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Uncertain Optimal Control Model for Management of Net Risky Capital Asset

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Abstract: *A new model of asset management for a business organization is proposed based on the uncertainty theory in which the capital assets are managed. Here, a continuous-time utility portfolio problem with the assumption of Hyperbolic Absolute Risk Aversion (HARA) utility function is examined from an investor whose income is generated by return and capital gains on investments in risky tangible assets with price and return on assets assumed to satisfy the Liu uncertain process. Thus, the problem is solved and the optimal controls are obtained.*

Keywords: *Optimal control, uncertainty theory, uncertain process, optimality, asset management*

I. Introduction

Optimal control theory is an optimization method used in deriving control policies which has been of great importance to the world as whole because of the way it deals with real life problems giving control laws to the problems in a way that certain optimality criterion is achieved. The study of optimal control extensively attained the attention of many researchers because of the need of strict expression form and control policies in optimal control theory. As a result of greater use of methods and results in mathematics, optimal control theory has achieved great developments and has been applied to many fields like Engineering, Medicine, Economics and Finance in developing models such as in Bamigbola and Aderinto (2009) for electric power generating system, Vincente et al. (2010) for the treatment of pathogenic disease, Arthur (1995) for analysing human postural balance, He (2010) for environmental economics regions, Ralf (1996) for dynamic portfolio management.

In 1971, Merton studied stochastic optimal control for finance which made the study of optimal control for finance greatly attract the attention of many researchers such as Karatzas (1989) which applied the study of optimal control in finance, Dixit and Pindych (1994) expressed the use of dynamic programming in optimization over Ito's process, Zhou and Li (2000) introduced the stochastic Linear Quadratic (LQ) control as a general framework to analyze the continuous-time mean variance portfolio optimization and hedging problem, Jensen (1998) studied stochastic optimal stopping problem in risk theory, Cairns (2000) formulated the optimal control models of stochastic pension fund, Komolov et al. (1979) presented the concept of optimal fuzzy control based on the fuzzy set theory and Stein (2003) modeled debt crisis and proposed a solution using stochastic optimal control.

However, the events we face in life are made uncertain in several ways due to the complexity in the world. A study case is that of a business which must possess enough funds to pay its financial obligations at a particular point in time to ensure continuity of the business operations. In order to manage such case, Optimal control approach is applied to the case to show what fraction of investor's business capital assets can and can't be used for current financial obligations. Understanding this control tool would help investors prevent their businesses from facing solvency problems that might be caused by unexpected events and sudden changes in business environment.

In 2007, Liu introduced an uncertain measure to assess the degree belief of an uncertain event. This led to Uncertainty theory based on the uncertain measure and refined by Liu (2016) which is now known as a branch of mathematics for behavior of uncertain phenomena to be studied. Thus, as corresponding function of stochastic process and Brownian motion, Liu (2008) introduced uncertain process and canonical process. Various researchers have worked based on the uncertain theory. Some are Gao (2009) who worked on some properties of continuous uncertain measure, Zhu (2007) obtained the principle of optimality for uncertain optimal control using Bellman's principle of optimality, Deng and Zhu (2012) used the principle of optimality for uncertain optimal control to solve an uncertain optimal control model with n jumps.

II. Preliminary

Uncertainty theory is a branch of mathematics for modelling belief degrees. This theory is based on some concepts which may be referred to Liu (2016).
For easy interpretation, some of the concepts are given.

Let Γ be a nonempty set and L a σ - algebra over Γ such that (Γ, L) be a measurable space. Each element $\Lambda \in L$ is called an event.

Definition 2.1 (Liu 2007): A set function M defined on the σ-algebra over L is called an uncertain measure if it satisfies the following axioms:
Axiom 1. *(Normality Axiom)*: $M\{\Lambda\} = 1$ *for the universal set* Γ.
Axiom 2. *(Duality Axiom)*: $M\{\Lambda\} + M\{\Lambda^c\} = 1$ *for any event* Λ.
Axiom 3. *(Subadditivity Axiom)*: For every countable sequence of events, $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\} \tag{2.1}$$

Axiom 4. *(Product Axiom)*: Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1,2,\cdots$ The product uncertain measure M is an uncertain measure satisfying

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \min_{1\leq k\leq\infty} M_k\{\Lambda_k\} \tag{2.2}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1,2,\cdots,$ respectively.

Definition 2.2 (Liu 2009): An uncertain process C_σ is said to be a canonical Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_σ has stationary and independent increments,

(iii) every increment $C_{s+\sigma} - C_s$ is a normal uncertain variable with expected value 0 and variance σ^2. The uncertainty distribution of C_σ is

$$\Phi_\sigma(x) = \left[1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right]^{-1}, \quad x \in \Re \tag{2.3}$$

and the inverse distribution is

$$\Phi_\sigma^{-1}(y) = \frac{\sigma\sqrt{3}}{\pi}\ln\frac{y}{1-y}, \quad y \in \Re \tag{2.4}$$

Definition 2.3 (Liu 2008): Suppose C_t is a canonical Liu process, and f and g are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{2.5}$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (2.3) and (2.4) identically in t.

Definition 2.4 (Liu 2007): Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^0 M\{\xi \leq x\}dx \tag{2.6}$$

provided that at least one of the two integrals is finite

III. Model Formulation

Under uncertainty, any decision must quantify its aims and its costs which implies that any asset management problem consists of two essential features: expected net value and possible risk.

3.1 Capital Asset Optimization Problem

An individual invests his wealth in capital asset of his business A_t from time t_0 to time t_n. He starts with a known initial net worth X_0. At time t, he must choose what fraction of his net worth to utilize on capital asset, ψ , what fraction of his net worth is incurred on liability of the business, τ and thus, determine the expected present net asset, E such that the net worth is maximized.

Table 3.1 Definition of Parameters

Parameter	Description
X_t	Net worth at time (state variable) t
K_t	Consumption at time t
A_t	Capital asset at time t
I_t	Investment at time t
T_t	Indirect tax at time t
D_t	Depreciation at time t
Z_t	Net Foreign supply at time t (less home supply from foreign supply)
L_t	Liability at time t
R_t	Net foreign factor revenue generated at time t
b_t	Return on capital asset at time t
τ	Liability ratio (control) $\tau > -1$
σ_r	Diffusion volatility of liability (with variance σ_r^2 per unit time)
ψ	Capital asset ratio (control) $\psi > 0$
σ_b	Diffusion volatility of asset (with variance σ_b^2 per unit time)
α	Capital gain on asset due to inflation
σ_p	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
β	Mean rate of return on asset
ω	Mean interest rate of liability
C_r	Liu canonical process of liability
C_p	Liu canonical process of capital asset
C_b	Liu canonical process of productivity of asset
μ	Consumption level
f	Investment ratio
j	Tax ratio
g	Depreciation ratio
h	Net foreign asset ratio
η	discount rate
λ	degree of relative risk
U	Utility function
E	Expectation operator

A dynamic optimization model of the expected present value of asset over a given the life cycle is herein presented.

3.2 Equations of the Uncertain Risk Asset Model

Suppose the investor's net worth with respect to capital asset is defined capital asset less the liability on the asset, i.e.,

$$X_t = A_t - L_t \quad (3.1)$$

$$dX_t = dA_t - dL_t \quad (3.2)$$

We assume uncertainty in the price change of capital asset, thus, we have

$$dA_t = \alpha A_t dt + \sigma_p dC_p A_t \quad (3.3)$$

The change in liability can be expressed as sum of liability service, consumption, investment and net foreign supply, less taxation, depreciation and revenue over a period of time. i.e.,

$$dL_t = L_t dt + (K_t + I_t + Z_t)dt - (T_t + D_t + R_t)dt \qquad (3.4)$$

Thus, interest payment of liability can be expressed also in the form of uncertain differential equation

$$r_t dL_t = \omega L_t + \sigma_r L_t dC_r$$

From equation (3.2), we thus have:

$$dX_t = \alpha A_t dt + \sigma_p A_t dC_p - [\omega L_t dt + \sigma_r L_t dC_r + (K_t + I_t + Z_t)dt - (T_t + D_t + R_t)dt] \qquad (3.5)$$

The constraint can now be expressed as:

$$dX_t = [(\alpha A_t + T_t + D_t + R_t) - (K_t + I_t + Z_t + \omega L_t)]dt + \sigma_p A_t dC_p - \sigma_r L_t dC_r \qquad (3.6)$$

Following Merton (1969) and (1971), we assume that the goal of the asset management is to choose the optimal utilization and asset allocation policies of liability for maximizing a value function which discounts exponentially future uncertain values of an utility function over a given time horizon. Thus, the following asset-liability optimal control problem of capital asset is considered.

We assume that the investor is interested in maximizing its expected discounted utility of a risk asset over a given time interval derived from consumption and liability. Thus, optimal liability-asset utilization problem with objective functional may be described as follows:

$$J(X) = \max E_C\left[\int_{t_0}^{t_n} e^{-\eta t} U(A(t))dt\right] \qquad (3.7)$$

where E_C denotes expectation, U is the utility function, $0 < \eta < 1$ is the arbitrary discount rate and $n = 1,2,\cdots$ (i.e, $t_n < +\infty$). The time interval is inversely proportional to the discount rate in the sense that a high discount rate implies a short time interval, Stein (2003) .

We consider the utility function as a special case of Hyperbolic absolute risk aversion (HARA) which helps to focus more on ratios and its assumption lowers the dimension of dynamic system in order for the model to be easily solved analytically compared to other utility functions.

$$U(A) = \begin{cases} \frac{1}{\lambda} A^{\lambda}, & 0 < \lambda < 1 \\ \ln A, & \lambda = 0 \end{cases} \qquad (3.8)$$

where λ measures the degree of investor's relative risk aversion. The larger λ, the more reluctant to own a risky asset. Then, the investor's optimal control model may be expressed as

$$J(X) = \max E_C\left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} A_t^{\lambda} dt\right] \qquad (3.9)$$

subject to

$$dX_t = [(\alpha A_t + T_t + D_t + R_t) - (K_t + I_t + Z_t + \omega L_t)]dt + \sigma_p A_t dC_p - \sigma_r L_t dC_r \qquad (3.10)$$

Thus, the constraint can be simplified further considering the following ratios. The return on asset is expressed as the ratio of asset revenue and capital asset, that is, let

$$b_t = \frac{R_t}{A_t} \qquad (3.11)$$

which implies

$$b_t dt = \frac{R_t dt}{A_t}$$

$$R_t dt = A_t b_t dt \quad (3.12)$$

which implies that the revenue is the product of capital investment and return of asset.

We assume that the risk asset gains an uncertain return $b_t dt$ in time interval $(t, t+dt)$ and generates a mean rate of return α along with a variance σ^2 per unit time.

Thus, the productivity of asset is also expressed as an uncertain differential equation

$$b_t dt = \beta dt + \sigma_b dC_b \quad (3.13)$$

where

$$R_t dt = A_t \beta dt + A_t \sigma_b dC_b \quad (3.14)$$

The asset efficiency can be expressed as the ratio of asset investment of asset and net asset worth

$$\psi = \frac{A_t}{X_t}$$

$$A_t = \psi X_t \quad (3.15)$$

By HARA utility function assumption, consumption, investment, indirect tax, depreciation and net foreign supply are proportional to net worth of asset. Likewise, the liability is proportional to the net worth of asset, Fleming and Stein (2004).

This implies that

$$K_t = \mu X_t, \quad I_t = fX_t, \quad T_t = jX_t, \quad D_t = gX_t, \quad Z_t = hX_t, \quad L_t = \tau X_t \quad (3.16)$$

We divide through equation (3.1) by X_t and derive

$$\frac{X_t}{X_t} = \frac{A_t}{X_t} - \frac{L_t}{X_t}$$

$$1 = \psi - \tau$$

$$\tau = \psi - 1 \quad (3.17)$$

which implies

$$A_t = (1+\tau) X_t \quad (3.18)$$

Therefore, we have

$$dX_t = [(\alpha+\beta)A_t + T_t + D_t - K_t - I_t - Z_t - \omega L_t]dt + \sigma_p A_t dC_p + \sigma_b A_t dC_b - \sigma_r L_t dC_r$$

$$dX_t = [(\alpha+\beta)\psi - (\omega(\psi-1) + \mu + f + h - j - g)]X_t dt + \psi\sigma_p X_t dC_p$$

$$+\psi\sigma_b X_t dC_b - (\psi-1)\sigma_r X_t dC_r$$

$$dX_t = [(\alpha+\beta)\psi - (\omega(\psi-1) + \mu + f + h - j - g)]X_t dt + [\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t dC_t$$

(3.19)

where C_t is the Liu canonical process for the whole system.

Hence, the model in a simplified form is

$$J(X) = \max_{\psi} E_C\left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} (\psi X_t)^{\lambda} dt\right] \quad (3.20)$$

subject to

$$dX_t = [(\alpha+\beta)\psi - (\omega(\psi-1) + \mu + f + h - j - g)]X_t dt + [\psi\sigma_p + \psi\sigma_b - \psi(\sigma_r - 1)]X_t dC_t$$

IV. Optimality of the Problem

Theorem 4.1 (Principle of Optimality, Zhu (2010)): For any $(t,x)\in[0,T)\times\Re$ and $\Delta t>0$ with $t+\Delta t<T$, we have

$$J(t,x)=\sup_D E\left[\int_t^{t+\Delta t} f(X_s,D,S)ds+J(t+\Delta t,x+\Delta X_t)\right] \quad (4.1)$$

where $x+\Delta X_t=X_{t+\Delta t}$

Theorem 4.2 (Equation of Optimality, Zhu (2010)) Let $J(t,x)$ be twice differentiable on $[0,T)\times\Re$, Then we have

$$-J_t(t,x)=\sup_D\left[f(x,D,t))+J_x(t,x)V(x,D)\right] \quad (4.2)$$

where $J_t(t,x)$ and $J_x(t,x)$ are the partial derivatives of the function $J(t,x)$ in t and X respectively.

4.1 Analytical solution to the problem

We apply the above equation of optimality to our uncertainty optimal control problem to obtain the optimal controls analytically.

By applying equation (4.2), we obtain

$$-J_t=\max_\psi\left\{\frac{1}{\lambda}e^{-\eta t}(\psi X)^\lambda-\psi(\alpha+\beta)XJ_X+(\mu+j+g+f+h-(\psi-1)\omega)J_X\right\}$$

$$=\max_\psi H \quad (4.3)$$

where H stands for terms in the braces.

$$\frac{\partial H}{\partial\psi}=0$$

(condition the optimal ψ satisfies)

$$\frac{\partial H}{\partial\psi}=e^{-\eta t}(\psi X)^{\lambda-1}X-(\alpha+\beta-\omega)XJ_x=0$$

$$(\psi X)^{\lambda-1}X=(\omega-\alpha-\beta)J_x e^{-\eta t}$$

$$(\psi X)=\left[(\omega-\alpha-\beta)J_X e^{-\eta t}\right]^{\frac{1}{\lambda-1}}$$

$$\psi=\frac{1}{X}\left[(\omega-\alpha-\beta)J_X e^{-\eta t}\right]^{\frac{1}{\lambda-1}} \quad (4.4)$$

Therefore, equation(4.3) becomes

$$-J_t=\frac{1}{\lambda}e^{-\eta t}\left[\frac{1}{X}\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}}X\right]^\lambda+(\alpha+\beta)\left[\frac{1}{X}(\omega-\alpha-\alpha)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}}XJ_X$$
$$+\left[(\mu+j+g+\omega-f-h)-\omega\left[\frac{1}{X}\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}}\right]\right]XJ_X$$

$$-J_t=\frac{1}{\lambda}e^{-\eta t}\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{\lambda}{\lambda-1}}+(\alpha+\beta)\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}}XJ_X$$
$$+(\mu+j+g+\omega-f-h)XJ_X-\omega\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}}J_X$$

$$-J_t = \frac{1}{\lambda}e^{-\eta t}\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{\lambda}{\lambda-1}} + ((\alpha+\beta)-\omega)\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}} J_X$$
$$+(\mu+j+g+\omega-f-h)XJ_X$$

$$-J_t e^{\eta t} = \frac{1}{\lambda}\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{\lambda}{\lambda-1}} - (\omega-\alpha-\beta)\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda-1}} J_X e^{\eta t}$$
$$+(\mu+j+g+\omega-f-h)XJ_X e^{\eta t}$$

$$-J_t e^{\eta t} = \frac{1}{\lambda}\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{\lambda}{\lambda-1}} - (\omega-\alpha-\beta)\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{\lambda}{\lambda-1}}$$
$$+(\mu+j+g+\omega-f-h)XJ_X e^{\eta t}$$

$$-J_t e^{\eta t} = (\frac{1}{\lambda}-1)\left[(\omega-\alpha-\beta)J_X e^{\eta t}\right]^{\frac{\lambda}{\lambda-1}} + (\mu+j+g+\omega-f-h)XJ_X e^{\eta t} \tag{4.5}$$

Thus, we conjecture that $J(t,X) = qX^{\lambda}e^{-\eta t}$ where

$$J_t = -q\eta X^{\lambda}e^{-\eta t} \qquad J_t e^{\eta t} = -q\eta X^{\lambda} \tag{4.6}$$

which implies

$$J_X = q\lambda X^{\lambda-1}e^{-\eta t} \qquad J_X e^{\eta t} = q\lambda X^{\lambda-1} \tag{4.7}$$

By substituting equation (4.6) and (4.7) into equation (4.5), we have

$$q\eta X^{\lambda} = (\frac{1}{\lambda}-1)\left[(\omega-\alpha-\beta)q\lambda X^{\lambda-1}\right]^{\frac{\lambda}{\lambda-1}} + (\mu+j+g+\omega-f-h)Xq\lambda X^{\lambda-1}$$

$$q\eta X^{\lambda} = (\frac{1}{\lambda}-1)\left[(\omega-\alpha-\beta)q\lambda\right]^{\frac{\lambda}{\lambda-1}} X^{\lambda} + (\mu+j+g+\omega-f-h)q\lambda X^{\lambda}$$

$$q\eta = (\frac{1}{\lambda}-1)\left[(\omega-\alpha-\beta)q\lambda\right]^{\frac{\lambda}{\lambda-1}} + (\mu+j+g+\omega-f-h)q\lambda$$

$$q\eta = (\frac{1-\lambda}{\lambda})\left[(\omega-\alpha-\beta)q\lambda\right]^{\frac{\lambda}{\lambda-1}} + (\mu+j+g+\omega-f-h)q\lambda$$

$$q\eta - (\mu+j+g+\omega-f-h)q\lambda = (\frac{1-\lambda}{\lambda})\left[(\omega-\alpha-\beta)q\lambda\right]^{\frac{\lambda}{\lambda-1}}$$

$$\frac{q\lambda\eta - (\mu+j+g+\omega-f-h)q\lambda^2}{1-\lambda} = \left[(\omega-\alpha-\beta)q\lambda\right]^{\frac{\lambda}{\lambda-1}}$$

$$\frac{q\lambda\left[\eta - (\mu+j+g+\omega-f-h)q\lambda\right]}{1-\lambda} = \left[(\omega-\alpha-\beta)q\lambda\right]^{\frac{\lambda}{\lambda-1}}$$

$$(q\lambda)^{\frac{1}{\lambda-1}} = \frac{\eta - (\mu+j+g+\omega-f-h)\lambda}{(1-\lambda)(\omega-\alpha-\beta)^{\frac{\lambda}{\lambda-1}}}$$

$$q\lambda = \left[\frac{\eta - (\mu + j + g + \omega - f - h)\lambda}{(1-\lambda)(\omega - \alpha - \beta)^{\frac{\lambda}{\lambda-1}}} \right]^{\lambda-1}$$

From equation (4.4), we have

$$\psi = \frac{1}{X}\left[(\omega - \alpha - \beta)q\lambda\right]^{\frac{1}{\lambda-1}}$$

$$\psi = (\omega - \alpha - \beta)^{\frac{1}{\lambda-1}}(q\lambda)^{\frac{1}{\lambda-1}}$$

$$\psi = (\omega - \alpha - \beta)^{\frac{1}{\lambda-1}} \left[\frac{\eta - (\mu + j + g + \omega - f - h)\lambda}{(1-\lambda)(\omega - \alpha - \beta)^{\frac{\lambda}{\lambda-1}}} \right]$$

Hence, we obtain the optimal ratio of the net worth in capital assets as

$$\psi = \frac{(\mu + j + g + \omega - f - h)\lambda - \eta}{(1-\lambda)(\alpha + \beta - \omega)}$$

However, we can obtain the liability ratio, τ also as control to the system.

Since $\tau = \psi - 1$

$$\tau = \left[\frac{(\mu + j + g + \omega - f - h)\lambda - \eta}{(1-\lambda)(\alpha + \beta - \omega)} \right] - 1$$

or

$$\tau = \frac{(\mu + j + g - f - h)\lambda - (1-\lambda)(\alpha + \beta) - \eta}{(1-\lambda)(\alpha + \beta - \omega)}$$

Remark 4.1: It is observed that the business's optimal capital asset ratio and optimal liability ratio do not depend on the net worth of the business.

Remark 4.2: A negative liability is a positive financial asset position, that is, the greater the liability ratio than -1, the more the financial risk of the business.

V. Conclusion

Based on the uncertainty theory, an uncertain optimal control problem in asset management is formed and studied where an investor is interested in determining the expected present net asset such that the net worth is maximized. The problem was solved analytically using principle of optimality by Zhu (2010) for uncertain optimal control.

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