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Fixed Point Theorems For Generalized Contraction In Ultra Metric Spaces

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Abstract: *In this paper, we prove some fixed point theorems for two and three maps of Jungck generalized contractive mappings on spherically complete Ultra metric space using generalized contractive mappings . Our results extend various known results in ultra metric space such as Pant and Mishra[13],Gajic[7] and others.*

I. Introduction

Generalization of metric space have been done in many ways such as 2-metric space,G- metric space, b- metric space, probabilistic metric space, Fuzymetric space etc. Rooji [1] introduced the concept ofultrametric space.Later on Petals and Vidalis [2] proved a fixed point theorem for contractive mappings on spherically complete ultrametric space X.

Petals and Vidalis [2] established the following fixed point theorem:
Theorem (1.1): Let (X,d) be a spherically complete ultrametric space and $T:X\rightarrow X$ a contractive mapping. Then T has a unique fixed point.
In 2001 Gajic [7] obtained the following generalization of the above theorm:
Theorem (1.2): Let (X,d) be a spherically complete ultrametric space and $T:X\rightarrow X$ a mapping such that for all $x,y \in X$, $x\neq y$,
$d(Tx,Ty) < \max \{ d(x,y), d(x,Tx), d(y,Ty)\}$
Then T has a unique fixed point.
Later on Rao and Kishore [5] extended the above result for a pair of maps of Jungck type as follows:
Theorem (1.3): Let (X,d) be a spherically complete ultrametric space. If f and T are self maps on X satisfying
$T\left(X\right)\subseteq f\left(X\right)$
$d(Tx,Ty)<\max\{d(f(f(x),f(y)), d(f(x),T(x)),d(f(y),T(y))\}$, $x,y \in x \neq y$.
then there exists $z \in X$ such that fz=Tz.
Further if f and T are coincidentally commutating at z then z is the unique common fixed point of f and T.
Further in 2014, Mishra and Pant [13] extended the result of Gajic [7] by introducing a more gereralized contractive mapping as follows:
Definition (1.4): A self-mapping T of a metric (resp. an ultrametric) space X is said to be generalized contractive mapping if
$d(Tx,Ty)<M(x,y)$for all $x,y \in X$ with $x\neq y$, where
$M(x,y) = \max\{d(x,y), d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$
and proved the following theorem:

Theorem (1.5): Let (X,d) be a spherically complete ultrametric space and $T:X \longrightarrow X$ a generalized contractive mapping. Then T has a unique fixed point.
In this paper we have generalized and extended the previous results by:
(i) Increasing the number of maps.
(ii) Increasing the number of terms in R.H.S.

II. Preliminaries

Definition (2.1): An ultrametric space is a set X together with a function $d:XxX \longrightarrow R_+$, which satisfies for all x,y and z in X
(U_1) $d(x,y) \geq 0$
(U_2) $d(x,y) = 0$ if $x=y$
(U_3) $d(x,y) = d(y,x)$ (Symmentry)
(U_4) $d(x,z) \leq \max\{d(x,y),d(y,z)\}$ (strong triangle or ultrametric inequality)
Example (2.2): The discrete metric is an ultrametric.
Example (2.3): the p-adic number form a complete ultrametric space.

Definition (2.4): An ultrametric space (X,d) is said to be spherically complete if every shrinking collection of balls in X has non-empty intersection.
Definition (2.5): A self mapping T of a metric (resp. an ultrametric) space X is said to be contractive (or, strictly contractive) mapping if
$d(Tx,Ty)<d(x,y)$ for all $x,y \in X$ with $x \neq y$.
Example (2.6): Let $X=(-\infty,\infty)$ endowed with the usual metric and $T:X \to X$ defined by

$$Tx = x + \frac{1}{1+e^{x}}$$

for all $x \in X$. Here X is complete and T is a contractive mapping but T does not have a fixed point.
Definition (2.7): A self – mapping T of a metric (resp. an ultrametric) space X is said to be generalised contractive mapping if:
$d(Tx,Ty) < M(x,y)$ for all $x,y \in X$ with $x \neq y$, where
$M(x,y) = \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$
Definition (2.8): For $x \in X$, $r>0$, $B_r(x) = \{y \in X : d(x,y)<r\}$ is called the ball (open) with centre x and radius r.

III. Main Results

Theorem (3.1): Let (X,d) be a spherically complete ultrametric space. Let f and T are self maps on X satisfying:
1. $T(x) \subseteq f(x)$
2. $d(Tx,Ty) < \max\{d(f(x),f(y)),d(f(x),T(x)),d(f(y),T(y)),d(f(x),T(y)),d(f(y),T(x))\}$ $\forall\, x,y \in X$, $x \neq y$.

then there exists $z \in X$ such that $f(z) = T(z)$.
Further if f and T are coincidently commutating at z then z is the unique common fixed point of f and T.
Proof:
Let $B_a = (f_a : d(fa,Ta)$ denote the closed sphere centred at fa with the radius d(fa,Ta) and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \le B_b$ if $B_b \subseteq B_a$ is a partial order on A. Let A_1 be a totally ordered sub family of A. Since (X,d) is spherically complete, we have $\bigcap_{B_a \in A_1}$
Let $f(b) \in B$ and $B_a \in A_1$. Then $f(b) \in B_a$. Hence
$d(f(b),f(a) \le d(f(a),T(a))$ ……………………….(i)
If a=b, then $B_a=B_b$. We assume that $a \neq b$.
Let $x \in B_b$, then

$$\begin{aligned}
d(x,f(b)) &\le d(f(b),T(b)) \\
&\le \max\{d(f(b),f(a)),d(f(a),T(a)),d(T(a),T(b))\} \\
&= \max\{d(f(a),T(a)),d(Ta,Tb)\} \text{ ………………………. from (i)} \\
&< \max\{d(f(a),T(a)),d(f(a),f(b)),d(f(b),T(b)),d(f(a),T(b)),d(f(b),T(a))\} \text{ from (2)} \\
&< d(f(a),T(a)) \text{ ………………………. (ii)}
\end{aligned}$$

Now, $d(x,f(a)) \le \max\{d(x,f(b)),d(f(b),f(a)) \le d(f(a),T(a))$ from (i) & (ii)
Thus, $x \in B_a$.Hence $B_b \subseteq B_a$ for any $B_a \in A_1$. Thus B_b is an upper bound in A and hence by Zorn's Lemma, A has a maximal element say B_z, $z \in X$.
Suppose that $f(z) \neq T(z)$. Since $Tz \in T(x) \subseteq f(x)$, there exists $w \in X$ such that $T(z)=f(w)$. clearly$z \neq w$. Now from (2) we have:

$$\begin{aligned}
d(f(w),Tw)) &= d(Tz,Tw) \\
&< \max\{d(f(z),f(w)),d(f(z),T(z)),d(f(w),T(w)),d(f(z),T(w)),d(f(w),T(z))\}
\end{aligned}$$

But by the strong triangle inequality, we have
$d(f(z),T(w)) < \max\{d(f(z),T(w)),d(f(w),T(w))\}$
and $d(f(w),T(z)) < \max\{d(f(w),T(z)),d(f(z),T(z))\}$

$$\begin{aligned}
\text{Thus, } d(f(w),Tw)) &= d(T(z),T(w)) \\
&< \max\{d(f(z),f(w)),d(f(z),T(z)),d(f(w),T(w))\} \\
&= d(f(z),f(w)).
\end{aligned}$$

Thus $fz \notin B_w$. Hence $fz \not\subseteq B_w$ .And this contradicts the maximality of B_z. Hence $f(z) = T(z)$.
Further assume that f and T are coincidentally commutating at z.
Then $f^2(z)=f(f(z))=f(T(z))=Tf(z)=T(T(z))=T^2(z)$
Suppose that $f(z) \neq z$. Now from (2) we have,

$$\begin{aligned}
d(Tf(z),T(z)) &< \max\{d(f^2(z),f(z)),d(f^2(z),Tf(z)),d(f(z),T(z)),d(f^2(z),T(z)),d(f(z),Tf(z))\} \\
&= d(Tf(z),T(z))
\end{aligned}$$

Hence, $f(z) = z$. Thus $z=f(z)=T(z)$
Uniqueness: Suppose (if possible) w is another common fixed point of f and T such that $z \neq w$.
Then $d(z,w) = d(T(z),T(w))$
$< \max\{d(f(z),f(w)),d(f(z),T(z)),d(f(w),T(w)),d(f(z),T(w)),d(f(w),T(z))\}$

=max{d(z,w),d(z,z),d(w,w)}

= d(z,w), which is not possible. Thus our supposition is wrong and hence z is the unique common fixed point of f and T.

Remarks (3.2): If we put f=I, (identity map) theorem (1.1.6) follows.

Example (3.3): Let X=R, and $d(x,y)=\begin{cases}0 \; if \; x=y\\ 1 \; if \; x\neq y\end{cases}$

We define T, f : X $\longrightarrow$X as Tx=1 and f(x) = $\frac{x+1}{2}\forall x\notin X$.

Then all conditions of theorem (1.3.6) are satisfied and 1 is the unique common fixed point of T and f.

Theorem (3.4): Let (X,d) be a spherically complete Ultra metric space. If S;T:X $\longrightarrow$X are mappings such that:

(i) d(Tx,Ty) <max{d(Sx,Sy),d(Sx,TSx),d(Sy,TSy),d(Sx,TSy),d(Sy,TSx)} $\forall x, y \in X, x\neq y$

(ii) d(Sx,Sy) < d(x,y)

(iii) TS(x) = ST(x) $\forall\, x \in X$

Then S and T have a unique common fixed point in X.

Proof: Using conditions (ii) and (iii) in (i) we have

d(Tx,Ty) < max{d(Sx,Sy),d(Sx,STx),d(Sy,STy),d(Sx,STy),d(Sy,STx)}

or d(Tx,Ty) < max{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)}

By theorem (1.5) T has unique fixed point i.e. there exists a point z€X such that z=Tz.

Now,

d(z,Sz) = d(Tz,STz) = d(Tz,TSz)

< max{d(Sz,S^2z),d(S^2z,TS^2z),d(S^2z,TS^2z),d(S^2z,TS^2z),d(S^2z,TS^2z)}

= max{d(Sz,S^2z),d(S^2z,S^2Tz),d(S^2z,S^2Tz),d(S^2z,S^2Tz),d(S^2z,S^2Tz)}

< max{d(z,Sz),d(Sz,Sz),d(Sz,Sz),d(Sz,Sz),d(Sz,Sz)}

i.e. d(z,Sz) < d(z,Sz) which is a contradiction. Hence z=Sz.

Uniqueness: If possible let z and w be two distinct fixed point of S and T. Then,

d(z,w) = d(Tz,Tw) < max{d(Sz,Sw),d(Sz,TS(w)),d(Sz,TS(w)),d(Sz,TS(w)),d(Sw,TS(z))}

= max{d(z,w),d(z,w),d(z,w),d(z,w),d(w,z)

i.e. d(z,w) < d(z,w) which is not possible and hence z=w. Therefore z is the unique common fixed point of S and T.

Remarks (3.5): If we put S=I (identity map), theorem (3.5) reduces to the theorem (1.6) due to Mishra and Pant[13].

Theorem (3.6): Let (X,d) be spherically complete ultrametric space. If T, f and g are self maps on X satisfying

(i) g(x)$\subseteq$f(x)

(ii) d(g(x),g(y)) <max{d(fT(x),fT(y)),d(fT(x),gT(x)),d(fT(y),gT(y)),d(fT(x),gT(y)),d(fT(y),gT(x))} $\forall x, y \in X, x\neq y$

(iii) d(Tx,Ty) < d(x,y)

(iv) Tf(x) = fT(x) and Tg(x) = gT(x) then z=Tz=f(z)=g(z). further

Proof: using condition (iii) and (iv) condition (i) becomes:

d(g(x),g(y)) < max{d(Tf(x),Tf(y)),d(Tf(x),Tg(x)),d(Tf(Tg(y)),d(Tf(y),Tg(x))}y),Tg(y)),d(Tf(x),

or, d(g(x),g(y)) < max{d(f(x),f(y)),d(f(x),g(x)),d(f(y),g(y)),d(f(x),g(y)),d(f(y),g(x))}

By theorem (1.5) there exists a unique common fixed point for f and g i.e. z=f(z)=g(z).

Now

d(z,Tz) = d(g(z),Tg(z)) = d(g(z),gT(z)

<max{d(fT(z),fT^2(z)),d(fT(z),gT(z)),d(fT^2(z),gT^2(z)),d(fT(z),gT^2(z)),d(fT^2(z),gT(z))}

< max{d(f(z),Tf(z)),d(f(z),g(z)),d(Tf(z),Tg(z)),d(f(z),Tg(z)),d(Tf(z),g(z))}

= max{d(z,Tz),d(z,z),d(Tz,Tz),d(z,Tz),d(Tz,z)}

i.e. d(z,Tz) < d(z,Tz) which is a contradiction , hence z=Tz and using the theorem (1.6), z is unique for T. Hence, T, f and g have a unique common fixed point.

Remarks (3.7): If we put T = I, identity map, theorem (3.6) reduces to the theorem (3.5).

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