$

Now let us consider the stability analysis of the logistic differentialequation:

$$f(y) = y(1 - y) = 0 \quad (9)$$

The equilibriumpoints are

$$y_1{}^* = 0 \,, y_2{}^* = 1 \quad (10)$$

Since $f^{'}(0) = 1 > 0$ and $f^{'}(1) = -1 < 0$, the equilibrium point $y_1{}^* = 0$ is unstable and the equilibrium point$y_2{}^* = 1$ is stable.

This can be deduced also by an alternativeway fromthe exact solution (5), by calculating its limit for $t \longrightarrow \infty$ the populationhas just $y = 1$ as its asymptotic value.

III. Finite Difference Approach

To transform a continuous-time model into a discrete one,the continuous variable $t \in [0, \infty)$ must be replaced by thediscrete variable $k \in N$and the variable y must take discretevaluesy_k. The result is a difference equation. In this studywe will consider both standard and nonstandard finite differenceschemes and finally their numerical solution will becompared at different time steps.

3.1Numerical solution of logistic differential equation using Forward Euler Scheme

This is one of the oldest way to derive a finite difference equationfrom a differential equation. Ifwe use a simple ForwardEuler scheme to approximate solutions of eq(4) by taking has the step size of the approximation and replace

$$\frac{dy}{dt} \longrightarrow \frac{y_{k+1} - y_k}{h}$$

We then have

$$\frac{y_{k+1} - y_k}{h} = y_k(1 - y_k) \quad (11)$$

Solving for y_{k+1} gives

$$y_{k+1} = y_k + hy_k(1 - y_k) \quad (12)$$

The approximate solutions of (4) using Forward Euler at different step lengths with the exact solution has been shown graphically in the following figures.

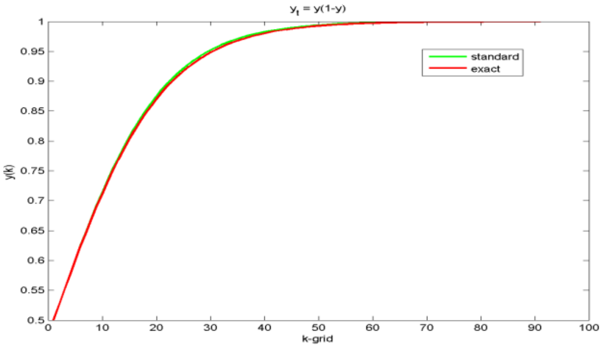

Figure: 1 Numerical solution of logistic differential equation using forward Euler scheme when*h* = 0.1

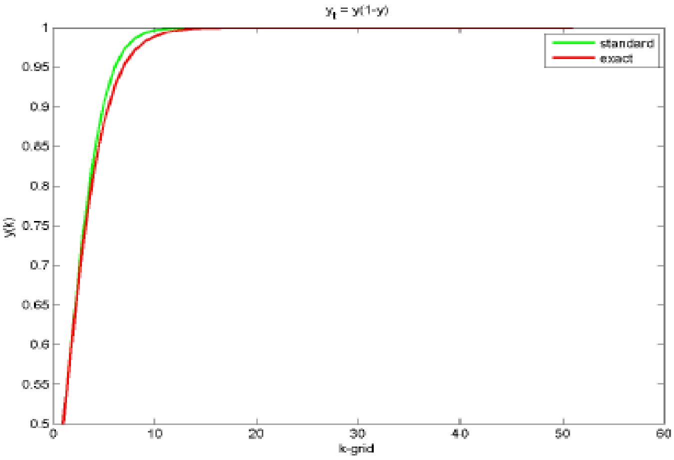

Figure: 2 Numerical solution of logistic differential equation using forward Euler scheme when$h$ = 0.7

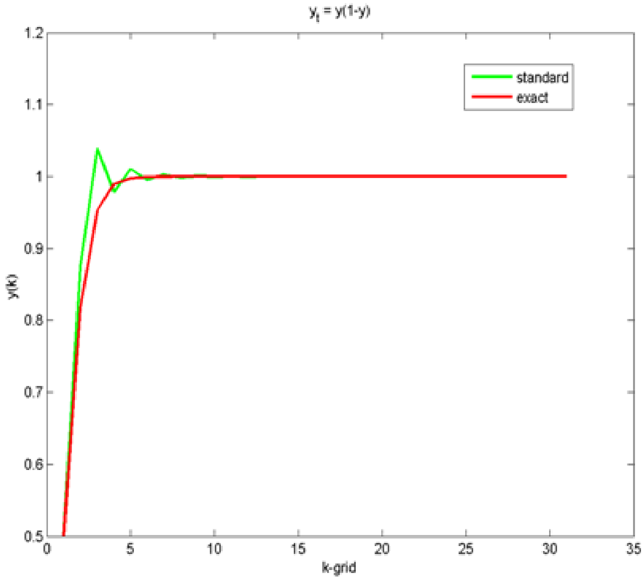

Figure: 3 Numerical solution of logistic differential equation using forward Euler scheme when$h$ = 1.5

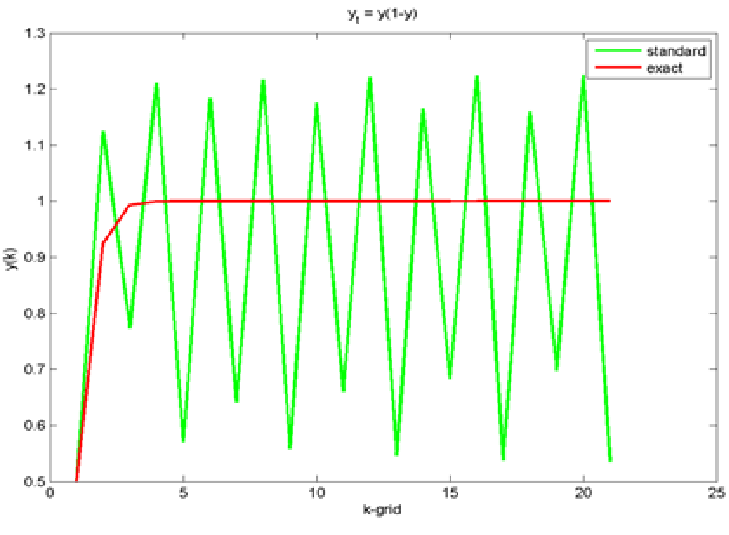

Figure: 4 Numerical solution of logistic differential equation using forward Euler schemewhen h = 2.5

3.2Numerical solution of logistic differential equation using central difference scheme

Applying popular standard finite central difference scheme to approximate the solutionof (4) by taking h as the step size of the approximationand by replacing

$\frac{dy}{dt} \longrightarrow \frac{y_{k+1}-y_{k-1}}{2h}$, wethen have

$$\frac{y_{k+1}-y_{k-1}}{2h} = y_k(1-y_k) \tag{13}$$

Solving for y_{k+1} gives

$$y_{k+1} = y_{k-1} + 2hy_k(1-y_k) \tag{14}$$

$$y_{k+1} = y_k - h + 2hy_k(1-y_k) \tag{15}$$

The approximate solution of (4) using central differencescheme at different step lengths and the exact solutionhas been shown graphically in the following figures.

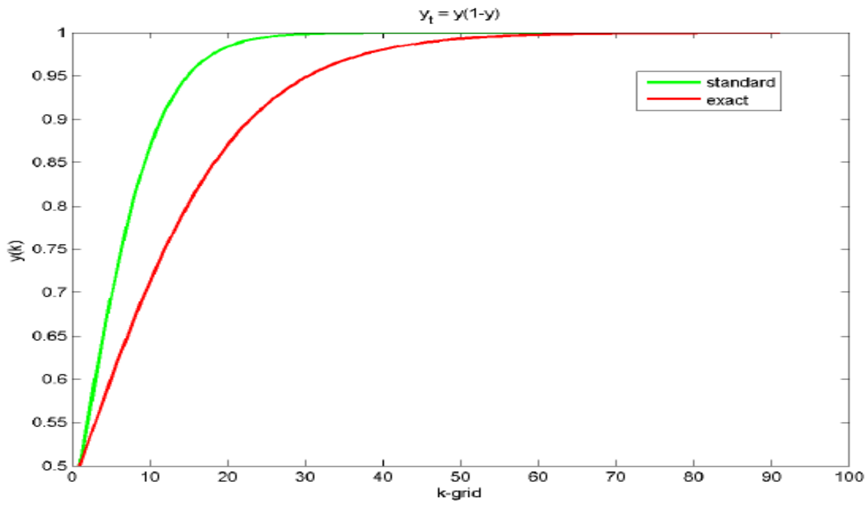

Figure: 5 Numerical solution of logistic differential equation using central difference scheme when$h$ = 0.1

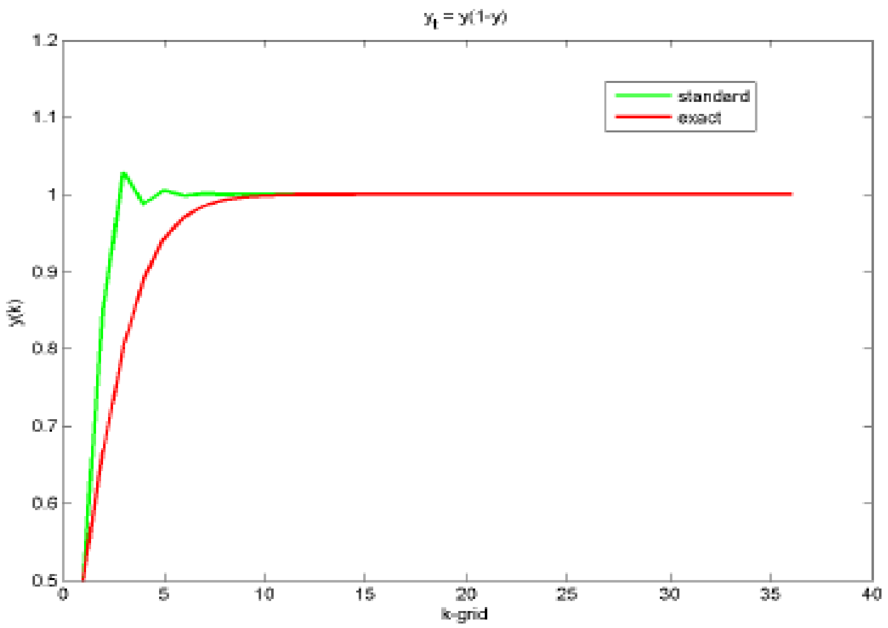

Figure: 6 Numerical solution of logistic differential equation using central difference scheme When$h$ = 0.7

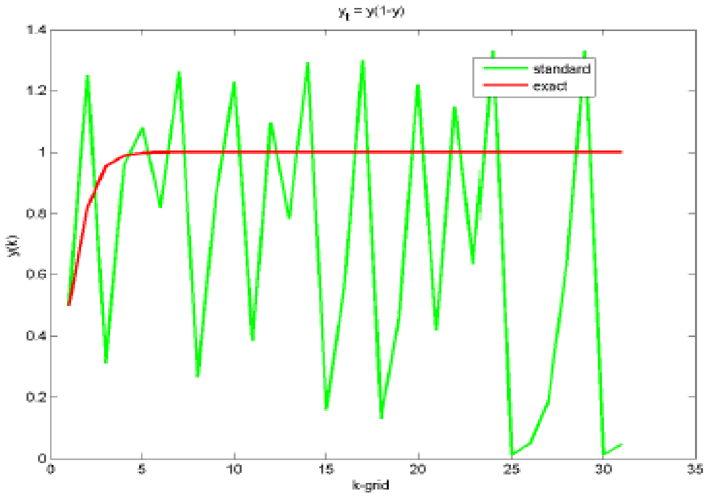

Figure: 7 Numerical solution of logistic differential equation using central difference scheme when$h$ = 1.5

IV. Nonstandard Finite Difference Method

Detailed study of well-known exact finite difference schemesgivesthe foundation to the NSFD methods. The extension andgeneralization of these results to special groups of differentialequations, for which exact schemes are not available, has also provided additional insight into the required structuralproperties of NSFD methods. Nonstandard finite difference schemes have emerged asan alternative method for solving a wide range of problemswhose mathematical models involve algebraic, differentialand biological models as well aschaotic systems. Thesetechniques have many advantages over classical techniquesand provide an efficient numerical solution.

4.1 Numerical solution of logistic differential equation using nonstandardfinite difference Scheme

We apply the NSFD rules to transform the model into thediscrete scheme.

$$\frac{\Delta y(t)}{\phi(\Delta t)} = y(t) - (y(t))^2 \tag{16}$$

Where

$\Delta y(t) = y_{k+1} - y_k$

and$\phi(\Delta t)$ is a function of the time step $\Delta t = h,$ which is the denominator function.
Now replacing the nonlinear term y^2 by nonlocal representationwith$y_k y_{k+1}$in (16) we get

$$\frac{y_{k+1} - y_k}{\phi(h)} = y_k - y_k y_{k+1} \tag{17}$$

now we derive a suitable denominator function$\phi(h)$for the above equation. One of the method in searching the denominator function is given by:

$$\phi(h, R^*) = \frac{\varphi(h, R^*)}{R^*} \tag{18}$$

where$\varphi$is the denominator of the exact solution of the differentialequation under study . As mentioned earlier the exact solution of our model is solved by variable separationmethod and is given by

$$y(t) = \frac{1}{1-e^{-t}} \tag{19}$$

Therefore,

$$\varphi(z) = 1 - e^{-z} \tag{20}$$

and$R^*$is calculated by the formula:

$$R^* = Max|R_i| , i = 1, 2 , \dots etc. \tag{21}$$

where$R_i$is :

$$R_i = \frac{df(y^*)}{dy} , f(y^*) = 0. \tag{22}$$

clearly$R^* = 1$ and this tells us that the suitable denominatorfunction $\phi$is given by

$$\phi(h,1) = \frac{\phi(h)}{1} = 1 - e^{-h} = \phi(h) \tag{23}$$

Substituting (23) into (17), we get

$$\frac{y_{k+1} - y_k}{1 - e^{-h}} = y_k - y_k y_{k+1} \tag{24}$$

Therefore y_{k+1} can be so explicited

$$y_{k+1} = \frac{(2 - e^{-h}) y_k}{1 + (1 - e^{-h}) y_k} \tag{25}$$

This equation can be transformed by letting

$$W_k = \frac{1}{y_k} \tag{26}$$

$$\frac{1}{W_{k+1}} = \frac{2 - e^{-h}}{W_k + 1 - e^{-h}} \tag{27}$$

After simple algebraic arrangements we found

$$W_{k+1} - \left(\frac{1}{2 - e^{-h}}\right) W_k = \frac{1 - e^{-h}}{2 - e^{-h}} \tag{28}$$

The approximate solutions of (4) using nonstandard finitedifference scheme at different step lengths and the exactsolution has been shown graphically in the following figures.

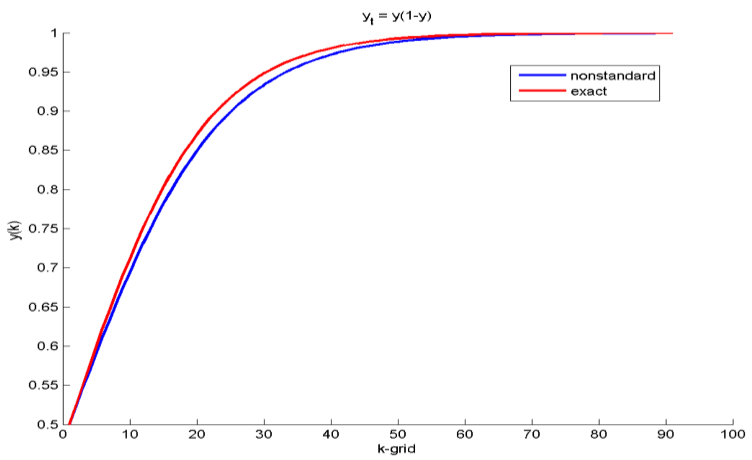

Figure 8: Numerical solution of logistic differential equation using nonstandard finite difference scheme when h = 0.1.

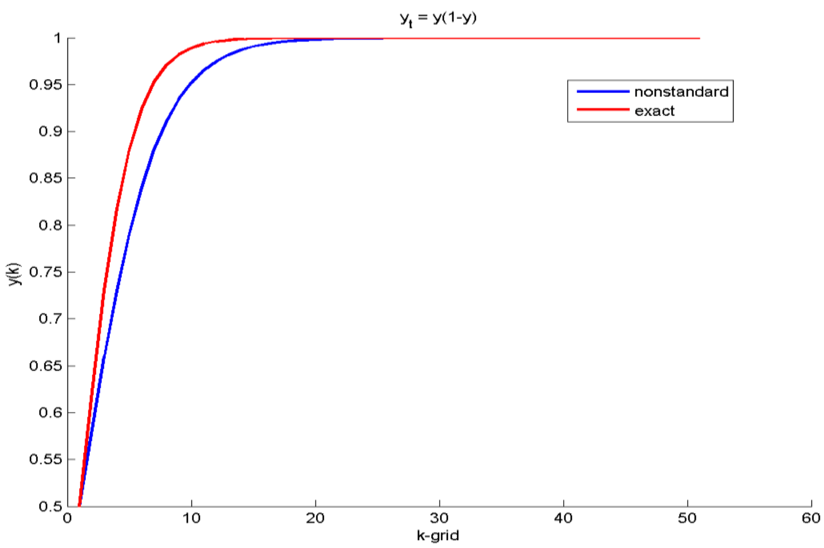

Figure 9: Numerical solution of logistic differential equation using nonstandard finite difference scheme when h = 0.5.

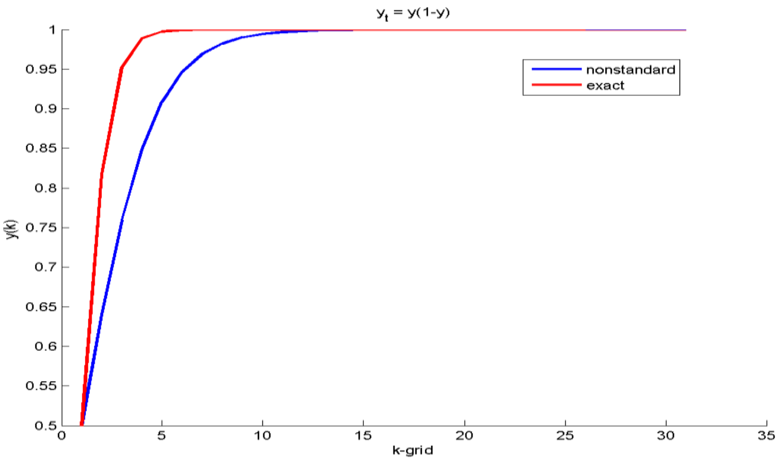

Figure 10: Numerical solution of logistic differential equation using nonstandard finite difference scheme when h = 1.5.

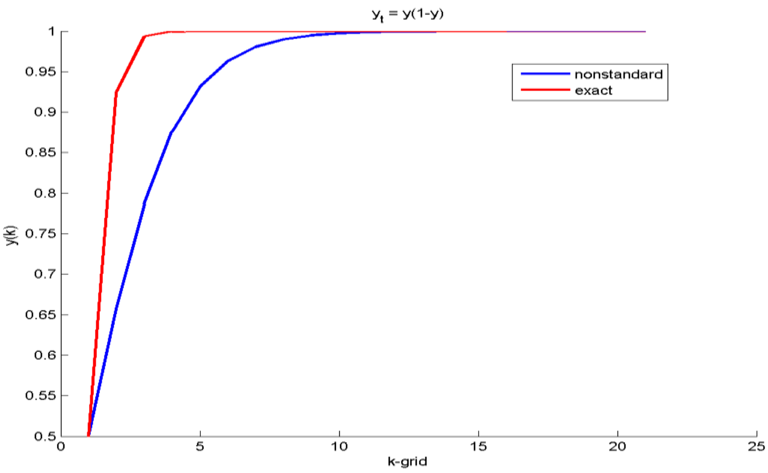

Figure 11: Numerical solution of logistic differential equation using nonstandard finitedifference scheme when h = 2.5

V. Comparison between Euler Forward and NSFD Scheme for Logistic Equation at different step lengths

Fromthe figures (1-7), we observed that the Euler Forwardscheme ismore stable than the Central difference schemewhenwe increase the step length. So in this section we aregiving the comparison between Euler Forward and the nonstandard finite difference scheme.

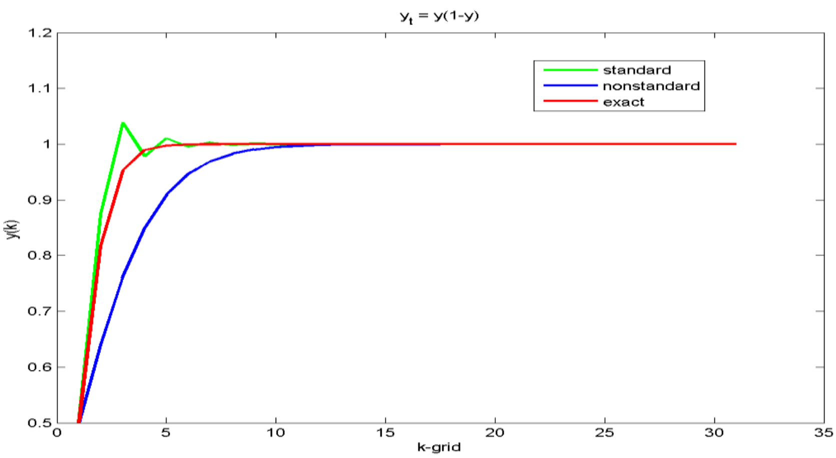

Figure 12 : Comparison between the numerical solution of logistic differential equation using nonstandard and Euler forward finite difference scheme when h = 1.5

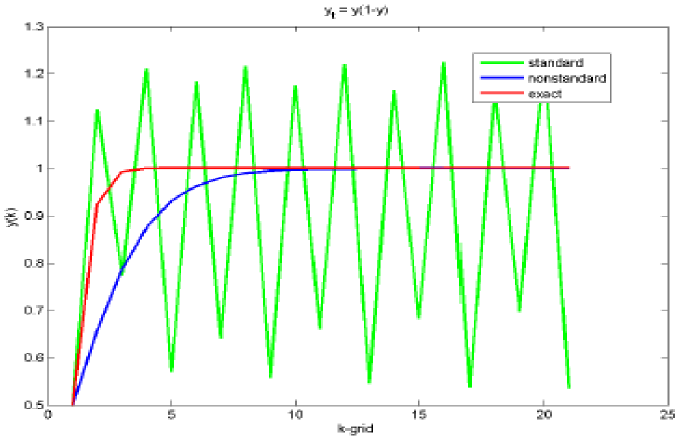

Figure 13: Comparison between the numerical solution of logistic differential equation using nonstandard and Euler forward finite difference scheme when h = 2.5

VI. Conclusion

We considered the logistic differential equation and found the numerical solution using NSFD scheme by applying Mickens rules. For the sake of comparison we have chosen the same problems and obtained the solution by using Euler Forward scheme and central difference scheme. From the figures (1-7) we observed that the Euler Forward method is relatively more stable than the central difference scheme, so we compared the solutions from NSFD scheme with the Euler Forward at different step lengths. Although for $h < 1$, both methods are in a good agreement, for $h > 1$, standard finite difference method exhibits the numerical instability but nonstandard discrete models do not exhibit numerical instabilities for all h. From all simulations we have been made, we conclude that the non-standard finite difference scheme is dynamically consistent and stable than the standard finite difference methods in solving the model in population Biology known as logistic differential equation.

References

[1]. L. Edelstein-Keshe .Mathematical Models in Biology.McGraw-Hill, 1988.
[2]. L. J. S. Allen. Some discrete-time SI, SIR, and SIS epidemic models MathematicalBiosciences, 124: 83-105, 1994.
[3]. Mickens R.E.2005, Advances in the applications of nonstandard finite difference schemes edited by Ronald E.Mickens.2005, pp.823-847.
[4]. H. Al-Kahby, F. Dannan, and S. Elaydi. Non-standard discretization methods for somebiological models. In R. Mickens, editor, Applications of Nonstandard Finite Difference Schemes, chapter 4, pages 155-180. World Scientific, New Jersey, 2000.
[5]. C.W. Chi, S.-B. Hsu, and L.-I. Wu. On the asymmetric May-Leonard model of threecompeting species. SIAM J. Appl.Math., 58(1): 211-226, 1998.
[6]. Patidar, K. (2005). On the use of nonstandard finite difference methods, Journal of Difference Equations and Applications 11(8): 735-758.
[7]. H. Jiang and T. D. Rogers. The discrete dynamics of symmetric competition in the plane. J.Math. Biol., 25:573-596, 1987.
[8]. J. Hale and H. KoCak, Dynamics and Bifurcations, Springer-Verlag, New York, 1991.
[9]. J. Hofbauer, V. Hutson, andW. Jansen. Co-existence for systems governed by difference equations of Lotka-Volterra type. J.Math. Biol., 25:553-570, 1987.
[10]. M. J. Gander and R. Meyer-Spasche. An introduction to numerical integrators preserving physical properties. In R. Mickens, editor, Applications of Nonstandard Finite Difference Schemes, chapter 5, pages 181 - 243. World Scientific New Jersey, 2000.
[11]. S. Jang and S. N. Elaydi. Difference equations from discretization of a continuousepidemic model with immigration of infectives. Canadian Applied MathematicsQuarterly, 11 (2), 2005.
[12]. Arenas, A., No, J.M. and CortÃ©s, J. (2008). Nonstandard numerical method for mathematical model of RSV epidemiological transmission, Computers and Mathematics with Applications 56(3): 670-678.
[13]. Ronald E. Mickens and TalithaM.Washington, anoteonannsfd scheme for amathematical model of respiratory virus transmission, arxiv: 1008.2314v1 [q.bio.pe].13, aug, 2010.
[14]. Gumel, A., Patidar, K. and Spiteri, R. (2005). Asymptotically consistent nonstandard finite difference methods for solving mathematical models arising in population biology, in R. Mickens (Ed.), Applications of Nonstandard Finite Difference Schemes, World Scientific, Singapore, pp. 385-421.
[15]. Villanueva, R., Arenas, A. and Gonzalez-Parra, G. (2008). A nonstandard dynamically consistent numerical scheme applied to obesity dynamics, Journal of Applied Mathematics, Article ID 640154, DOI: 10.1155/2008/640154.
[16]. Karcz-DulË›eba, I. (2004). Asymptotic behavior of a discrete dynamical system generated by a simple evolutionary process, International Journal of Applied Mathematics and Computer Science 14(1): 79-90.
[17]. Rauh et al. (2009). Verification controllers for nonlinear dynamic systems with un certainties, International Journal of Applied Mathematics and Computer Science19 (3): 425-439, DOI: 10.2478/v10006/009/0035/1.
[18]. Zhai, G. and Michel, A.M. (2004). Generalized practical stability analysis of discontinuous dynamical systems, International Journal of Applied Mathematics and Computer Science 14(1):5-12.
[19]. Dimitrov,D. and Kojouharov, H. (2006). Positive and elementary stable nonstandardnumerical methods with applications to predator prey models, Journal of Computational and Applied Mathematics 189(1): 98-108.