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# Post – Newtonian Approximation and Maclaurin Spheroids

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***Abstract:*** *There is a natural relationship to the post-Newtonian expansion scheme that is used to describe sources of gravity which are not too far from Newtonian, i.e, not too relativistic. Here the expantion can lead to the solutions for the exterior fields of such sources. Till now the post-Newtonian expansion known to high order, in some cases 8th order. This paper has provided a high order expansion of the vacuum axisymmetric equations and also shows how some of our expanded terms relate to the known post-Newtonian ones for the particular case of homogeneous rotating bodies (generalized Maclaurin spheroids).*
***Keywords:*** ***P****ost-Newtonian approximation, Maclaurin spheroids.*

## I. Introduction

We have mentioned in 'Paper-III' [1], Chandrasekhar, in collaboration with colleagues and on his own, has done extensive work on homogeneous rotating bodies, both in Newtonian gravitation and general relativity [2 – 8]. Bardeen carried out a 're-examination' of Chandrasekhar's work on Maclaurin spheroids, which the latter referred to as 'penetrating'[9, 10]. The earlier paper (Paper-III) [1], the present Paper and forthcoming paper may be considered as complementary to the work of Chandrasekhar and others, at a very modest level. In this paper (hereinafter referred to as "Paper IV"), we have extended the problem in another direction [10].

We begin with some introductory remarks on post-Newtonian approximations, taken from the papers by Chandrasekhar and by Bardeen with minor changes in notation [6,9,10]. In the Newtonian limit Eulerian equations of hydrodynamics studied by Chandrasekhar are given by [6,10]:

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial}{\partial x^j}(\rho_0 v^j) = 0, \tag{1a}$$

$$\frac{\partial}{\partial t}(\rho_0 v^i) + \frac{\partial}{\partial x^j}(\rho_0 v^i v^j) = -\frac{\partial p}{\partial x^i} + \rho_0 \frac{\partial U}{\partial x^i}\,. \tag{1b}$$

Here $x^i = (x^1, x^2, x^3) = (x, y, z)$ are cartesian co-ordinates, $\rho_0$ is the mass-density, $v^i = (v^1, v^2, v^3)$ are velocity components of the fluid in cartesian co-ordinates ( $\rho_0$ is distinct from the cylindrical polar co-ordinate $\rho$ used in Paper-III) [1, 10]). In general relativity $\rho_0$ is replaced by ε, the mass-energy density. The pressure and the Newtonian gravitational potential are given respectively by $p, U$, with $U$, $\rho_0$ satisfying Poisson's equations [6, 10]:

$$\nabla^2 U = -4\pi G \rho_0\,, \tag{1c}$$

$G$ being Newton's gravitational constant. In a suitable limit, Eqs. (1a,b,c) derive from the energy-momentum tensor given by

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p g^{\mu\nu}, \tag{2}$$

where ε is the mass-energy density, $u^\mu$ the contravariant four-velocity, $g^{\mu\nu}$ is the metric tensor. Chandrasekhar writes ε as follows :

$$\varepsilon = \rho_0 c^2 (1 + \Pi / c^2), \tag{3}$$

where $c$ is the velocity of light, $\rho_0 c^2$ is associated with material density $\rho_0$, while $\rho_0 \Pi$ is connected to the internal energy of the first and second laws of thermodynamics. Einstein's equations are :

$$R^{\mu\nu} = -(8\pi G c^{-4})(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu}), \tag{4}$$

$R^{\mu\nu}$ being the Ricci tensor, and $T$ the trace of $T^{\mu\nu}$ : $T = g_{\mu\nu} T^{\mu\nu} = T_\mu^{\ \mu}$. For the post-Newtonian approximations, through a standard procedure we can set

$$g_{00} = 1 - 2c^{-2}U + O(c^{-4})\ ;\ g^{00} = 1 + 2c^{-2}U + O(c^{-4})\,; g_{ij} = -(1 + 2c^{-2}U)\delta_{ij} + O(c^{-4})\ ;$$

$$g^{ij} = -(1-2c^{-2}U)\delta_{ij} + O(c^{-4}) \; ; \; g^{0j} = O(c^{-3}) \; ; \tag{5}$$

where indices *i*, *j* = 1,2,3 and Greek indices $\mu, \nu$ = 0,1,2,3. $\delta_{ij}$ is the kronecker delta, = 0 if $i \neq j$, = 1 if $i = j$ (no summation) [6]. Chandrasekhar carries out an extensive analysis and solves the equations in the post-Newtonian approximations [6,10,11]. Here we give a brief review of this work in order to provide the basis of Chandrasekhar's work on homogeneous rotating bodies [10].

In section II equations of the post-Newtonian approximation have been given. The post-Newtonian equations for a rotating star are given in section III, which display the behaviour of the solution in the post-Newtonian approximation. Finally the general relativistic field for a rotating homogeneous mass is given in section IV.

## II. Equations of the post-Newtonian Approximation

Chandrasekhar generalizes the Eulerian equations of hydrodynamics and the equation of continuity in a consistent manner to Einstein's field equations, expanded in an approximation to $O(c^{-2})$, the so-called post-Newtonian approximation. He also shows that the post-Newtonian equations allow integrals of motion that are analogous to the Newtonian integrals that express the conservation of mass, linear momentum, angular momentum and energy. These new conservation laws enable a consistent definition of mass, momentum and energy in the framework of the post-Newtonian theory. A suitable tensor form of the virial theorem is also shown to be valid [6,10].

The formalism of post-Newtonian theory can be used to show that phenomena occur in general relativity that are distinct to those expected from the Newtonian theory, such as the stability of spherical gaseous masses; these become unstable to spherically symmetric radial oscillations well before the Schwarzschild limit. Some of these results can also be derived exactly (see, e.g., Thorne and Campolattaro 1967, Islam 1970, in addition to earlier work of Chandrasekhar 1964, and others) [12–15]. An essential simplification occurs at the post-Newtonian level in that gravitational radiation, well known to require complicated mathematical and physical considerations, plays no role. This is particularly relevant to homogenous rotating bodies.

One of the important consequences of Einstein's equations (4) is that the covariant divergence of $T^{\mu\nu}$ vanishes [6,10]:

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{6}$$

Chandrasekhar considers this equation in detail in the post-Newtonian approximation and derives interesting conservation equations.

Even for this brief summary, it is necessary to introduce some notation. We set

$$g_{00} = 1 + h_{00}, g_{0i} = h_{0i}, g_{ij} = -\delta_{ij} + h_{ij} \; ; \tag{7a}$$

$$h_{00} = -2c^{-2}U + O(c^{-4}),$$
$$h_{0i} = O(c^{-3}), h_{ij} = -2c^{-2}U\delta_{ij} + O(c^{-4}). \tag{7b}$$

At the post-Newtonian approximation, it is adequate to raise or lower indices with the Minkowski tensor $\eta^{\mu\nu} = 0 = \eta_{\mu\nu}, \quad \mu \neq \nu, \eta^{00} = \eta_{00} = 1, \quad \eta_{ii} = \eta^{ii} = -1$ (no summation), $i$ = 1,2,3. The contravariant components of the four velocity

$$u^{\mu} = \frac{dx^{\mu}}{ds} \; ; \; ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu},$$

may be written as

$$u^0 = 1 + c^{-2}(\frac{1}{2}v^2 + U) + O(c^{-4}),$$

$$u^i = [1 + c^{-2}(\frac{1}{2}v^2 + U)]\, v^i c^{-1} + O(c^{-5}),$$

with $(x^0 = ct)$

$$v^i = \frac{dx^i}{dt}, \quad ds = cdt\,[1 - \{c^{-2}(\frac{1}{2}v^2 + U)\} + O(c^{-4})]. \tag{8}$$

These equations enable us to evaluate the energy momentum tensor (2) and the 'conservation equation' (6), at the post-Newtonian level. The following 'gauge condition' is useful :

$$\frac{1}{2}\frac{\partial h_k{}^k}{\partial x^0} - \frac{\partial h_0{}^k}{\partial x^k} = 0,\ h_k{}^k = \eta^{jk} h_{kj},$$

reflecting essentially the freedom to carry out a residual spatial transformation.

After some manipulations, Chandrasekhar derives from the Einstein equations (4) and the conservation equation (6), the following useful relations (with the use of (7 a,b), (8)) [6,10]:

$$R_{00} = -8\pi\, G c^{-4} (T_{00} - \frac{1}{2} g_{00} T), \tag{9a}$$

reduces to (there should be no confusion between Chandrasekhar's use of $\phi$ here and our earlier used of $\phi$ as an azimuthal angle (Paper-III) [1]) :

$$\nabla^2 (-\frac{1}{2} h_{00} - c^{-2} U + c^{-4} U^2) = -8\pi\, G\, c^{-4} \varepsilon\, \phi \tag{9b}$$

$$\phi = v^2 + U + \frac{1}{2}\Pi + \frac{3}{2}\frac{p}{\varepsilon}, \tag{9c}$$

with $\Pi$ given by (3). If we define a "potential" $\Phi$ by the Poisson equation

$$\nabla^2 \Phi = -4\pi\, G \rho_0 \phi, \tag{10a}$$

then (9b) has the solution

$$h_{00} = -2c^{-2} U + c^{-4} (2U^2 - 4\Phi) + O\,(c^{-6}). \tag{10b}$$

Another set of Einstein equations

$$R_{0i} = -8\pi G c^{-4} (T_{0i} - \frac{1}{2} T g_{0i}), \tag{11a}$$

has the solution :

$$h_{0\,i} = c^{-3} (4U_i - \frac{1}{2}\frac{\partial^2 \chi}{\partial t \partial x^i}), \tag{11b}$$

where $\chi, U_i$ are the 'superpotential' and another potential:

$$\nabla^2 \chi = -2U, \quad \nabla^2 U_i = -4\pi\, G \varepsilon v_i \tag{11c}$$

Equation (6) leads to the post-Newtonian form of the continuity equation :

$$\frac{\partial \sigma}{\partial t} + \frac{\partial (\sigma v^i)}{\partial x^i} + c^{-2} (\varepsilon \frac{\partial U}{\partial t} - \frac{\partial p}{\partial t}) = 0, \tag{12a}$$

$$\sigma = \varepsilon\, [1 + c^{-2} (v^2 + 2U + \Pi + \frac{p}{\varepsilon})]\,. \tag{12b}$$

Equation (12a) can be cast more into the form of a continuity equation as follows :

$$\frac{\partial \varepsilon^*}{\partial t} + \frac{\partial}{\partial x^i} (\varepsilon^* v^i) = 0, \quad \varepsilon^* = \varepsilon [1 + c^{-2} (\frac{1}{2} v^2 + 3U)]\,. \tag{12c}$$

(Since at the post-Newtonian approximation indices are raised and lowered by the Minkowski metric tensor, the difference between a covariant and contravariant spatial index involves a trivial change of sign ; we may ignore the difference).

Chandrasekhar defines certain auxiliary expressions. Define

$$U_{ijk} = G \int \varepsilon(x') \upsilon_i (x') \frac{(x_j - x'_j)\ (x_k - x'_k\ )}{|\, x - x'|^3} dx'. \tag{13}$$

(Chandrasekhar writes this as $U_{i\,;jk}$ but we omit the semicolon). One can show that

$$\frac{\partial^2 \chi}{\partial t \partial x_i} = U_i - U_{jij}\,. \tag{13a}$$

With the use of $U_{ijk}$, define $W_j$ as follows:

$$W_j(\boldsymbol{x}) = v_i \frac{\partial}{\partial x_i}(U_j - U_{kjk}). \tag{13b}$$

Chandrasekhar goes on to show that the following is a reasonable definition of the linear momentum of the system :

$$\pi_i = \sigma v_i + \frac{1}{2} c^{-2} \varepsilon (U_i - U_{jij}) + 4c^{-2} \varepsilon (v_i U - U_i). \tag{14}$$

Although the volume integral of the last term vanishes, it is needed to give an adequate definition of the angular momentum of the system, which is given as follows:

$$M_{ij} = \int_v (x_i \pi_j - x_j \pi_i) d\boldsymbol{x}, \tag{15}$$

with $\pi_i$ given by (14).

If in the above equations one confines to steady rotational motion, in the sense that the motion is unchanged in time, all the time derivatives vanish, and the velocity takes a circular (around an axis) cross- radial form. These attributes can be defined in the post-Newtonian situation as well ; Chandrasekhar makes extensive use of the formalism for his study of Maclaurin spheroids.

Chandrasekhar's discussion and treatment of hydrodynamical post-Newtonian equations may be considered as complementary to the discussion of Einstein's field equations by Einstein, Infeld and Hoffmann in which the motion of singularities in the field are derived [16]. Chandrasekhar makes the point that the hydrodynamical post-Newtonian equations are physically more transparent and closer to astrophysical situations such as stellar configurations. There is clearly a great deal of further work to be done in this direction. Chandrasekhar has laid firm foundations for this possible future work.

Chandrasekhar gives the following interesting expression for the energy per unit volume of fluid :

$$\mathcal{E} = (\sigma - \frac{1}{2}\varepsilon^*) v^2 + \varepsilon^* \Pi - \frac{1}{2}\varepsilon^* U^* + c^{-2} \varepsilon \Big( -\frac{1}{8} v^4 + \frac{1}{2} U^2 - \Pi U - \frac{1}{2} v^2 \Pi + \frac{5}{2} v^2 U - \frac{7}{4} v_i U_i - \frac{1}{4} v_i U_{jij} \Big), \tag{16}$$

where $U^* = U + c^{-2} Q,$ with $\nabla^2 Q = -4\pi G \varepsilon (\frac{1}{2} v^2 + 3U).$

The appearance of some of the terms, and the co-efficients, seem somewhat surprising. But as Chandrasekhar remarks, one is led to this expression in a direct manner from Einstein's field equations. These points to the completeness and beauty of Einstein's equations about which Chandrasekhar has often remarked [6,10].

## III. Post- Newtonian equations for a rotating star

We consider in this section an extension of the paper by Bardeen (1971) on post-Newtonian Maclaurin spheroids [9]. As mentioned, Chandrasekhar speaks highly of Bardeen's paper, which is primarily concerned with the rest mass, angular momentum and the binding energy, and various associated parameters, of an axisymmetric, uniformly rotating, incompressible perfect fluid star in the post- Newtonian approximation. Bardeen makes extensive use of numerical methods.

In this section, starting from some of Bardeen's equations, we will carry out an extension of the equations, partly to illustrate some of the techniques introduced earlier paper, as a 'comprehension exercise', as it were, which may add to the overall motivation of the paper, explained in some detail elsewhere. The main results of this section are, firstly, the reduction of two of Bardeen's equations to a single equation, and an approximate solution of this equation in the neighbourhood of the centre of the rotating body.

To conform to the notation of the paper, and at the same time retain some convenient aspects of Bardeen's formulation, we proceed as follows. Taking $c\neq 1$, and write the metric with different signature [1,10] :

$$ds^2 = e^{\mu}(d\rho^2 + dz^2) + \rho^2 e^{2\lambda}(d\phi - w dt)^2 - e^{2\nu} c^2 dt^2, \tag{17}$$

where Bardeen has $2\mu$ instead of $\mu$, the symbol $\varpi$ instead of $\rho$, and $\omega$ instead of $w$ (The $f$ used here, e.g. in eq. (49) ref. [1], is here written as $-c^2 e^{2\nu}, -\rho^2 f^{-1}$ written as $\rho^2 e^{2\lambda}$, etc.). The energy- momentum tensor is for a perfect fluid in uniform rotation, which has uniform density and is incompressible; the rest-mass energy density is $\varepsilon c^2$ (we use $\varepsilon$ instead of Bardeen's $\rho$), the latter being independent of position and discontinuous at the surface. The coordinate angular velocity $\Omega = \frac{d\phi}{dt}$ is constant, the shear being zero. Relative to a locally nonrotating observer the linear velocity of rotation is [10] ;

$$v = (\Omega - w)\,\rho\, e^{\lambda - \nu}. \tag{18}$$

In the post-Newtonian approximation Einstein equations are relatively simple, with $\lambda = \frac{1}{2}\mu = -\nu$, which are as follows :

$$\nabla^2 \nu = \alpha[1 + \frac{3p}{\varepsilon c^2} + 2\frac{v^2}{c^2} - 2\nu], \tag{19a}$$

$$\nabla.(\rho^2 \nabla w) = -4\alpha\,\Omega\,\rho^2; \alpha = \frac{4\pi G\varepsilon}{c^2}. \tag{19b}$$

For our required purpose we write here the Laplacian and gradient operators $\nabla^2, \boldsymbol{\nabla}$ in cylindrical polar coordinates are as follows [10, 17]:

$$\nabla = \left( \mathbf{i}^{(\rho)} \frac{\partial}{\partial \rho} + \mathbf{i}^{(\phi)} \rho^{-1} \frac{\partial}{\partial \phi} + \mathbf{k} \frac{\partial}{\partial z} \right), \tag{20a}$$

and

$$\nabla^2 = \left( \frac{\partial^2}{\partial \rho^2} + \rho^{-1} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right). \tag{20b}$$

Eq. (20b) being valid if the function on which $\nabla^2$ is acting is independent of $\phi$, which is a function of $\rho$ and *z* only.For the evaluation of the left hand side of (19b), we note that (from eq. (2.25) ref. [10]):

$$\frac{\partial\, \mathbf{i}^{(\rho)}}{\partial \phi} = -\mathbf{i} \sin\phi + \mathbf{j} \cos\phi = \mathbf{i}^{(\phi)}, \tag{21a}$$

$$\frac{\partial\, \mathbf{i}^{(\phi)}}{\partial \phi} = -\mathbf{i} \cos\phi - \mathbf{j} \sin\phi = -\mathbf{i}^{(\rho)}. \tag{21b}$$

Inserting the operator (20a) on the left hand side of (19b) and using (21a,b), we find, after some manipulation, that (19b) reduces to the following equation [10]:

$$w_{\rho\rho} + \frac{3}{\rho} w_\rho + w_{zz} = -4\alpha\Omega, \tag{22}$$

with $w_\rho = \frac{\partial w}{\partial \rho}$, etc., as before. Eqs. (17), (18), (19 a,b) are Bardeen's Eqs. (1), (2), (3), (4) respectively [9]. There is an error in Bardeen's Eq. (3) in that *p* is the written as *P*. The exact equation of hydrostatic equilibriam is

$$\left(1 + \frac{p}{\varepsilon c^2}\right) e^{\nu} \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = \text{constant}$$

$$= 1 - \gamma, \tag{23}$$

which is Bardeen's Eq. (5). We write this as follows:

$$\frac{p}{\varepsilon c^2} = (1-\gamma) e^{-\nu} \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1, \tag{23a}$$

and expand the factor $\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ to get the following equations :

$$\frac{p}{\varepsilon c^2} = (1-\gamma)e^{-\nu}\left(1+\frac{v^2}{2c^2}\right)-1. \tag{23b}$$

Substitution in (19a) yields the following equation :

$$\nabla^2\nu + 2\alpha\nu + 2\alpha = 3\alpha(1-\gamma)e^{-\nu}\left(1+\frac{v^2}{2c^2}\right)+2\alpha\frac{v^2}{c^2}. \tag{24}$$

From (18), we get (with $\lambda = -\nu$ ) :

$$v^2 = (\Omega - w)^2\rho^2 e^{-4\nu}. \tag{25}$$

Note that, because of the factor $\rho^2$, provided the other factors are finite, as $\rho \to 0, v^2 (or\ v)$ tends to zero i.e., on the axis of rotation. This is to be expected for otherwise there would be some form of singularity (perhaps a source or a sink) on the axis $\rho = 0$. Before substituting for $v^2$ from (25) into (24), we find an exact solution for *w* given by the equation (22). Indeed, it is readily verified that the following expression for *w* gives an exact solution for (22) :

$$w = w_0 + a\rho^2 - 2(2a + \alpha\,\Omega)z^2 \tag{26}$$

where *a*, $w_0$ are arbitrary constants. We now substitute for *w* given by (26) in (25), and insert the resulting $v^2$ into (24), to get the following equation for $\nu$ alone :

$$\nabla^2\nu + 2\alpha\nu + 2\alpha = 3\alpha(1-\gamma)e^{-\nu} + \alpha\rho^2c^{-2}\left[\frac{3}{2}(1-\gamma)c^{-\nu}+2\right]\left[\Omega - w_0 - a\rho^2 + (4a+2\alpha\Omega)z^2\right]^2 e^{-4\nu}. \tag{27}$$

Thus we have reduced the coupled equations (22), (24) for $\nu$ and *w* (via (18)) to a single equation (27) for $\nu$ alone. We can go a little further if we seek an approximate solution of (27) for $\nu$ in the neighbourhood of the origin $\rho = z = 0$, i.e., near the centre of the rotating spheroid. To this effect, we expand $\nu$ in a power series in $\rho^2, z^2$ (we omit odd powers, a justification being that this leads to a consistent solution), as follows :

$$\nu = \nu_0 + \nu_1\rho^2 + \nu_1' z^2 + \nu_2\rho^4 + \nu_2'\rho^2 z^2 + \nu_2'' z^4 + \dots, \tag{28a}$$

where $\nu_0, \nu_1,$ etc. are constant coefficients. For $e^{-\nu}, e^{-4\nu}$ we use the expansion

$$e^{-\nu} = e^{-\nu_0}(1 - \nu_1\rho^2 - \nu_1' z^2 + \dots), \tag{28b}$$

$$e^{-4\nu} = e^{-4\nu_0}(1 - 4\nu_1\rho^2 - 4\nu_1' z^2 + \dots). \tag{28c}$$

We will not need higher powers in (28 a,b). Substituting from (28 a,b,c) into (27) and equating the constant terms, and coefficients of $\rho^2, z^2$ respectively on both sides, we find :

$$4\nu_1 + 2\nu_1' + 2\alpha\nu_0 + 2\alpha = 3\alpha(1-\gamma)e^{-\nu_0}, \tag{29a}$$

$$16\nu_2 + 2\nu_2' + 2\alpha\nu_1 = -3\alpha(1-\gamma)e^{-\nu_0}\nu_1 + \alpha\left[\frac{3}{2}(1-\gamma)e^{-\nu_0}+2\right](\Omega - w_0)^2, \tag{29b}$$

$$4\nu_2' + 12\nu_2'' + 2\alpha\nu_1' = -3\alpha(1-\gamma)e^{-\nu_0}\nu_1'. \tag{29c}$$

Taking $\nu_0 \neq 0$ gives a more general solution, but there appears to be no inconsistency if $\nu_0 = 0$, which we take to be the case. Then (29 a,b,c) reduce to the following equations:

$$4\nu_1 + 2\nu_1' = \alpha(1-3\gamma), \tag{30a}$$

$$16\nu_2 + 2\nu_2' = -\alpha(5-3\gamma)\nu_1 + \frac{1}{2}\alpha(7-3\gamma)(\Omega - w_0)^2 \tag{30b}$$

$$4\nu_2' + 12\nu_2'' = -\alpha(5-3\gamma)\nu_1'. \tag{30c}$$

Multiplying (30b) by 2 and adding to (30c), and replacing in the resulting equation the combination $(2\nu_1 + \nu_1')$

$(=\frac{1}{2}\alpha(1-3\gamma))$ from (30a), we get a certain combination of the fourth order coefficients in (28a) in terms of the basic constants $\alpha, \gamma$ :

$$4(8\nu_2+\nu_2')+12\nu_2''=-\frac{1}{2}\alpha^2(5-3\gamma)(1-3\gamma)+\frac{1}{2}\alpha(7-3\gamma)(\Omega-w_0)^2. \qquad (31)$$

Eqs. (30a,b,c), (31) thus constitute a solution of $\nu$ given by (28a) to the order given, with some remaining arbitrary constants. The process can clearly be continued to arbitrary high orders in $\rho^2, z^2$, and yields a solution to Bardeen's equations (19a,b) which displays, in particular, the behaviour of the solution in the post-Newtonian approximation in the neighbourhood of the centre. As indicated, Bardeen's paper considers the problem in a somewhat different direction. The present section of the paper can be considered as an extension in another direction.

## IV. On the General Relativistic Field for a Rotating Homogeneous Mass.

As stated in Islam's book [17], it has not been possible to find an exact solution representing a homogeneous rotating body in general relativity, either interior or exterior. It is already mentioned that Chandrasekhar has done extensive work on the Maclaurin spheroids in the post-and post-post-Newtonian approximations, for the interior field. Finding an exact solution in this case, either interior or exterior is clearly a formidable problem, if indeed such a solution can be found in closed form. In this section of the paper we present one possible approach to this problem, which may be of some use for further work in this direction [10].

To begin with, we go back to the approximate solution eq. (21) (ref. [1]), and taking $\beta=0$, $\zeta=\sigma$, we have

$$h^{(1)}=\sigma,\ w^{(1)}=\rho\sigma_\rho. \qquad (32)$$

We will explain later the reasons for this choice. Similarly from eqs. (24), (25), (28), (29a,b), (31a,b) of ref. [1], we have

$$\nabla^2 h^{(2)}=\sigma_\rho^2+\sigma_z^2+\sigma_{\rho z}^2+\sigma_{zz}^2, \qquad (33a)$$

$$h^{(2)}=\frac{1}{2}\sigma^2+\frac{1}{2}\sigma_z^2. \qquad (33b)$$

$$\Delta w^{(2)}=2\rho(\sigma_z\sigma_{\rho z}-\sigma_\rho\sigma_{zz}), \qquad (34a)$$

$$w_\rho^{(2)}=\rho(\sigma_z^2-\sigma\sigma_{zz}), \qquad (34b)$$

$$w_z^{(2)}=\rho(\sigma\sigma_{\rho z}-\sigma_\rho\sigma_z). \qquad (34c)$$

$$\nabla^2 h^{(3)}=\sigma(\sigma_{\rho z}^2+\sigma_{zz}^2)+\sigma(\sigma_\rho^2+\sigma_z^2), \qquad (35a)$$

which has the partial solution [10] :

$$h^{(3)}=h'^{(3)}+\frac{1}{6}\sigma^3, \qquad (35b)$$

$$\nabla^2 h'^{(3)}=\sigma(\sigma_{\rho z}^2+\sigma_{zz}^2). \qquad (35c)$$

$$\Delta w^{(3)}=2\rho\sigma(\sigma_z\sigma_{\rho z}-\sigma_\rho\sigma_{zz}). \qquad (36)$$

It does not seem possible to solve (35c), (36) in terms of a general harmonic function $\sigma$ [10,18,19]. As indicated in this paper and the next, exact solutions to orders 4 and 5 can be found for specific choice of harmonic functions such as $\sigma=\frac{a}{r}$ [1,10]. However, if (35c), (36) can be solved if $\sigma$ represents the exterior Newtonian gravitational field of a homogeneous rotating fluid mass, this may be a step in the direction of finding a solution of this problem in general relativity. We proceed to take some steps in this direction.

he following demonstration that the exterior Newtonian gravitational field of a homogeneous rotating body satisfies Laplace's equation is very standard, but nevertheless, we believe it may help to provide some ingredients in the solution of the very difficult problem of the exterior general relativistic field of a homogeneous rotating body, either exact or approximate. These calculations have some similarities with the extensive but essentially incomplete calculations of this problem by Chandrasekhar. We proceed to these calculations[10].

Consider first a homogeneous ellipsoid whose equation is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, \tag{37a}$$

where *a*,*b*,*c* are the axes of the ellipsoid and $a > b > c$ (here *c* is not to be confused with the velocity of light). Then the Newtonian potential of the field at an arbitrary point outside (exterior) the body is given by the formula ( Landau and Lifshitz, 1975, Eq. (99.11)) [20]:

$$\phi = -\pi\mu\, abck \int_{\xi}^{\infty} \left(1-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}-\frac{z^2}{c^2+s}\right)\frac{ds}{R_s}, \tag{37b}$$

where $R_s = \sqrt{(a^2+s)(b^2+s)(c^2+s)}$ , $\mu$ is the mass density and $\xi$ is the positive root of the equation

$$\frac{x^2}{a^2+\xi}+\frac{y^2}{b^2+\xi}+\frac{z^2}{c^2+\xi} = 1. \tag{37c}$$

For an oblate ellipsoid of rotation putting $a = b$, $x^2+y^2=\rho^2$, we get from (37b)

$$\phi = -\pi\mu\, a^2ck \int_{\xi}^{\infty} \left(1-\frac{\rho^2}{a^2+s}-\frac{z^2}{c^2+s}\right)\frac{ds}{R_s};\ R_s = (a^2+s)\sqrt{c^2+s} \tag{37d}$$

From (37c), we get

$$\frac{\rho^2}{a^2+\xi}+\frac{z^2}{c^2+\xi} = 1. \tag{38}$$

Now we have to show that

$$\phi_{\rho\rho}+\phi_{zz}+\rho^{-1}\phi_{\rho} = 0 = V_{\rho\rho}+V_{zz}+\rho^{-1}V_{\rho},$$

where $\phi \equiv -\pi\mu\, a^2ckV$. (39)

For this purpose putting $f \equiv \left(1-\frac{\rho^2}{a^2+s}-\frac{z^2}{c^2+s}\right)\frac{1}{(a^2+s)\sqrt{c^2+s}}$

$$= f(\rho, z, s), \tag{40}$$

and comparing (37d) and (39), we find

$$V = \int_{\xi}^{\infty}\left(1-\frac{\rho^2}{a^2+s}-\frac{z^2}{c^2+s}\right)\frac{ds}{(a^2+s)\sqrt{c^2+s}}$$

$$= \int_{\xi}^{\infty} f(\rho, z, s)\, ds. \tag{41}$$

Here $V_{\rho} = \int_{\xi}^{\infty}\frac{\partial f}{\partial\rho}ds + (-f(\rho, z, \xi))\frac{\partial\xi}{\partial\rho}$

$$= \int_{\xi}^{\infty}\frac{\partial f}{\partial\rho}ds\ , \tag{42a}$$

by using the definition of $\xi$, that is,

$$f(\rho,z,\xi) = \left(1-\frac{\rho^2}{a^2+\xi}-\frac{z^2}{c^2+\xi}\right)\frac{1}{(a^2+\xi)\sqrt{c^2+\xi}}$$

$= 0.$ by using (38) (42b)

$$V_{\rho\rho} = \int_{\xi}^{\infty}\frac{\partial^2 f}{\partial\rho^2}ds+(-\frac{\partial}{\partial\rho}f(\rho,z,\xi))\frac{\partial\xi}{\partial\rho}. \qquad (42c)$$

Similarly $V_z = \int_{\xi}^{\infty}\frac{\partial f}{\partial z}ds$ , (42d)

$$V_{zz} = \int_{\xi}^{\infty}\frac{\partial^2 f}{\partial z^2}ds+(-\frac{\partial}{\partial z}f(\rho,z,\xi))\frac{\partial\xi}{\partial z}. \qquad (42e)$$

From (40), we have

$$\frac{\partial f}{\partial\rho}=-\frac{2\rho}{(a^2+s)^2\sqrt{c^2+s}};\quad \frac{\partial^2 f}{\partial\rho^2}=-\frac{2}{(a^2+s)^2\sqrt{c^2+s}};$$
$$\frac{\partial f}{\partial z}=-\frac{2z}{(a^2+s)(c^2+s)^{\frac{3}{2}}};\quad \frac{\partial^2 f}{\partial z^2}=-\frac{2}{(a^2+s)(c^2+s)^{\frac{3}{2}}}. \qquad (43)$$

Making use of (42b), (43) into (42a), (42c) and (42e), we have by simple simplifications

$$V_{\rho\rho}+V_{zz}+\rho^{-1}V_{\rho} = -\left[\int_{\xi}^{\infty}\frac{4ds}{(a^2+s)^2\sqrt{c^2+s}}+\int_{\xi}^{\infty}\frac{2ds}{(a^2+s)(c^2+s)^{\frac{3}{2}}}\right]$$
$$+\frac{2\rho}{(a^2+\xi)^2\sqrt{c^2+\xi}}\frac{\partial\xi}{\partial\rho}+\frac{2z}{(a^2+\xi)(c^2+\xi)^{\frac{3}{2}}}\frac{\partial\xi}{\partial z}. \qquad (44)$$

Again making use of (38) and by some manipulation we have from (44) :

$$\frac{2\rho}{(a^2+\xi)^2\sqrt{c^2+\xi}}\frac{\partial\xi}{\partial\rho}+\frac{2z}{(a^2+\xi)(c^2+\xi)^{\frac{3}{2}}}\frac{\partial\xi}{\partial z}$$
$$= \frac{4\rho^2(c^2+\xi)^{\frac{3}{2}}}{(a^2+\xi)[\rho^2(c^2+\xi)^2+z^2(a^2+\xi)^2]}+\frac{4z^2(a^2+\xi)}{\sqrt{c^2+\xi}\,[\rho^2(c^2+\xi)^2+z^2(a^2+\xi)^2]}$$
$$= \frac{4}{(a^2+\xi)\sqrt{c^2+\xi}}. \qquad (45)$$

Again let

$$I_1 = \int_{\xi}^{\infty}\frac{4ds}{(a^2+s)^2\sqrt{c^2+s}}, \qquad I_2 = \int_{\xi}^{\infty}\frac{2ds}{(a^2+s)(c^2+s)^{\frac{3}{2}}}. \qquad (46)$$

Putting $c^2+s=t^2$, $\therefore\ ds=2tdt$ .

When $s=\infty$, $t=\infty$, when $s=\xi$, $t=\sqrt{c^2+\xi}$.

$$\therefore\ I_1=\int_{\sqrt{c^2+\xi}}^{\infty}\frac{8dt}{\{(a^2-c^2)+t^2\}^2}.$$

Again putting $t=\sqrt{a^2-c^2}\,\tan\theta\ \therefore\ dt=\sqrt{a^2-c^2}\,\sec^2\theta\ d\theta$.

When $t=\infty$, $\theta=\frac{\pi}{2}$; when $t=\sqrt{c^2+\xi}$, $\theta=\tan^{-1}\sqrt{\frac{c^2+\xi}{a^2-c^2}}$.

$$\therefore\ I_1=\int_{\tan^{-1}\sqrt{\frac{c^2+\xi}{a^2-c^2}}}^{\frac{\pi}{2}}\frac{8(\sqrt{a^2-c^2})\sec^2\theta\ d\theta}{\{(a^2-c^2)(1+\tan^2\theta)\}^2}$$

This integral finally reduces to

$$I_1=\frac{4}{(a^2-c^2)^{\frac{3}{2}}}\left[\frac{\pi}{2}-\tan^{-1}\sqrt{\frac{c^2+\xi}{a^2-c^2}}-\frac{\sqrt{(c^2+\xi)(a^2-c^2)}}{(a^2+\xi)}\right]. \qquad (47a)$$

Similarly

$$I_2=\int_{\xi}^{\infty}\frac{2ds}{(a^2+s)(c^2+s)^{\frac{3}{2}}}$$

$$=\int_{\sqrt{c^2+\xi}}^{\infty}\frac{4dt}{\{(a^2-c^2)+t^2\}t^2}$$

$$=\int_{\sqrt{c^2+\xi}}^{\infty}\frac{4}{(a^2-c^2)}\left[\frac{1}{t^2}-\frac{1}{(a^2-c^2)+t^2}\right]dt$$

This integral finally leads to

$$I_2=\frac{4}{(a^2-c^2)}\left[\frac{1}{\sqrt{c^2+\xi}}-\frac{\pi}{2\sqrt{a^2-c^2}}+\frac{1}{\sqrt{a^2-c^2}}\tan^{-1}\sqrt{\frac{c^2+\xi}{a^2-c^2}}\right]. \qquad (47b)$$

Using foregoing eqs. (47a), (47b) and by some simplifications, we get

$$-(I_1+I_2)=-\frac{4}{(a^2+\xi)\sqrt{c^2+\xi}}. \qquad (48)$$

Again using aforesaid eqs. (45) and (48), we have the following :

$$V_{\rho\rho}+V_{zz}+\rho^{-1}V_{\rho}=\frac{4}{(a^2+\xi)\sqrt{c^2+\xi}}-\frac{4}{(a^2+\xi)\sqrt{c^2+\xi}}$$
$$=0. \qquad (49)$$

Substituting the value of (49) into (39), we have

$$\phi_{\rho\rho}+\phi_{zz}+\rho^{-1}\phi_{\rho}=0=V_{\rho\rho}+V_{zz}+\rho^{-1}V_{\rho}.$$

Hence the exterior Newtonian gravitational field of a homogeneous rotating body satisfies Laplace's equation.

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