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The Fixed Point Theorem of Volterra Applied To the Cauchy Problem

Choucha Abdelbaki[1], Guerbati Kaddour[2]
[1,2] *(Laboratory of Mathematics and Applied Sciences/ University of Ghardaia, Algerie)*

Abstract: In this work we can apply the fixed point theorem of Volterra to the Cauchy problem associated with 1st-order equations:

$$y' = f(x,y) \qquad (1)$$

Keywords: fixed point of Volterra, Cauchy problem, Cauchy sequence, linear application.

I. Introduction

The Italian mathematician Vito Volterra (May 3, 1860 - October 11, 1940 in Rome), he puts a fixed point theorem, it uses linear applications and transformation then gives the fixed point in the form of a convergent series.

There are theorems of existence and uniqueness of the Cauchy problem associated with 1st-order equations (Cauchy - Lipchitz, BiCart ...) [2,3,6].

In this work we want to apply the fixed point theorem of Volterra to the Cauchy problem associated with 1st-order equations.

II. Fixed Point Of Volterra

1- Theorem: (fixed point of Volterra)

Let E Banach space, $z_0 \in E, and\ S \in L(E)$, as :

$$1 + \sum_{n=1}^{+\infty} \|S^n\| \equiv A < \infty \qquad (2)$$

Then: T transformation defined by:

$$T(x) = z_0 + S(x)$$

has a unique fixed point x_0 defined by:

$$x_0 = z_0 + \sum_{n=1}^{+\infty} S^n(z_0) \qquad (3)$$

2- Proof :

- ***Existence :***

Suppose : $x_1 \in E$, take a suite $(x_n)_{n\geq 1}$ as :

$$x_{n+1} = T(x_n) \qquad (4)$$

We have :

$$x_2 = T(x_1) = z_0 + S(x_1)$$
$$x_3 = T(x_2) = z_0 + S(x_2)$$
$$x_3 = z_0 + S\big(z_0 + S(x_1)\big) = z_0 + S(z_0) + S^2(x_1)$$
$$\vdots$$
$$x_n = z_0 + \sum_{k=1}^{n-2} S^k(z_0) + S^{n-1}(x_1) \qquad (5)$$

By recurrence: (assume (5) are verified for n)

$$\begin{aligned} x_{n+1} &= T(x_n) \\ &= z_0 + S(x_n) \\ &= z_0 + S\left(z_0 + \sum_{k=1}^{n-2} S^k(z_0) + S^{n-1}(x_1)\right) \end{aligned}$$

$$= \quad z_0 + S(z_0) + S(\sum_{k=1}^{n-2} S^k(z_0)) + S^n(x_1)$$

$$= \quad z_0 + \sum_{k=1}^{n-1} S^k(z_0) + S^n(x_1)$$

Therefore (5) is satisfied for $(n+1)$,

Finally (5) is verified for every $n \in \mathbb{N}$, and the existence of suite (x_n).

- We will study the convergence of the sequence (x_n) (Cauchy sequence).

By using (5) we have:

$$\begin{aligned}
\|x_{n+p} - x_n\| &= \left\| z_0 + \sum_{k=1}^{n+p-2} S^k(z_0) + S^{n+p-1}(x_1) - z_0 - \sum_{k=1}^{n-2} S^k(z_0) - S^{n-1}(x_1) \right\| \\
&= \left\| \sum_{k=n-1}^{n+p-2} S^k(z_0) + S^{n+p-1}(x_1) - S^{n-1}(x_1) \right\| \\
&\leq \sum_{k=n-1}^{n+p-2} \|S^k(z_0)\| + \|S^{n+p-1}\| \|x_1\| - \|S^{n-1}\| \|x_1\| \\
&\leq \sum_{k=n-1}^{n+p-2} \|S^k(z_0)\| (\|z_0\| + \|x_1\|) \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

Because the condition (2) we have:

$$\sum_{k=n-1}^{n+p-2} S^k(z_0) = \sum_{k=0}^{n+p-2} S^k(z_0) - \sum_{k=0}^{n-1} S^k(z_0) \xrightarrow[n\to\infty]{} A - A = 0$$

And thereafter (x_n)Cauchy in space complete E (Banach).

So: the sequence (x_n) converges to $x_0$defined by:

$$\begin{aligned}
x_0 &= \lim_{n\to\infty} x_n \\
&= \lim_{n\to\infty} (z_0 + \sum_{k=1}^{n-2} S^k(z_0) + S^{n-1}(x_1)) \\
&= z_0 + \sum_{k=1}^{\infty} S^k(z_0) + \lim_{n\to\infty} S^n(x_1)
\end{aligned}$$

So: $(\lim_{n\to\infty} S^n(x_1) = 0)$

$$x_0 = z_0 + \sum_{k=1}^{\infty} S^k(z_0)$$

Furthermore we have:

$$\begin{aligned}
x_0 = \lim_{n\to\infty} x_n &= \lim_{n\to\infty} T(x_{n-1}) \\
&= T(\lim_{n\to\infty} x_{n-1}) \\
&= T(x_0)
\end{aligned}$$

Therefore x_0 is a fixed point of application T .

- **Uniqueness:**

We assume two fixed points of T : x_0 and y_0

$$\begin{aligned} x_0 - y_0 &= T(x_0) - T(y_0) \\ &= z_0 + S(x_0) - z_0 - S(y_0) \\ &= S(x_0 - y_0) \end{aligned}$$

And it was.:

$$\begin{aligned} S(x_0 - y_0) &= S(S(x_0 - y_0)) = S^2(x_0 - y_0) \\ &= S^2(x_0 - y_0) = \cdots \\ &= \quad . \\ &= S^n(x_0 - y_0) \xrightarrow[n \to \infty]{} 0 \end{aligned}$$

So :

$$x_0 - y_0 = 0 \Leftrightarrow x_0 = y_0$$

Hence the uniqueness of the fixed point.

Note :

The fixed point x_0 is a solution of the equation:

$$\begin{aligned} & x_0 = T(x_0) = z_0 + S(x_0) \\ \Leftrightarrow \quad & x_0 - S(x_0) = z_0 \\ \Leftrightarrow \quad & (I - S)(x_0) = z_0 \\ \Leftrightarrow \quad & x_0 = (I - S)^{-1}(z_0) \end{aligned}$$

($(I - S)$ is reversible, because $\|S\| < 1 \Leftrightarrow \sum S^n < \infty$)

So :

$$x_0 = z_0 + \sum_{k=1}^{\infty} S^k(z_0) = (I - S)^{-1}(z_0) \qquad (6)$$

- The Condition (2) is equivalent to the existence of the inverse of the operator $(I - S)$, or $\|S\| < 1$.

Therefore the theorem Volterra applied to the fixed point equations in the form:

$$x = z_0 + S(x)$$

With : $S \in L(E)$ and $\|S\| < 1$, (E is a Banach space).

III. Cauchy Problem

Consider the following problem:

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \qquad (7)$$

- **Theorem (Cauchy-Lipchitz)**[3]

If $f \in \mathcal{C}([a, b])$ and if f verified the condition:

$$\forall x \in [a, b], \forall y_1, y_2 \in \mathcal{C}([a, b]), \exists k > 0$$

As:

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

Then the Cauchy problem has a solution which on $[a, b]$.

- We use the lemma Gronwall [9], to ensure la'unicité of the solution.
- **Theorem** :[2]

Let the problem (7)

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

If $f(x, y) and$ $\frac{df}{dy}(x, y)$ are continued and bounded at all points (x, y) of the rectangle R:

$$R = \{(x,y) \in \mathbb{R}^2, |x - x_0| < a\,, |y - y_0| < b\}$$

i.e:

$$\forall (x,y) \in \mathbb{R}^2, \exists M > 0, |f(x,y)| < M, et \ \left|\frac{df}{dy}(x,y)\right| < \mathrm{M}\,.$$

Then the problem (7) has a unique solution on the interval:

$$|x - x_0| < \propto, tel\ que \quad \propto = \min(a, \frac{b}{\propto})$$

And after recurrence converges to the exact solution of the problem.

IV. Application

Volterra fixed-point theorem is applied to the Cauchy problem associated with the first order equations:
We have :

$$y' = f(x,y)$$

$$\Leftrightarrow \int_{x_0}^{x} y'(s)ds = \int_{x_0}^{x} f(s, y(s))ds$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s))ds$$

Applying the fixed point theorem of Volterra to:

$$\begin{cases} z_0 = y(x_0) \\ S(y) = \int_{x_0}^{x} f(s, y(s))ds \end{cases} \tag{8}$$

- ***Lemma :***

If $S \in \mathcal{L}(E), and\ \|S\| < 1$, then the problem (7) has a unique solution defined by:

$$y(x) = z_0 + \sum_{1}^{\infty} S^n(z_0) = y(x_0) + \sum_{1}^{\infty} (\int_{x_0}^{x} f(s, y_0(s))ds)^n$$

V. Examples

1- ***Example 1*** :

- Consider the following problem:

$$\begin{cases} y' = y \\ y(0) = 1 \end{cases} \tag{9}$$

We have:

$$y' = y \quad \Leftrightarrow \quad y(x) - y(0) = \int_{0}^{x} y(t)dt$$

$$\Leftrightarrow \quad y(x) = \underbrace{y(0)}_{z_0} + \underbrace{\int_{0}^{x} y(t)dt}_{S(x)}$$

$$\forall 0 \le x \le a, and \ \ E = \mathcal{C}([a\,,b]), z_0 = y(0) = 1 \in E, \qquad S(y) = \int_{0}^{x} y(t)dt\,.$$

By Volterra theorem equation $x = z_0 + S(x)$ has a unique solution given by:

$$\begin{aligned} y &= z_0 + \sum_{1}^{\infty} S^n(z_0) \\ &= 1 + \sum_{1}^{\infty} (\int_{0}^{x} y(0)dt)^n \\ &= 1 + \sum_{1}^{\infty} (\int_{0}^{x} 1.dt)^n \\ &= \sum_{0}^{\infty} \frac{x^n}{n!} = e^x \end{aligned}$$

- ***Direct solution:***

$$y' = y \quad \Leftrightarrow \quad \frac{y'}{y} = 1 \Rightarrow \ln y = x + k$$

$$\Rightarrow \quad \begin{cases} y(x) = c.e^x \\ y(0) = 1 \end{cases} \Rightarrow c = 1$$

So the solution is:

$$y(x) = e^x$$

2- ***Example 2*** :

- Consider the following problem::

$$\begin{cases} y' = x + y \\ y(0) = 1 \end{cases}$$

We have :

$$y' = x + y \overset{\int}{\Rightarrow} y(x) = y(0) + \int_{0}^{x} \big(t + y(t)\big)dt$$

We set :

$$\begin{cases} z_0 = y(0) \\ S(y) = \int_{0}^{x} \big(t + y(t)\big)dt \end{cases}$$

And according to the Volterra theorem given by the solution:

$$\begin{aligned} y &= z_0 + \sum_{1}^{\infty} S^n(z_0) \\ &= y(0) + \sum_{1}^{\infty} (\int_{0}^{x} \big(t + y(t)\big)dt)^n \end{aligned}$$

By integrating n times:

$$S^n(z_0) = \frac{x^{n+1}}{(n+1)!} + \frac{x^n}{n!}$$

And the solution given by:

$$\begin{aligned} y(x) &= 1+\sum_{1}^{\infty}\left(\frac{x^{n+1}}{(n+1)!}+\frac{x^{n}}{n!}\right) \\ &= 1+\sum_{2}^{\infty}\frac{x^{n}}{n!}+\sum_{1}^{\infty}\frac{x^{n}}{n!} \\ &= 1+(-x-1+e^{x})+(-1+e^{x}) \end{aligned}$$

finally:

$$y(x)=2e^{x}-x-1$$

- ***Direct solution :***

Let:

$$\begin{cases} y'=x+y \\ y(0)=1 \end{cases}$$

homogeneous equation:

$$y'-y=0 \Leftrightarrow y'=y \Leftrightarrow y(x)=ke^{x}$$

the method of variation of the constant is used, it is:

$$y(x)=-x-1+ke^{x}$$

With the initial condition $y\ (0)\ =\ 1$, the problem solution given by:

$$y(x)=-x-1+2e^{x}$$

VI. Conclusion

The principle of the fixed point is crucial in the area of applications. He took part in the resolution of several differential equations for the problems of existence and uniqueness.
In this work we approach the application in particular the fixed point theorem of Volterra for solving Cauchy problems associated with 1st -order equations, with application examples.
But there are obstacles (if the nonlinear operator S, Volterra theorem does not apply, and the application $S^{n}\ (z_0)$ in general is difficult to calculate, so we can use a numerical method (a algorithm) to calculate a value approach with error ε).

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