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Measure of Finite Matrix Set

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Abstract: *This paper extends the concept of measure to a finite matrix set. Concrete constructions of measurable functions for finite matrix set are presented and their properties identified. In the process some theorems emerged.*
Keywords: *measure, matrix set, measurable function, sigma algebra*.

I. Introduction

In mathematical analysis, a measure on a set is a systematic way to assign a positive number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume. A particularly important example is the Lebesgue measure on a Euclidean Space as in Bogachev, (2006), which assigns the conventional length, area and volume of Euclidean geometry to suitable subsets of the n-dimensional Euclidean space R^n. For instance, the Lebesgue measure of the closed bounded interval [a, b] of real numbers is its length in the everyday sense of the word which is specifically the number b - a.

Technically, a measure is a function that assigns a non-negative real number to subsets of a set X. It must assign 0 to the empty set and be countably additive: the measure of a 'large' subset that can be decomposed into a finite (or countable) number of 'smaller' disjoint subsets, is the sum of the measures of the "smaller" subsets Bogachev, (2006). In general, if one wants to associate a consistent size to each subset of a given set while satisfying the other axioms of a measure, one only finds trivial examples like the counting measure. This problem was resolved as stated in Bogachev, (2006) by defining measure only on the collection of all subsets called measurable subsets, which are required to form a σ- algrbra. This means that countable unions, countable intersections and complements of measurable subsets are measurable.

Matrix as in Hohn F. E. (1958) is a rectangular array of numbers. A matrix set therefore is a set consisting of matrices as the elements. Matrices have wide range of applications in algebra and analysis.

II. Preliminaries

The following definitions will serve in the discussion in subsequent sections.

Sigma σ-Algebra - A collection F of subsets of a set X is said to be a σ-algebra (sigma algebra) on X if F has the following properties;
i) X, ∅ ∈ F
ii) If A ∈ F, then A^c ∈ F where A^c is the complement of A with respect to X
iii) If the sets A_1, A_2, A_3, …,A_n ∈ F then;

$$A = \bigcup_{i=1}^{n} A_i \in F$$

Measure $(\mathcal{M})$ - A measure $(\mathcal{M})$ as in [1] is a function defined on a $\sigma -$ algebra (sigma-algebra) F over the set X that takes values within the interval $[0, \infty)$ such that the following conditions are satisfied:
i) $\mathcal{M}(F_i) \geq 0$
ii) $\mathcal{M}$ (∅) = 0 if ∅ is an empty set
iii) Countable additivity: if A_i is countable sequence of pair-wise disjointed measurable sets in F then;

$$\mathcal{M}\left(\bigcup_{i=1}^{\infty}(A_i)\right) = \sum_{i=1}^{\infty} \mathrm{M}(\mathrm{A_i})$$

(Bogachev, 2006)

III. Construction Of Measure Over Finite Matrix Set

We construct some measures over a finite matrix set (X) and demonstrate with examples.

Sum of Square Difference (SSD)

Sum of Square Difference as in Keith C. (2000) is the norm of the difference between two matrices say X_1 and X_2 defined as $SSD(X_1, X_2) = \|X_1 - X_2\|_2^2 = \sum_{i,j=1}^{n}\left[\left(\mathrm{a}_{\mathrm{i,j}}\right)_{X_1} - \left(\mathrm{a}_{\mathrm{i,j}}\right)_{X_2}\right]^2$ where n is the size of each of the matrices.

***Sum of Square Difference measure* (SSD_M).** Given the finite set $X = \{P_1, P_2\}$ where $P_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $P_2 = \begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}$ then the Sum of Square Difference measure of X is;

$$SSD_M(X) = \sum (P_1 - P_2)^2$$

Now $P_1 - P_2 = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$,

Therefore;

$$SSD_M\ (X) = (-2)^2 + (-1)^2 + 1^2 + 0^2 = 6$$

Theorem- Let X be a finite matrix set. Let 2^X be the collection of all subsets of X. Let $2_{\mathrm{i}}^{\mathrm{X}}$ be the members of 2^{X} such that $1 \leq i \leq 2^{|X|}$, where $|X|$ denote the cardinality of X. Then the function $SSD_M\colon 2^X \longrightarrow \mathbb{R}^+$, defined by

$$SSD_M\left(2_i^X\right) = \sum_{i,j=1}^{n}\left[\left(\mathrm{a}_{\mathrm{i,j}}\right)_{X_1} - \left(\mathrm{a}_{\mathrm{i,j}}\right)_{X_2}\right]^2$$

is a measure function.

Proof- From the hypothesis, X is a finite matrix set. 2^X is the power set of X. In tabular form;
$2^X = \{2_1^{\mathrm{X}}, 2_2^{\mathrm{X}}, 2_3^{\mathrm{X}}, 2_4^{\mathrm{X}}, \ldots, 2_{2^{|X|}}^{\mathrm{X}}\}$, where $|X|$ denote the cardinality of X. So 2^X satisfies all the axioms of a sigma-algebra numbered from i to iii above. Furthermore, the function $SSD_M\colon 2^X \longrightarrow \mathbb{R}^+$, defined by

$$SSD_M\left(2_i^X\right) = \sum_{i,j=1}^{n}\left[\left(\mathrm{a}_{\mathrm{i,j}}\right)_{X_1} - \left(\mathrm{a}_{\mathrm{i,j}}\right)_{X_2}\right]^2$$

where $1 \leq i \leq 2^{|X|}$ is well-defined, since if $2_i^X = \left\{[b_{ij}]_1, [b_{ij}]_2, \ldots, [b_{ij}]_{|2_i^X|}\right\}$, then the sum of square difference measure function on X can be represented pictorially as below:

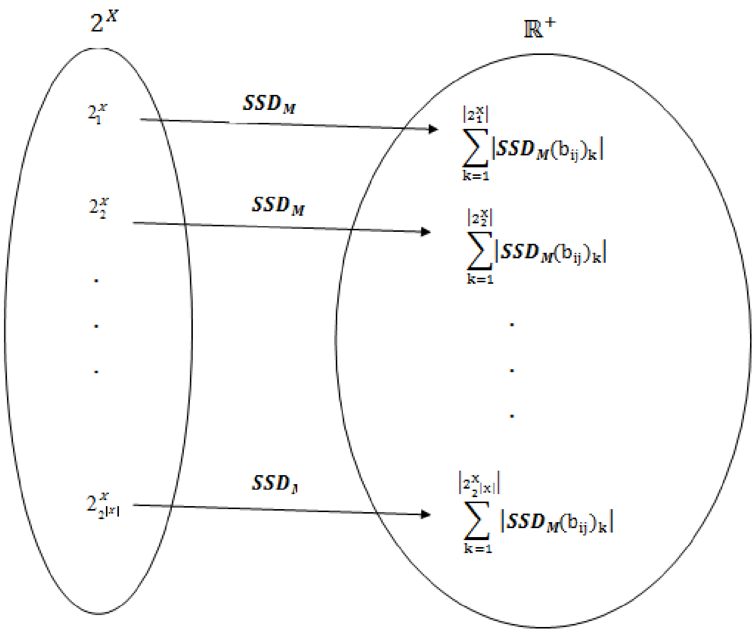

Fig. 1: Sum of Square Difference Measure of a finite matrix set X

Now, to show that the function SSD_M satisfies all the axioms of a measure proceed as follows:

Suppose X = ϕ, it follows that the power set $2^X = \{\phi\}$.

Therefore, $\mathcal{M}(\phi) = 0$.

Next, assuming that $X \neq \phi$, it implies $2^X = \{2_1^X, 2_2^X, 2_3^X, 2_4^X, \dots, 2_{2^{|X|}}^X\}$.

Let $SSD_M(2_1^X) = x_1,\ SSD_M(2_2^X) = x_2,\ SSD_M(2_3^X) = x_3,\ SSD_M(2_4^X) = x_4, \dots, SSD_M\left(2_{2^{|X|}}^X\right) = x_{2^{|X|}}$ where $x_1, x_2, x_3, x_4, \dots, x_{2^{|X|}} \in \mathbb{R}^+$, then;

$$SSD_M\left(\bigcup_{i=1}^{2^{|X|}}(2_i^X)\right) = SSD_M\ \left(2_1^X \cup 2_2^X \cup 2_3^X \cup 2_4^X \cup \dots \cup 2_{2^{|X|}}^X\right)$$
$$\leq SSD_M(2_1^x) + SSD_M(2_2^x) + SSD_M(2_3^X) + SSD_M(2_4^X) \dots + SSD_M(2_{2^{|X|}}^X)$$
$$= x_1 + x_2 + x_3 + x_4 + \dots + x_{2^{|X|}} \in \mathbb{R}^+$$

Hence, SSD_M is a measure function on the sigma algebra 2^x over finite matrix set X.

Mean Square Error (MSE)

Mean square error as in Keith C. (2000) is the average of the norm of the difference between two matrices say X_1 and X_2 defined as $MSE(X_1, X_2) = \frac{1}{n}\|X_1 - X_2\|_3^2 = \dfrac{1}{n}\left(\sum_{i,j=1}^{n}\left[\left(a_{i,j}\right)_{X_1} - \left(a_{i,j}\right)_{X_2}\right]^2\right)$ where n is the size of each of the matrices.

Mean Square Error measure (MSE_M). Let the set X = {P_1, P_2} where $P_1 = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $P_2 = \begin{bmatrix}3 & 3\\2 & 4\end{bmatrix}$ then the Mean Square Error measure of X is;

$$MSE_M(\mathrm{X}) = \frac{1}{n}\sum\nolimits_i (P_1 - P_2)^2$$

now P1 – P2 = $\begin{pmatrix}-2 & -1\\1 & 0\end{pmatrix}$,

therefore

$MSE_M(\mathrm{X}) = \frac{1}{2}[(-2)^2 + (-1)^2 + 1^2 + 0^2]$

= ½ of 6 = 3

Lemma - Let X be a finite matrix set. Let 2^X be the collection of all subsets of X. Then the function MSE_M from 2^X to the set of positive real numbers defined by;

$$MSE_M(2_i^X) = \frac{1}{n}\sum\nolimits_i \left(2_1^X - 2_2^X\right)^2 \text{ is a measurable function.}$$

Euclidean Distance (ED)

Euclidean distance is the norm of the difference between two matrices say X_1 and X_2 that depicts the distance between two vectors defined as $ED(X_1, X_2) = \|(x_1 - x_2)\|_E = \sqrt{\left(\sum_{i,j=1}^{n}\left[\left(a_{i,j}\right)_{X_1} - \left(a_{i,j}\right)_{X_2}\right]^2\right)}$, where n is the size of each of the matrices.

Euclidean distance measure (ED_M) - Let the set X = {P_1, P_2} where $P_1 = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $P_2 = \begin{bmatrix}3 & 3\\2 & 4\end{bmatrix}$ then the Euclidean distance measure on X is;

$$ED_M(X_i) = \sqrt{\sum\nolimits_i (P_1 - P_2)^2}$$

Now $P_1 - P_2 = \begin{pmatrix}-2 & -1\\1 & 0\end{pmatrix}$

Therefore;

$ED_M(X_i) = \sqrt{(-2)^2 + (-1)^2 + 1^2 + 0^2}$

$= \sqrt{6} = 2.4495$.

Lemma - Let X be a finite matrix set. Let 2^X be the collection of all subsets of X. Then the function (ED_M) from 2^X to the set of positive real numbers defined by;

$$ED_M(2_i^X) = \sqrt{\sum_i (2_1^X - 2_2^X)^2} \textbf{ is a measurable function.}$$

Euclidean norm of a matrix EN ($\|\mathbf{X}\|_E$)

Euclidean norm of a matrix is the square root of the sum of the square of each element of the matrix expressed as;

$$\|\boldsymbol{X}\|_{\boldsymbol{E}} = \sqrt{\sum_{i,j=1}^{n} (\boldsymbol{a_{ij}})^2}$$

Euclidean norm measure $(\boldsymbol{EN_M})$ - Let X be a finite matrix set. The Euclidean norm measure of X denoted as $\boldsymbol{EN_M(X)}$, is defined by;

$$\boldsymbol{EN_M(X)} = \sum_{i=1}^{m} \left(\sqrt{\sum_{i,j=1}^{n} (\boldsymbol{a_{ij}})^2} \right)$$

Example - Given that set X = $\{P_1, P_2\}$ where $P_1 = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$, $P_2 = \begin{bmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$ then a Euclidean norm measure of set X by definition is;

$$\boldsymbol{EN_M(X_i)} = \sum_{i=1}^{2} (\|\boldsymbol{P_i}\|_{\boldsymbol{E}}) = \|\boldsymbol{P_1}\|_{\boldsymbol{E}} + \|\boldsymbol{P_2}\|_{\boldsymbol{E}}$$

$$= \sqrt{\mathbf{1^2 + 2^2 + 3^2 + 4^2}} + \sqrt{\mathbf{3^2 + 3^2 + 2^2 + 4^2}}$$

$$= \sqrt{\mathbf{30}} + \sqrt{\mathbf{38}}$$

= 5.4772 + 6.1644

= 11.6416.

Lemma - Let X be a finite matrix set. Let 2^X be the collection of all subsets of X. Then the function $\boldsymbol{EN_M}$ from 2^X to the set of positive real numbers defined by;

$$\boldsymbol{EN_M(2_i^X)} = \sum_{i=1}^{m} \|\mathbf{2_i^X}\|_{\boldsymbol{E}} \textbf{ is a measurable function.}$$

Condition Number (CN)

The condition number of a matrix X is the product of the norm of the matrix and the norm of its inverse i.e. $\|\boldsymbol{X}\|.\|\boldsymbol{X}^{-1}\|$. We note that the norm of $\boldsymbol{X}$ can be determined using any of the defined matrix norms like Euclidean, Frobenius, etc. Thus the condition number of $\boldsymbol{X}$ depends on the type of norm used.

Condition Number measure $(\boldsymbol{CN_M})$ - We define Condition Number measure of set X using Euclidean norm as;
$\boldsymbol{CN_M(X)} = \|\boldsymbol{X_i}\|_{\boldsymbol{E}}.\|(\boldsymbol{X_i})^{-1}\|_{\boldsymbol{E}}$ i= 1,2,….n

Example - Let the set X = $\{P_1, P_2\}$ where $P_1 = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$, $P_2 = \begin{bmatrix} \mathbf{3} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$, then ;

$$\boldsymbol{CN_M(X)} = \sum_{i=1}^{2} (\|\boldsymbol{P_i}\|_E.\|(P_i)^{-1}\|_E)$$

i.e. $CN_M(X) = \|P_1\|_E.\|(P_1)^{-1}\|_E + \|P_2\|_E.\|(P_2)^{-1}\|_E$

Since from previous example, $\|P_1\|_E = 5.4772$ *and* $\|P_2\|_E = 6.1644$. We proceed to get the inverse of each of P_1 and P_2 and subsequently calculate their respective Euclidean norms before determining the Condition Number measure of set X.

Now, $P_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow (P_1)^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \therefore \|(P_1)^{-1}\|_E = 3.2404.$

Also;

$P_2 = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \Rightarrow (P_2)^{-1} = \begin{pmatrix} 2/3 & -1/2 \\ -1/3 & 1/2 \end{pmatrix} \therefore \ \|(P_2)^{-1}\|_E = 1.0274.$

Therefore; $CN_M(X)$is;

5.4772 x 3.2404+ 6.1644 x 1.0274
= 17.7483 + 6.3333
= 24.0816

Lemma - Let X be a finite matrix set. Let 2^X be the collection of all subsets of X. Then the function CN_M from 2^X to the set of positive real numbers defined by $CN_M\left(2_i^X\right) = \left\|2_{\ i}^{X}{}_{i}\right\|_E . \left\|\left(2_{\ i}^{X}{}_{i}\right)^{-1}\right\|_E$ is a measurable function.

IV. Conclusion

In this paper, some measures over a given finite matrix set have been developed and illustrated with concrete examples. It has also been proved that given any finite matrix set X, a function defined on the collection of all subsets of X, (2^x) that returns positive real numbers is a measurable function. It is recommended for further studies that this measurability study be extended to various matrix norms.

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