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Rate of Equi-Summability of Some Series Associated with Fourier series and Certain Integrals

SanghamitraBeuria[1] ,G.Das and B.K.Ray[2]
Institute of Mathematics and Applications, Andharua, Bhubaneswar-3, Odisha,India.
[1]Dept. of Mathematics, College of Basic Science and Humanities, OUAT. Bhubaneswar,Odisha, India.
[2]Plot.No-102,Saheed Nagar,Bhubaneswar-7, Odisha, India.

Abstract: *Salem and Zygmund have studied the equi-summability problem of factored Fourier series and conjugate series the factor being n^{γ}, $0 < \gamma < 1$. In the present work we study the rate of equi- summability of same series forfunctions belonging to $H_p^{(\omega)}$ space ,p$\geq$ 1.*

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I. Definition and Notation

Let f be 2π-periodic function and $f \in L_p[0,2\pi], p \geq 1$. Let the Fourier series of f at x be given by

$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x)$ (1.1)

The series conjugate to (1.1) is given by

$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x)$ (1.2)

We write

$\emptyset^*(x,t)=$ f(x + t) + f(x − t) ,

$\emptyset(x,t)= \emptyset^*(x,t) - 2f(x)$

ψ_x(t) = $f(x+t) - f(x-t)$

$\tilde{f}(x;\epsilon)= -\frac{1}{\pi}\int_{\epsilon}^{\pi} \psi_x(t)\frac{1}{2}cot\frac{1}{2}tdt$,

$$\tilde{f}(x) = \lim_{\epsilon\to 0^+} \tilde{f}(x;\epsilon), \text{ whenever the limit exists.}$$

f is said to be belong to (see Zygmund [8])$lip\alpha$ if $|f(x+t) - f(x)| = o(t^{\alpha})$ and to Lipα if$|f(x+t) - f(x)| = O(t^{\alpha})$,$0 < \alpha \leq 1$.

It was Prossdorf [5], who first studied the degree of approximation problems of Fourier series in$H_{\alpha}(0 < \alpha \leq 1)$ space in the $H\ddot{o}$ ldermetric. Generalizing the H¨ older metric, Leindler [4] introduced the space H^{ω} $space$ replacing $t^{\alpha}$by an arbitrary function ω which is given by

$H^{\omega} = \{f \in C_{2\pi} | \omega(\delta,f) = O(\omega(\alpha))\}$

where ω is a modulus of continuity. The norm $\|.\|_{\omega}$ in H^{ω} is defined by

$\|f\|_{\omega} = \|f\|_c + \sup_{x,y} \frac{|f(x)-f(y)|}{\omega(|x-y|)}, x \neq y$

In the case $\omega(\delta) = \delta^{\alpha}, 0 < \alpha \leq 1$ the space H^{ω} reduces to H_{α} space (the norm$\|.\|_{\omega}$being replaced by $\|.\|_{\alpha}$introduced earlier by Prossdrof[5]. It is known that [4]

$H_{\alpha} \subseteq H_{\beta} \subseteq C_{2\pi}$, $0 \leq \beta < \alpha \leq 1$.

Das, Nath and Ray [3] have studied a further generalization of H^{ω}space which is defined as follows.

For $f \in L_p[0,2\pi], p \geq 1$, we write

$$A(f,\omega) = \sup_{t \neq 0} \frac{\|f(.+t) - f(.)\|_p}{\omega|t|}$$

whereω is a modulus of continuity. We say that $f \in Lip\ (\omega,p)$ if

$$\|f(.+t) - f(.)\|_p = O(\omega(t))$$

Define

$$H_p^{(\omega)} = \{f \in [0,2\pi], p \geq 1 | A(f,\omega) < \infty\}.$$

and $\|f\|_p^{(\omega)} = \|f\|_p + A(f,\omega)$.

It can be easily verified that $\|f\|_p^{(\omega)}$ is a norm in $H_p^{(\omega)}$. If we put $\omega(t) = t^{\alpha}$, $0 < \alpha \leq 1$ then $H_p^{(\omega)}$ reduces to $H(\alpha,p)$ space with the norm $\|f\|_p^{(\omega)}$ replaced by $\|f\|_{(\alpha,p)}$ introduced earlier by Das, Ghosh and Ray [2]. If further $p = \infty$,then $H(\alpha,p)$ reduces to H_{α} space introduced earlier By Prossdorf.If $\frac{\omega(t)}{t} \to 0$ as $t \to 0$ then f'(x) exists and zero everywhere, and f is constant. Given the space $H_p^{(\omega)}$and $H_p^{(v)}$if $\frac{\omega(t)}{v(t)}$is non decreasing then

$H_p^{(\omega)} \subseteq H_p^{(v)} \subseteq L_{p,}\ \ p \geq 1$ (1.5)

since$\|f\|_p^{(v)} \leq \max(1, \frac{\omega(2\pi)}{v(2\pi)})\ \|f\|_p^{(\omega)}$. If $\omega(t) = t^\alpha$and $\omega(t) = t^\beta$ then (1.5) reduces to the following Inclusion relation

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_{p,}\ p \geq 1$$

II. Introduction

For $0 < \gamma < 1$ and$0 < \epsilon < \infty$ we write

$I_\gamma(x;\epsilon) = -\frac{1}{\pi}\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}\int_\varepsilon^\infty \frac{\Psi(x,t)}{t^{1+\gamma}}$ (2.1)

$J_\gamma(x;\epsilon) = -\frac{1}{\pi}\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}\int_\varepsilon^\infty \frac{\emptyset(x,t)}{t^{1+\gamma}}$ (2.2)

We define

$I_\gamma(x) = \lim_{\epsilon\to 0+} I_\gamma(x;\epsilon)andJ_\gamma(x) = \lim_{\epsilon\to 0+} J_\gamma(x;\epsilon)$ (2.3)

when ever the limits exist.

Salem and Zygmund ([6],p.30) have studied he equi-summability of

$\sum_{n=1}^\infty n^\alpha B_n(x)$ (2.4)

and

$\sum_{n=1}^\infty n^\alpha A_n(x)\,, 0 < \alpha < 1$ (2.5)

respectively with the integrals $I_\gamma(x)$ and $J_\gamma(x)$.

Let $U_\gamma(r,x) and\ V_\gamma(r,x)$ respectively denote the Abel transforms of the series(2.4)and (2.5)that is

$$U_\gamma(r,x) = \sum_{n=1}^\infty n^\gamma B_n(x) r^n \quad ,0 < r < 1, \qquad 0 < \gamma < 1$$

$$V_\gamma(r,x) = \sum_{n=1}^\infty n^\gamma A_n(x) r^n \quad ,0 < r < 1, \qquad 0 < \gamma < 1$$

Salem and Zygmund proved the following.

Theorem A([6],p.30)

Let the function $f(x)$ be 2π –periodic and belong to $lip\ \gamma$, where $0 < \gamma < 1$. Then the difference

$I_\gamma(x;1-r) - U_\gamma(r,x)$ (2.6) tends to zero uniformly in x as $r \to 1^-$. If $f \in Lip\ \gamma, 0 < \gamma < 1$ the above expression in (2.6) is uniformly bounded.

Theorem B([6],p.31)

Let the function $f(x)$ be defined as in Theorem A. Then

$J_\gamma(x;1-r) - V_\gamma(r,x)$ (2.7)

tends to zero uniformly in x as $r \to 1^-$. If $f \in Lip\ \gamma, 0 < \gamma < 1$ the above expression in (2.7) is uniformly bounded.

The object of the present paper is to obtain the degree at which the expression (2.6) and (2.7) converges for functions in $H_p^{(\omega)}$ space.

III. Main Result

We prove the following theorems.

Theorems 1: Let $v\ and\ \omega$ be moduli of continuity such that ω/v is non-decreasing . If $f \in H_p^{(\omega)}, p \geq 1$ and, $0 < \gamma < 1$ then

$\|I_\gamma(.;1-r) - U_\gamma(r,.)\|_p^{(v)}$=$O(1)(1-r)^{1-\gamma}\int_{1-r}^\pi \frac{\omega(u)}{v(u)u^2}du$

Theorems 2: Let $v\ and\ \omega$ be moduli of continuity such that ω/v is non-decreasing . If $f \in H_p^{(\omega)}, p \geq 1$ and, $0 < \gamma < 1$ then

$\|J_\gamma(.;1-r) - V_\gamma(r,.)\|_p^{(v)}$=$O(1)(1-r)^{1-\gamma}\int_{1-r}^\pi \frac{\omega(u)}{v(u)u^2}du$

RemarkSee([3], p.47) and ([1], p.25)where the authors have obtained the degree of approximationin $H_p^{(\omega)}, (p \geq 1)$space respectively for the partial sum and Rogosinski mean.

IV. We need following notations and lemmas for the proof of Theorem 1.

For $0 < \gamma < 1$,we write

$B(r,x,t) = -\sum_{n=1}^{\infty} 2\sin nt B_n(x) r^n$, (4.1)

$h(x,t) = B(r,x.t) - \Psi(x,t)$ (4.2)

$i(x,t,u) = \Psi(x,t+u) - \Psi(x,t-u) - 2\Psi(x,t)$ (4.3)

$j(x,t,u) = \Psi(x,t+u) - \Psi(x,t-u)$ (4.4)

$P(r,u) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos nu = \frac{1}{2}\left(\frac{1-r^2}{1-2rucosu+r^2}\right)$ (4.5)

$f(r,x) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n(x) r^n$ (4.6)

We need the following lemmas for the proof of Theorem 1.

Lemma 1: let$\omega(t)$ and$v(t)$ be defined as in Theorem 1

If$f \in H_p^{(\omega)}, p \geq 1$, then for $0 < u \leq \pi$

(i) $\|\Psi(.,t+u) - \Psi(.,t)\|_p = O(\omega(u))$

(ii) $\|\Psi(.,t) - \Psi(.+y,t)\|_p = O(\omega(|y|))$

(iii) $\|i(.,t,u)\|_p = O(\omega(u))$

(iv) $\|i(.,t,u) - i(.+y,t,u)\|_p = O(1)\begin{cases}\omega(u)\\ \omega(|y|)\end{cases}$

(v) $\|i(.,t,u) - i(.+y,t,u)\|_p = O(1)v(|y|)\frac{\omega(u)}{v(u)}$

(vi) $\|j(.,t,u)\|_p = O(\omega(u))$

(vii) $\|j(.,t,u) - j(.+y,t,u)\|_p = O(1)\begin{cases}\omega(u)\\ \omega(|y|)\end{cases}$

(viii) $\|j(.,t,u) - j(.+y,t,u)\|_p = O(1)v(|y|)\frac{\omega(u)}{v(u)}$

Proof: We have

$$\Psi(x.t+u) - \Psi(x,t) = f(x+t+u) - f(x-t-u) - f(x+t) + f(x-t)$$
$$= \{f(x+t+u) - f(x+t)\} - \{f(x-t-u) - f(x-t)\}$$

It follows from the Minkowski's inequality for $0 < u \leq \pi$

$$\|\Psi(.,t+u) - \Psi(.,t)\|_p \leq \|\Psi(.+t+u) - \Psi(.+t)\|_p + \|\Psi(.-t-u) - \Psi(.-t)\|_p$$
$$= O(\omega(u))$$

By using the fact that $f \in lip\ (\omega,p)$. It is easy to see that the result holds when $\Psi(x.t+u)$ is replaced by $\Psi(x.t-u)$.

For the proof of (ii) , writing

$$\Psi(x.t) - \Psi(x+y,t) = \{f(x+t) - f(x+y+t)\} - \{f(x-t) - f(x-t-y)\}$$

and proceeding as above , we have

$$\|\Psi(.,t) - \Psi(.+y,t)\|_p = O\big(\omega(|y|)\big).$$

Writing

$$i(x,t,u) = \{\Psi(x.t+u) - \Psi(x,t)\} + \{\Psi(x.t-u) - \Psi(x,t)\}$$

and applying Minkowski's inequality and part(i) we obtain (iii). First part of (iv) follows from (iii).

We write

$$i(x,t,u) - i(x+y,t,u) = \{\Psi(x,t+u) - \Psi(x+y,t+u)\}$$
$$+\{\Psi(x,t-u) - \Psi(x+y,t-u)\}$$
$$-2\{\Psi(x,t) - \Psi(x+y,t)\}$$

from which the second part of (iv) follows by applying Minkowski's inequality and (ii).

Now, by first part of (iv)

$$\|i(.,t,u) - i(.+y,t,u)\|_p = O(\omega(u))$$
$$= O\left(v(u)\frac{\omega(u)}{v(u)}\right) = O\left(v(|y|)\frac{\omega(u)}{v(u)}\right)$$

for$u \leq |y|$ as v is non-decreasing.

If $u \geq |y|$, then

$$\frac{\omega(u)}{v(u)} \geq \frac{\omega(|y|)}{v(|y|)}\left(as\ \frac{\omega(u)}{v(u)}\ is\ non-decreasing\right) \quad (4.7)$$

so that

$$\|i(.,t,u) - i(.+y,t,u)\|_p = O(\omega|y|)(by\ second\ part\ of\ (iv))$$

$= O\left(v(|y|)\frac{\omega(u)}{v(u)}\right)$ (using (4.7))

This completes the proof of (v). Proceeding as above the estimates given in (vi) ,(vii) and (viii) can be derived.

Lemma 2: Let $B(r,x,t)$ and $h(x,t)$be defined as in (4.1) and (4.2). Then under the hypothesis of Theorem 1 as $r \to 1^-$

(i) $\|h(.,t) - h(.+y,t)\|_p = O(1)v(|y|)(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^2}du$

(ii) $\|h(.,t)\|_p = O(1)(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{u^2}du$

(iii) $\|h(.,t) - h(.+y,t)\|_p^{(v)} = O(1)(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^2}du$

Proof: Now from (4.1)

$$B(r,x,t) = -\sum_{n=1}^{\infty} 2\sin nt B_n(x)r^n, 0 < r < 1$$

It can easily verify that

$$B(r,x,t) = \sum_{n=1}^{\infty} A_n(x+t) - \sum_{n=1}^{\infty} A_n(x-t)$$

which by ((7)p.96) equal to

$$f(r,x+t) - f(r,x-t)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x+t+u) - f(x-t+u)\}P(r,u)du$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\{\Psi(x,t+u) - \Psi(x,t-u)\}P(r,u)du \quad (4.8)$$

where the Poisson's kernel P is defined in (4.5). As

$$\int_0^{\pi} P(r,u)\,du = \pi/2$$

from (4.2), we get

$$h(x,t) = B(r,x.t) - \Psi(x,t)$$

$$= \frac{1}{\pi}\int_0^{\pi}\{\Psi(x,t+u) + \Psi(x,t-u) - 2\Psi(x,t)\}P(r,u)du$$

$$= \frac{1}{\pi}\int_0^{\pi} i(x,t,u)P(r,u)\,du \quad (4.9)$$

from which it follows that

$$h(x,t) - h(x+y,t) = \frac{1}{\pi}\int_0^{\pi}\{i(x,t,u) - i(x+y,t,u))\}P(r,u)du.$$

Applying generalised Minkowski's inequality, we get for $p \geq 1$

$$\pi\|h(.,t) - h(.+y,t)\|_p \leq \int_0^{\pi}\|i(.,t,u) - i(.+y,t,u)\|_p P(r,u)du$$

$$= \left(\int_0^{1-r} + \int_{1-r}^{\pi}\right)\|i(.,t,u) - i(.+y,t,u)\|_p P(r,u)du \quad (4.10)$$

We know ([7], Vol.I,p.96) that for $0 < u \leq \pi$ and $0 < r < 1$

$$P(r,u) = O\left(\frac{1}{1-r}\right) \quad (4.11)$$

$$P(r,u) = O\left(\frac{1-r}{u^2}\right) \quad (4.12)$$

Now applying Lemma 1(v) and the estimates (4.11)and (4.12) respectively for the integrals $\int_0^{1-r}$ *and* $\int_{1-r}^{\pi}$ *in* (4.10), we obtain

$$\pi\|h(.,t) - h(.+y,t)\|_p$$

$$= O(1)v(|y|)\left[\frac{1}{1-r}\int_0^{1-r}\frac{w(u)}{v(u)}du + (1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^2}du\right] \quad (4.13)$$

As $\frac{w(u)}{v(u)}$ is monotonic non -decreasing , we have

$$\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^2}du \geq \frac{w(1-r)}{v(1-r)}\int_{1-r}^{\pi}\frac{du}{u^2} \geq \frac{k}{1-r}\frac{\omega(1-r)}{v(1-r)} \quad (4.14)$$

where k is some positive constant and hence Lemma2(i) follows from(4.13)and (4.14).

As $f \in Lip\,(\omega,p), p \geq 1$, proceeding as above Lemma2(ii)can be proved. Lemma2(iii) is an immediate consequence of (i) and (ii) of Lemma2 as by definition.

$$\|h(.,t) - h(.+y,t)\|_p^{(v)} = \|h(.,t)\|_p + \sup_{y \neq 0}\frac{\|h(.,t) - h(.+y,t)\|_p}{v(|y|)}$$

Lemma 3: Let $B(r,x,t)$ be defined as in (4.1). Then under the hypothesis of Theorem 1, we have as $r \to 1^-$for $p \geq 1$

(i) $\left\|\frac{\partial}{\partial t}(B(r,.,t)-B(r,.+y,t))\right\|_p = O(1)v(|y|)(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^3}du$

(ii) $\left\|\frac{\partial}{\partial t}(B(r,.,t)\right\|_p = O\ (1)(1-r)\ \int_{1-r}^{\pi}\frac{\omega(u)}{u^3}du$

(iii) $\left\|\frac{\partial}{\partial t}(B(r,.,t)-B(r,.+y,t)\right\|_p^{(v)} = O(1)(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^3}du$

Proof: Proceeding as in the proof of Lemma 2 and using the integral given in (4.6) for $f(r,x)$, we get from (4.8)
that

$$B(r,x,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\{P(r,u-x-t)-P(r,u-x+t)\}du$$

which on partial differentiation with respect to t gives , after simplification ,

$$\frac{\partial}{\partial t}B(r,x,t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\{-P'(r,u-x-t)-P'(r,u-x+t)\}du$$
$$= -\frac{1}{\pi}\int_0^{\pi} j(x,t,u)P'(r,u)du$$

where$j(x,t,u)$ has been defined in (4.4)

Therefore

$$\frac{\partial}{\partial t}\{B(r,x,t)-B(r,x+y,t)\} = -\frac{1}{\pi}\int_0^{\pi}\big(j(x,t,u)-j(x+y,t,u)\big)P^{'}(r,u)du$$

Applying generalized Minkowski's inequality, we have for $p \geq 1$

$$\left\|\frac{\partial}{\partial t}\{B(r,.,t)-B(r,.+y,t)\}\right\|_p \leq \int_0^{\pi}\|j(.,t,u)-j(.+y,t,u)\|_p|P^{'}(r,u)|du$$
$$= \left[\int_0^{\pi}+\int_{1-r}^{\pi}\right]\|j(.,t,u)-j.+y,t,u\|_p|P^{'}(r,u)|du \quad (4.15)$$

We know ([6],p.31)that for $0 < u \leq \pi\ and\ 0 < r < 1$

$$P^{'}(r,u) = \frac{-(1-r^2)rsinu}{(1-2rcos\ u+r^2)^2} \quad (4.16)$$

$$P^{'}(r,u) = O\left(\frac{u}{(1-r)^3}\right)$$

(4.17)

$$P^{'}(r,u) = O\left(\frac{1-r}{u^3}\right)$$

(4.18)

Applying Lemma 1(viii) and the estimates (4.17) and (4.18) respectively for the integrals $\int_0^{1-r}$ *and* $\int_{1-r}^{\pi}$ *in* (4.15)we obtain

$$\left\|\frac{\partial}{\partial t}\{B(r,.,t)-B(r,.+y,t)\}\right\|_p = O(1)v(|y|)\left[\frac{1}{(1-r)^3}\int_0^{1-r}\frac{u\omega(u)}{v(u)}du+(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^3}du\right]$$
$$= O(1)v(|y|)\left[\frac{\omega(1-r)}{(1-r)v(1-r)}+(1-r)\int_{1-r}^{\pi}\frac{\omega(u)du}{v(u)u^3}\right]$$
$$= O(1)v(|y|)\left[(1-r)\int_{1-r}^{\pi}\frac{\omega(u)du}{v(u)u^3}\right],$$

as by the monotonicity of $\frac{\omega(u)}{v(u)}$

$$\int_{1-r}^{\pi}\frac{\omega(u)du}{v(u)u^3} \geq \frac{K}{(1-r)^2}\frac{\omega(1-r)}{v(1-r)}$$

where K is some positive constant and this completes the proof of (i). As by the hypothesis $f \in Lip(\omega,p)$, $p \geq 1$ proceeding as above part(ii) can be proved.

By definition

$$\left\|\frac{\partial}{\partial t}B(r,.,t)\right\|_p^{(v)} = \left\|\frac{\partial}{\partial t}B(r,.,t)\right\|_p + \sup_{y \neq 0}\frac{\left\|\frac{\partial}{\partial t}B(r,\ .\ ,\ t)-B(r\ ,\ .+y,\ t)\right\|_p}{v(|y|)}$$

and hence part (iii) follows from (i) and (ii).

Remark: In what follows , in the proof of Theorem 1 we do not need part (iii) of Lemma 2 . However , part (iii) has been included for the sake of completeness of Lemma 2 . Similar remarks also applies to part (iii) of Lemma 3.

V. Proof of Theorem

From (4.1), we have for $0 < \gamma < 1$

$$\frac{B(r,x,t)}{t^{1+\gamma}} = -2\sum_{n=1}^{\infty}\frac{\sin nt}{t^{1+\gamma}}B_n(x)r^n \quad ,t \neq 0$$

The above serieson the right is uniformly convergent for $t > \epsilon > 0$.Hence integrating term by term over the interval $(\epsilon,T)and$ observing that (see[6] p.31)

$$\left|\int_0^{\epsilon} \frac{\sin nt}{t^{1+\gamma}} dt\right| < C\, n\epsilon^{1+\gamma}, \left|\int_T^{\infty} \frac{\sin nt}{t^{1+\gamma}} dt\right| < \frac{C}{T^{1+\gamma}}$$

(where the constant C depending on γ $only$,we obtain by integrating the series (5.1))

$$\int_0^{\infty} \frac{B(r,x,t)}{t^{1+\gamma}} dt = -2\sum_{n=1}^{\infty} B_n(x) r^n \int_0^{\infty} \frac{\sin nt}{t^{1+\gamma}}\, dt \qquad (5.2)$$

It is known ([6], p.31) that

$$\int_0^{\infty} \frac{\sin nt}{t^{1+\gamma}}\, dt = \frac{n^{\gamma}\pi}{\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}\ , 0 < \gamma < 1$$

(5.3)

Using (5.3) in (5.2), we get after simple computation ,

$$\frac{-\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}{\pi} \int_0^{\infty} \frac{\Psi(x,t)}{t^{1+\gamma}} dt = \sum_{n=1}^{\infty} n^{\gamma} B_n(x) r^n = U_{\gamma}(r,x)$$

which ensures that

$$\frac{-\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}{\pi} \int_{1-r}^{\infty} \frac{\Psi(x,t)}{t^{1+\gamma}} dt - U_{\gamma}(r,x)$$
$$= \frac{\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}{\pi} \left\{\int_0^{1-r} \frac{B(r,x,t)}{t^{1+\gamma}} dt + \int_{1-r}^{\infty} \frac{B(r,x,t)-\Psi(x,t)}{t^{1+\gamma}} dt\right\}$$

By the notations of (2.1),(4.2)the above result takes the following form.

$$I_{\gamma}(x;1-r) - U_{\gamma}(r,x) = \frac{\Gamma(\gamma+1)\cos\frac{\pi\gamma}{2}}{\pi} D(r,x)$$

(5.4)

where

$$D(r,x) = \int_{1-r}^{\infty} \frac{h(x,t)}{t^{1+\gamma}} dt + \int_0^{1-r} \frac{B(r,x,t)}{t^{1+\gamma}} dt\ ,0 < \gamma < 1 \qquad (5.5)$$

For the proof of Theorem 1, it is sufficient to show that as $r \to 1^-$

$$\|D(r,.) - D(r,.+y)\|_p^{(v)} = O(1)(1-r)^{1-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du \qquad (5.6)$$

Now

$$D(r,x) - D(r,x+y) = \int_0^{1-r} \frac{B(r,x,t)-B(r,x+y,t)}{t^{1+\gamma}} dt + \int_{1-r}^{\pi} \frac{h(x,t)-h(x+y,t)}{t^{1+\gamma}} dt$$

By generalised Minkowski'sinequality , we get for $p \geq 1$

$$\|D(r,.) - D(r,.+y)\|_p \leq \int_0^{1-r} \frac{\|B(r,\ .\ ,t)-B(r,\ .+y,t)\|_p}{t^{1+\gamma}} dt$$
$$+ \int_{1-r}^{\pi} \frac{\|h(.\ ,t)-h(.+y,t\|_p}{t^{1+\gamma}} dt$$
$$= I_1 + I_2 \text{ , say}$$

(5.7)

By Lemma2(i)

$$I_2 = O(1)v(|y|)(1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du \int_{1-r}^{\infty} \frac{dt}{t^{1+\gamma}}$$
$$= O(1)v(|y|)(1-r)^{1-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du \qquad (5.8)$$

As $B(r,x,0) = 0 = B(r,x+y,0), by\ Mean\ value\ Theorem$ we have for some θ $with$ $0 < \theta < 1$.

$$B(r,x,t) - B(r,x+y,t) = t\left[\frac{\partial}{\partial t}\{B\ (r,x,t) - B(r,x+y,t)\}\right]_{t=\theta}$$

Hence by Lemma3

$$\|\{B\ (r,.,t) - B(r,.+y,t)\}\|_p = t\left\|\frac{\partial}{\partial t}\{B\ (r,.,t) - B(r,.+y,t)\}\right\|_p t = \theta$$
$$= O(1)tv(|y|)(1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^3} du$$

(5.9)

Using (5.9), we get

$$I_1 = \int_0^{1-r} \frac{\|B(r,\ .\ ,t)-B(r,\ .+y,t)\|_p}{t^{1+\gamma}} dt$$
$$= O(1)v(|y|)(1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^3} du \int_0^{1-r} \frac{dt}{t^{1+\gamma}}$$
$$= O(1)v(|y|)(1-r)^{2-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^3} du \qquad (5.10)$$

Collecting the results from (5.7),(5.8) and (5.10), we obtain

$$\|D(r,.) - D(r,.+y)\|_p = O(1)v(|y|)(1-r)^{1-\gamma} \left[\int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du + (1-r)\int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^3} du\right]$$
$$= O(1)v(|y|)(1-r)^{1-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du$$

(5.11)

from which it follows that

$$\sup_{y \neq 0} \frac{\|D(r,\ .)-D(r,\ .+y)\|_p}{v(|y|)} = O(1(1-r)^{1-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du)$$

(5.12)

As $f \in Lip(\omega, p), p \geq 1$ by the hypothesis, proceeding as above it can be proved that

$$\|D(r,.)\|_p = O(1)(1-r)^{1-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{u^2} du$$

(5.13)

From (5.12) and (5.13), it follows that

$$\|D(r,.) - D(r,.+y)\|_p^{(v)} = O(1)(1-r)^{1-\gamma} \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du$$

and this completes the proof of Theorem 1.

VI. Additional notations and Lemmas for the proof of Theorem 2

We know ([7], Vol.I,p.50) that

$$\emptyset^*(x,t) \sim 2 \sum_{n=1}^{\infty} \cos nt\, A_n(x)$$

(6.1)

$Let\ A(r,x,t)$denote the Abel mean of the series

$$- 2 \sum_{n=1}^{\infty} (1-\cos nt)\, A_n(x)$$

(6.2)

that is ;

$$A(r,x,t) = -2 \sum_{n=1}^{\infty} (1-\cos nt)\, A_n(x)\, r^n, 0 < r < 1$$

(6.3)

We write

$$H(x,t) = A(r,x,t) - \emptyset(x,t)$$

(6.4)

$$I(x,t,u) = \emptyset(x,t+u) + \emptyset(x,t-u) - 2\emptyset(x,u) - 2\emptyset(x,t) \qquad (6.5)$$

$$J(x,t,u) = \emptyset(x,t+u) - \emptyset(x,t-u) \qquad (6.6)$$

$$\Delta(r,x) = \int_{1-r}^{\infty} \frac{H(x,t)}{t^{1+\gamma}} dt + \int_0^{1-r} \frac{A(r,x,t)}{t^{1+\gamma}} dt, 0 < \gamma < 1 \qquad (6.7)$$

We need the following Lemmas for the proof of Theorem2.

Lemma 4 Let$\omega(t)$ and $v(t)$ be defined as in Theorem 2 . If $f \in H_p^{(\omega)}, p \geq 1$ then for 0< $u \leq \pi$

(i) $\|\emptyset(.,y)\|_p = O(\omega(u))$

(ii) $\|\emptyset(.,t+u) - \emptyset(.,t)\|_p = O(\omega(u))$

(iii) $\|\emptyset(.,t) - \emptyset(.+y,t)\|_p = O(\omega(|y|))$

(iv) $\|I(.,t,u)\|_p = O(\omega(u))$

(v) $\|I(.,t,u) - I(.+y,t,u)\|_p = O\begin{cases}\omega(u)\\ \omega(|y|)\end{cases}$

(vi) $\|I(.,t,u) - I(.+y,t,u)\|_p = O(1)v(|y|)\frac{\omega(u)}{v(u)}$

(vii) $\|J(.,t,u)\|_p = O(\omega(u)$

(viii) $\|J(.,t,u) - J(.+y,t,u)\|_p = O(1)\begin{cases}\omega(u)\\ \omega(|y|)\\ v(|y|)\frac{\omega(u)}{v(u)}\end{cases}$

We omit the proof of Lemma 4 as its proof is similar to Lemma1.

Lemma 5 Let A(r,x,t) and $H(x,t)$ be defined as in section 3. Let $\omega(t)$ and $v(t)$ be defined as in Theorem 2. If $f \in H_p^{(\omega)}, p \geq 1 then\ r \to 1^-$

(i) $\|H(.,t) - H(.+y,t)\|_p = O(1)v(|y|)(1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du$

(ii) $\|H(.,t)\|_p = O(1)(1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{u^2} du$

(iii) $\|H(.,t) - H(.+y,t)\|_p^{(v)} = O(1)(1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du$

Proof : We have

$$\begin{aligned} A(r,x,t) &= -2 \sum_{n=1}^{\infty} (1-\cos nt)\, A_n(x)\, r^n \\ &= \sum_{n=1}^{\infty} [A_n(x+t) + A_n(x-t) - A_n(x)]\, r^n \\ &= f(r,x+t) + f(r,x-t) - f(r,x) \end{aligned}$$

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$$=\frac{1}{\pi}\left[\int_{-\pi}^{\pi}\{f(x+t+u)+f(x-t+u)-2f(x+u)\}P(r,u)du\right]$$

(6.8)

using (4.6), adopting the argument used in deriving the integral representation of $B(r,x,t)$ in section 3,

It can be shown that

$$A(r,x,t)=\frac{1}{\pi}\left[\int_{0}^{\pi}\{\emptyset(x,t+u)+\emptyset(x,t-u)-2\emptyset(x,u)\}P(r,u)du\right]$$

(6.9)

$$A(r,x,t)-\emptyset(x,t)=\frac{1}{\pi}\int_{0}^{\pi}I(x,t,u)P(r,u)du$$

(6.10)

The remaining part of the proof of the Lemma is similar to the proof of Lemma2.

Lemma 6 Let $A(r,x,t)$ be defined as in section 5. Let $\omega(t)$ and $v(t)$ be defined as in Theorem 2.

If $f\in H_p^{(\omega)}, p\geq 1\ then\ r\to 1^-$

(i) $\left\|\frac{\partial}{\partial t}\{A(r,.,t)-A(r,.+y,t)\}\right\|_p = O(1)v(|y|)(1-r)\ \int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^3}du$

(ii) $\left\|\frac{\partial}{\partial t}\ A(r,.,t)\right\|_p = O(1)(1-r)\ \int_{1-r}^{\pi}\frac{\omega(u)}{u^3}du$

(iii) $\left\|\frac{\partial}{\partial t}\{A(r,.,t)-A(r,.+y,t)\}\right\|_p^{(v)} = O(1)(1-r)\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^3}du$

Proof: We know that

$$A(r,x,t)=f(r,x+t)+f(r,x-t)-2f(r,x).$$

Using (4.6) we get after simple computation

$$\begin{aligned}\frac{\partial}{\partial t}A(r,x,t) &= \frac{\partial}{\partial t}\left[\frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\{P(r,u-x-t)+P(r,u-x+t)-2P(r,u-x)\}du\right]\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(u)\{-P'(r,u-x-t)+P'(r,u-x+t)\}du\\ &= -\frac{1}{\pi}\int_{0}^{\pi}[\emptyset(x,t+u)-\emptyset(x,t-u)]P'(r,u)du\\ &= -\frac{1}{\pi}\int_{0}^{\pi}J(x,t,u)P'(r,u)du\end{aligned}$$

which has similarity with j(x,t,u) replaced by J(x,t,u) replaced by $J(x,t,u)$the integral representation of $\frac{\partial}{\partial t}B(r,x,t)$ in the proof of Lemma 3 and hence proceeding as in the proof of Lemma 3the results of Lemma 6 can be established.

Proof of Theorem 2

Proceeding as in Theorem 1 and making use of the fact that ([6],p.32)

$$\int_0^{\infty}\frac{sin^2\frac{nt}{2}}{t^{1+\gamma}}=\frac{n^{\gamma}\pi}{4\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}}, 0<\gamma<1.$$

We have

$$\frac{-\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}}{\pi}\int_0^{\infty}\frac{A(r,x,t)}{t^{1+\gamma}}dt=\sum_{n=1}^{\infty}n^{\gamma}A_n(x)\,r^n=V_{\gamma}(r,x)$$

which further ensures that(using the notation given in(2.2), (6.4)and (6.7))

$$J_{\gamma}(x,1-r)-V_{\gamma}(r,x)=\frac{\Gamma(\gamma+1)\sin\frac{\pi\gamma}{2}}{\pi}\Delta(r,x) \tag{7.1}$$

For the proof of Theorem 2, it is enough to show that as$r\to 1^-$

$$\|\Delta(r,.)-\Delta(r,.+y)\|_p^{(v)}=O(1)(1-r)^{1-\gamma}\int_{1-r}^{\pi}\frac{\omega(u)}{v(u)u^2}du \tag{7.2}$$

Using Lemma5 and Lemma6 (in place of Lemma2 and Lemma3) and adopting the technique used in proving(5.6)

(see proof of Theorem 1) the validity of (7.2) can be established and this completes the proof of Theorem 2

VII. Corollaries

By taking $\omega(t)=t^{\alpha} and\ v(t)=t^{\beta}, 0\leq\beta<\alpha\leq 1$, in Theorem 1, we obtain the following corollaries:

Corollary 1

If $f\in H(\alpha,p), p\geq 1, 0\leq\beta<\alpha\leq 1, and\ 0<\gamma<1$ then

$$\left\|I_{\gamma}(.;1-r)-U_n(r\,;\,.)\right\|_{(\beta,p)}=O(1)\begin{cases}(1-r)^{\alpha-(\beta+\gamma)}, & 0\leq\beta<\beta+\gamma\leq\alpha<1\\ (1-r)^{1-(\beta+\gamma)}, & 0<\beta<\beta+\gamma\leq\alpha=1\\ (1-r)^{1-\gamma}\ log\frac{1}{1-r}\ , & \beta=0,\alpha=1, 0<\gamma<1\end{cases}$$

Putting $p = \infty$,in Corollary 1 , we get

Corollary 2
If $f \in H_\alpha$,$0 \leq \beta < \alpha \leq 1$, and $0 < \gamma < 1$ then

$$\left| I_\gamma(.\,;1-r) - U_n(r\,;.) \right|_\beta = O(1) \begin{cases} (1-r)^{\alpha-(\beta+\gamma)}, & 0 \leq \beta < \beta+\gamma \leq \alpha < 1 \\ (1-r)^{1-(\beta+\gamma)}, & 0 < \beta < \beta+\gamma \leq \alpha = 1 \\ (1-r)^{1-\gamma} log\frac{1}{1-r} & ,\beta = 0, \alpha = 1, 0 < \gamma < 1 \end{cases}$$

Corollary 3 If $f \in Lip\alpha, 0 < \alpha \leq 1$ and $0 < \gamma < 1$ then

$$\left\| I_\gamma(.\,;1-r) - U_n(r\,;\,.) \right\|_c = O(1) \begin{cases} (1-r)^{\alpha-\gamma}, & 0 < \gamma \leq \alpha < 1 \\ (1-r)^{1-\gamma} log\frac{1}{1-r}, & \beta = 0, \alpha = 1,\ 0 < \gamma < 1 \end{cases}$$

In the case $\alpha = \gamma$, $0 < \gamma < 1$, Corollary 3 reduces to the second part of Theorem A due to Salem and Zygmund.

Analogous results can be obtained for Theorem 2.

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