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Different Types Of Fuzzy Path Connectedness On Fuzzy Banach Manifold

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Abstract: *In this paper, we define three different types of fuzzy path connectedness on fuzzy Banach manifold and we show that these fuzzy path connectedness forms an equivalence relations on fuzzy Banach manifold. Further, we show that the image of fuzzy path connected fuzzy Banach manifold under smooth fuzzy map is fuzzy path connected and also we study some of its properties.*

Keywords: *Locally fuzzy path connectedness, Internally fuzzy path connectedness, Maximally fuzzy path connectedness, Smooth fuzzy network structure.*

I. Introduction

The concept of fuzzy sets was first introduced by L. A. Zadeh[7] during 1968. Further C. L. Chang[2], R. Lowen[10] and others have developed a theory of fuzzy topological space. The concept of Fuzzy vector space and fuzzy topological vector space was introduced by A. K. Katsaras and D. B. Liu [1] in 1977.

One of the intrinsic characteristic of fuzzy topological space is path connectedness. In 1984 C. Y. Zheng has introduced fuzzy path and fuzzy connectedness[3] and in 1987 D. M. Ali introduced the new concept of fuzzy path connected topological spaces and studied its various characteristics [4].

In consequence to the development of fuzzy sets and fuzzy topology, many authors like T. Bag, A. K. Samant[16], R. S. Saadati, S. M .Vaezpour[11] and G. Rano, T. Bag [6] have introduced the concept of fuzzy metric and fuzzy norm in different perception and in 1993 Mario Ferraro and David H. Foster have introduced the concept of C^1 fuzzy manifold [8] and in 2003 Erdal Guner introduced tangent bundle on C^1 fuzzy manifold [5]. With the same approach in our previous paper [14] we have introduced the concept of Fuzzy Banach Manifold.

When the path connectedness property of a topological space is introduced on manifolds covered by charts and atlases the connectedness can be studied in three different types which was first introduced by S. C. P. Halakatti in [12][13] using such perception we gave new definition of path connectedness on fuzzy Banach manifold in [15]. In this paper, we extend this study by defining three different types of fuzzy path connectedness on fuzzy Banach manifold and study some of its properties. The advantage of such study in path connectedness leads us to develop the concepts of smooth fuzzy network structure on the fuzzy Banach manifold.

II. Preliminaries

We use the following basic definitions and results to define different types of fuzzy path connectedness on fuzzy Banach manifold.

Definition 2.1: Let (X,T) be a fuzzy topological space. A fuzzy path in X is a fuzzy continuous function $f:(I,\bar{e}) \to (X,T)$ where $I = [0,1]$ and e be the euclidean subspace topology on I. The crisp singleton $f(0)$ and $f(1)$ are respectively called the initial and terminal points of the fuzzy path f.

Definition 2.2: Any fuzzy topological space (X,T) is said to be fuzzy path connected iff for each $x, y \in X$ there exists a fuzzy path f in X such that $f(0) = x$ and $f(1) = y$.

Definition 2.3: Let M and N be differentiable manifolds with corresponding maximal atlases A_M and A_N. We say that a map $f: M \to N$ is of class C^r (r times continuously diffrentiable) at $p \in M$ if there exists a chart (V,ψ) in A_N with $f(p) \in V$, and a chart (U,ϕ) from A_M with $p \in M$, such that $f(U) \subset V$ and such that $\psi \circ f \circ \phi^{-1}$ is of class C^r. If f is of class C^r at every point $p \in M$, then we say f is of class C^ron M.

III. Main Results

In this section, we define local, internal and maximal fuzzy path connectedness on fuzzy Banach manifold by referring[3][4][12][13][15].

Definition 3.1: Let M be fuzzy Banach manifold and $p, q \in U_i \in A^k(M)$. If there exists fuzzy continuous mapping $f:(I,\bar{e}) \to A^k(M)$ such that

$$f(0) = p \in U_i,$$

$$f(1) = q \in U_i,$$

then p is locally fuzzy path connected to q. If it is true for all $p, q \in U_i \in M$ then M is locally fuzzy path connected.

Definition 3.2: Let M be fuzzy Banach manifold and $p \in U_i, q \in U_j$ where $U_i, U_j \in A^k(M)$. If there exists fuzzy continuous mapping $f: (I, \bar{e}) \to A^k(M)$ such that

$$f(0) = p \in U_i,$$

$$f(1) = q \in U_j,$$

then p is internally fuzzy path connected to q. If it is true for all $p, q \in U_i, U_j$ respectively $\forall\ i, j \in I$ then M is internally fuzzy path connected.

Definition 3.3: Let M be fuzzy Banach manifold and $p \in U_i, q \in U_j$ and $r \in U_k$ where $U_i, U_j, U_k \in A^k(M)$. If there exists fuzzy continuous mapping $f: (I, \bar{e}) \to A^k(M)$ such that

$$f(0) = p \in U_i,$$

$$f\left(\frac{1}{2}\right) = q \in U_j,$$

$$f(1) = r \in U_k,$$

then p is maximally fuzzy path connected to r. If it is true for all $p, q, r \in U_i, U_j, U_k$ respectively $\forall\ i, j, k \in I$ then M is maximally fuzzy path connected.

Definition 3.4: Let M be fuzzy Banach manifold. We say that any two points $p, r \in M$ are related that is, $p \sim r$ if p is maximally fuzzy path connected to r.
Now we shall show that the maximally fuzzy path connected relation is an equivalence relation.

Theorem 3.1: Maximally fuzzy path connected relation on fuzzy Banach manifold is an equivalence relation.

Proof: Let M be fuzzy Banach manifold. Now we shall show that maximal fuzzy path connectedness is an equivalence relation.
i) Reflexive: Reflexive relation is trivial by considering a constant fuzzy paths $f: (I, \bar{e}) \to A^k(M)$ such that $f(t) = p\ \forall\ t \in [0,1]$.
Therefore maximal fuzzy path connected relation is reflexive.

ii) Symmetry: Suppose f_1 is a fuzzy path from p to r we define $f_2: (I, \bar{e}) \to A^k(M)$ such that:

$$f_2 = f_1(1 - t)\ \forall\ t \in [0,1],$$

then by the definition of f_2 it is clear that it is fuzzy continuous mapping and hence a fuzzy path in M from r to p such that:

$$f_2(0) = f_1(1 - 0) = f_1(1) = r$$

$$f_2\left(\frac{1}{2}\right) = f_1\left(1 - \frac{1}{2}\right) = f_1\left(\frac{1}{2}\right) = q$$

$$f_2(1) = f_1(1 - 1) = f_1(0) = p$$

Therefore maximal fuzzy path connected relation is symmetric.

iii) Transitive: Let f_1 is a fuzzy path from p to r, f_2 is a fuzzy path from r to t.
Let $f_3: (I, \bar{e}) \to A^k(M)$ defined as,

$$f_3(x) = \begin{cases} f_1(2t), & \text{if } 0 \leq \text{t} \leq \frac{1}{2}, \\ f_2(2t - 1), & \text{if } \frac{1}{2} \leq \text{t} \leq 1, \end{cases}$$

Then f_3 is well defined since $f_1(1) = r = f_2(0)$ is continuous such that:

$$f_3(0) = f_1(0) = p,$$

$$f_3\left(\frac{1}{2}\right) = \begin{cases} f_1\left(2\left(\frac{1}{2}\right)\right) = f_1(1) = r, \text{ if } 0 \le \text{t} \le \frac{1}{2} \\ f_2\left(2\left(\frac{1}{2}\right) - 1\right) = f_2(0) = r, \text{ if } \frac{1}{2} \le \text{t} \le 1 \end{cases}$$

$$f_3(1) = f_2(2(1) - 1) = f_2(1) = t,$$

Therefore f_3 is a fuzzy path from p to t.
Hence maximally fuzzy path connected relation is transitive.
Therefore maximal fuzzy path connectedness is an equivalence relation. ■

The above relation $(M, \sim)$ on fuzzy Banach manifold M induces a smooth fuzzy network structure on M. Similarlly, we can show that locally fuzzy path connected and internally fuzzy path connected relations are also equivalence relations.

Corollary 3.1: The locally fuzzy path connected relation on fuzzy Banach manifold is an equivalence relation.
Proof: Proof is similar to Theorem 3.1. ■

The above relation induces a smooth fuzzy network structure locally on any fuzzy Banach chart of M.

Corollary 3.2: The internally fuzzy path connected relation on fuzzy Banach manifold is an equivalence relation.
Proof: Proof is similar to Theorem 3.1. ■

The above relation induces a smooth fuzzy network structure internally between any two fuzzy Banach charts of M.

Now we shall define smooth fuzzy maps on fuzzy Banach manifolds as follows.

Definition 3.4: Let M and N be fuzzy Banach manifolds with corresponding maximal fuzzy Banach atlases A_M and A_N. We say that a map $f: M \to N$ is of class C^r (r times continuously fuzzy diffrentiable) at $p \in M$ if there exists a fuzzy Banach chart (V, ψ) in A_N with $f(p) \in V$, and a fuzzy Banach chart (U, ϕ) from A_M with $p \in M$, such that $f(U) \subset V$ and such that $\psi \circ f \circ \phi^{-1}$ is of class C^r(i.e., fuzzy differentiable of class $r > 0$). If f is of class C^r at every point $p \in M$, then we say f is of class C^ron M. Maps of class C^∞ are called smooth fuzzy maps.

Theorem 3.2: The image of maximally fuzzy path connected fuzzy Banach manifold under smooth fuzzy map is fuzzy path connected.

Proof: Let M_1 and M_2 be fuzzy Banach manifolds where M_1 is maximally fuzzy path connected. If $g: M_1 \to M_2$ is a smooth fuzzy map then we show that M_2 is maximally fuzzy path connected.
Let $p \in U_i, q \in U_j$ and $r \in U_k$ where $U_i, U_j, U_k \in A^k(M)$. Since g is a smooth map we have $g(p) \in g(U_i) \subset V_i$, $g(q) \in g(U_j) \subset V_j$ and $g(r) \in g(U_k) \subset V_k$ where V_i, V_j and $V_k \in A^k(M_2)$.
We know that M_1 is maximally fuzzy path connected therefore there exists fuzzy path f such that:

$$f(0) = p \in U_i,$$

$$f\left(\frac{1}{2}\right) = q \in U_j,$$

$$f(1) = r \in U_k,$$

Since g is smooth fuzzy map between M_1 and M_2, we get a fuzzy path $g \circ f: [0, 1] \to A^k(M_2)$ such that:

$$g \circ f(0) = g(p) \in g(U_i) \subset V_i,$$

$$g \circ f\left(\frac{1}{2}\right) = g(q) \in g(U_j) \subset V_j,$$

$$g \circ f(1) = g(r) \in g(U_k) \subset V_k,$$

That is there exists a fuzzy path $g \circ f$ in M_2 for every fuzzy path f in M_1 satisfying above conditions and is true for all $g(p), g(q)$ and $g(r) \in M_2$. Therefore M_2 is maximally fuzzy path connected.

Hence image of a maximally fuzzy path connected fuzzy Banach manifold under smooth fuzzy map is fuzzy path connected. ■

Corollary 3.3: The image of locally fuzzy path connected fuzzy Banach manifold under smooth fuzzy map is fuzzy path connected.
Proof: Proof is similar to Theorem 3.2. ■

Corollary 3.4: The image of internally fuzzy path connected fuzzy Banach manifold under smooth fuzzy map is fuzzy path connected.
Proof: Proof is similar to Theorem 3.2. ■

Theorem 3.3: Let $\{U_i: i \in I\}$ be a family of locally fuzzy path connected fuzzy Banach charts of M. If $\cap_{\mathrm{i}\in\mathrm{I}} \mathrm{U_i} \neq \emptyset$ then $\cap_{\mathrm{i}\in\mathrm{I}} \mathrm{U_i}$ is locally fuzzy path connected.

Proof: Let $\{U_i: i \in I\}$ be a family of locally fuzzy path connected fuzzy Banach charts of M and $\cap_{i\in I} U_i \neq \emptyset$ then we show that $\cap_{i\in I} U_i$ is also locally fuzzy path connected.
Let $p, q \in \cap_{i\in I} U_i, \Rightarrow p, q \in U_i \ \forall \ i \in I$.
Since each U_i's are compatible and locally fuzzy path connected there exists a fuzzy path $f: (I, \bar{e}) \to A^k(M)$ such that

$$f(0) = p \in U_i,$$

$$f(1) = q \in U_i,$$

that is p is locally fuzzy path connected to q. Since $p, q \in \cap_{i\in I} U_i$ are arbitrary we say that for every $p, q \in \cap_{i\in I} U_i$ there exists a fuzzy path such that p is locally fuzzy path connected to q. Therefore $\cap_{i\in I} U_i$ is locally fuzzy path connected.

Hence if $\{U_i: i \in I\}$ be a family of locally fuzzy path connected fuzzy Banach charts of M and $\cap_{i\in I} U_i \neq \emptyset$ then $\cap_{i\in I} U_i$ is locally fuzzy path connected. ■

Corollary 3.5: Let $\{A_i: i \in I\}$ be a family of internally fuzzy path connected fuzzy Banach atlases of M. If $\cap_{i\in I} A_i \neq \emptyset$ then $\cup_{i\in I} A_i$ is internally fuzzy path connected.
Proof: Let $\{A_i: i \in I\}$ be a family of internally fuzzy path connected fuzzy Banach atlases of M and $\cap_{i\in I} A_i \neq \emptyset$ then we show that $\cup_{i\in I} A_i$ is internally fuzzy path connected. ■

Let $p, q \in U_i, U_j$ respectively, where $U_i, U_j \in \cup_{i\in I} A_i$ for any $i \in I$, that is, $U_i, U_j \in A_i$ for any $A_i \in \cup_{i\in I} A_i$.
Since each A_i is internally fuzzy path connected, there exists a fuzzy path $f: (I, \bar{e}) \to A^k(M)$ such that

$$f(0) = p \in U_i,$$

$$f(1) = q \in U_j,$$

that is $p \in U_i$ is internally fuzzy path connected to $q \in U_j$ which is true for all A_i. Since A_i is any arbitrary member of $A^k(M)$ it is true for all A_i. Therefore $\cup_{i\in I} A_i$ is internally fuzzy path connected.

Hence if $\{A_i: i \in I\}$ be a family of internally fuzzy path connected fuzzy Banach atlases of M and $\cap_{i\in I} A_i \neq \emptyset$ then $\cup_{i\in I} A_i$ is internally fuzzy path connected. ■

***Corollary 3.6*:** Let $\{A_i: i \in I\}$ be a family of maximally fuzzy path connected fuzzy Banach charts of M. If $\bigcap_{i\in I} A_i \neq 0$ then $\bigcup_{i\in I} A_i$ is maximally fuzzy path connected.
Proof: Proof is similar to Corollary 3.5. ■

IV. Conclusion

This paper deals with three different types of fuzzy path connectedness inducing a smooth network structure on fuzzy Banach manifold at three different levels, that is locally on any fuzzy Banach chart, internally between any two fuzzy Banach charts and maximally between any three fuzzy Banach charts. The fuzzy path connectedness at three different levels generates smooth fuzzy network structure of first order inducing smooth fuzzy network manifold.

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