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Elementary Considerations on Prime Numbers and On the Riemann Hypothesis

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Abstract: *The Riemann Hypothesis states the non trivial zeros of a mathematical function, the Riemann zeta function, are, all of them, points pertaining to a vertical line in the Argand-Gauss plane. Technically, this implies there exists an intrinsical order, a balance, within the structure of natural numbers, related to the building blocks, these the prime numbers. We prove the natural numbers are free to exist, i.e., the Riemann Hypothesis is false, which is the main purpose of this paper: to provide an elementary disproof of the Riemann Hypothesis. This version contains an appendix that should suffice for explanation on the cardinality of , , which is important here, emphasizing denumerable means countable [countably infinity], from which the scope of combinatorics is justified.*

Keywords: *Riemann hypothesis, prime number, number theory, disproof*

I. Combinatorics on Partial Sum of the Liouville Function up to a Prime Superior Quota, Limiting and Subsidiary Complements

The Liouville function [1] depending on the variable (in this paper,) is given by:

$$\lambda(n) = (-1)^{\omega(n)}, \tag{1}$$

where:

$$\omega(n) = \text{ Number of prime factors of } n, \tag{2}$$

Being these prime factors not necessarily distinct, counted with multiplicity. Hence, the image of . E.g.: , since 1 has not got any prime factors (); , since 2 has got just one prime factor (itself, since 2 is a prime number), from which ; , since, counted with multiplicity, 20 has got 3 prime factors (), from which . Of course, all the numbers , except the number 1, will have prime factors, being such quantity of prime factors, exactly , either odd or even. A prime number has got itself as its prime factor, hence , i.e., being prime: .

By virtue of these considerations, one may be interested in a generic factorization with an even number of prime factors and also be interested in a factorization with an odd number of prime numbers. The presence of 1 as a factor in a given factorization does not matter, as well as for any quantity of the number 1 as a factor, except for the unique case of the factorization of the number 1, this latter being trivial and unique, once a consideration of 1 factor(s) in a given natural number contributes nothing to , viz.:

$$\omega(n) = \omega(1 \times 1 \times \cdots \times 1 \times \cdots \times 1 \times n), \tag{3}$$

By the very elementary fact that we just count the quantity of prime factors of an . Also, once a factorization is unique, except for the order of the prime factors of a given natural number, a given even (or odd) factorization with (or), , factors turns out to be unique under a combinatorial (combination) consideration with repetition with (or) elements. As a matter of fact and clarification, here, we should start to put these assertions under a more mathematically generalized sound.

Let be the set of all the first prime numbers, so that is the set of the prime numbers. One may consider , , slots filled with any element, , from the set, e.g.:

$$\underbrace{\boxed{p_3} \times \boxed{p_5} \times \boxed{p_5} \times \boxed{p_2} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_8} \times \cdots \times \boxed{p_2}}_{(2k)-slots} \tag{4}$$

Of course, any permutation of (4) generates the very same configuration, generates the very same number due to the multiplication (between slots). One should infer that repetition is allowed, viz., the elements in (4) do not need to be different. However, for an instantaneously fixed order of (4), a change in a given slot element (choosing a different one from), with the elements within the remaining slots not changed, would lead to a new configuration for (4). A permutation of this latter new configuration does not change it. A given configuration represents a unique natural number with an even number of prime factors, since, in spite of permutation, its factorization (configuration) is unique. Similarly, one may consider , , slots filled with any element, , from the set, e.g.:

$$\underbrace{\boxed{p_3} \times \boxed{p_5} \times \boxed{p_5} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_8} \times \cdots \times \boxed{p_2}}_{(2k-1)-slots} \tag{5}$$

Of course, any permutation of (5) generates the very same configuration, generates the very same number due to the multiplication (between slots). One should infer that repetition is allowed, viz., the elements in (5) do not need to be different. However, for an instantaneously fixed order of (5), a change in a given slot element (choosing a different one from), with the elements within the remaining slots not changed, would lead to a new configuration for (5). A permutation of this latter new configuration does not change it. A given configuration represents a unique natural number with an odd number of prime factors, since, in spite of permutation, its factorization (configuration) is unique.

To consider the totality of numbers in , one needs to impose , hence , and to consider the totality of slots and to consider the totality of slots, these latter totalities by completely considering (viz.:).

Since a given configuration does not change with a permutation of its elements, we are dealing with a problem of combination [2] (the order does not matter). However, elements may be repeated, hence the combinatorics involved is neither nor . To uniquely represent a configuration, we may define a convention. In fact, considering an hypothetical factorization with factors, e.g., as represented below:

$$\underbrace{\boxed{p_N} \times \boxed{p_5} \times \boxed{p_1} \times \boxed{p_2} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_8} \times \cdots \times \boxed{p_2}}_{(q)-slots}, \tag{6}$$

it may be rearranged respecting the order of the indexes:

$$\underbrace{\boxed{p_1} \times \boxed{p_2} \times \boxed{p_2} \times \boxed{p_3} \times \boxed{p_3} \times \boxed{p_5} \times \boxed{p_8} \times \cdots \times \boxed{p_N}}_{(q)-slots}, \tag{7}$$

which is the very same factorization preserving its same elements, now written in increasing order of indexes, which, to uniquely be represented, may have each index increased by an amount exactly equal to the quantity of previous elements (which is a unique characteristic per element), also changing the label to , viz.:

$$\underbrace{\boxed{x_1} \times \boxed{x_3} \times \boxed{x_4} \times \boxed{x_6} \times \boxed{x_7} \times \boxed{x_{10}} \times \boxed{x_{14}} \times \cdots \times \boxed{x_{N+q-1}}}_{(q)-slots} \tag{8}$$

With this convention, each -combination turns out to have got all of its elements with different indexes, with the maximum index occurring when a -combination has got its last slot occupied by , from which we turn out to have a problem of simple -combination of elements. Hence, the quantity of unique factorizations having prime, with multiplicity allowed, factors taken from is:

$$Q_q^N = C_{N+q-1,\,q} = \frac{(N+q-1)\,!}{(N-1)\,!\,q\,!}, \tag{9}$$

And the quantity of unique factorizations having prime factors turns out to be:

$$Q_q^{\mathbb{N}} = \lim_{N\to\infty} C_{N+q-1,\,q} = \lim_{N\to\infty} \frac{(N+q-1)\,!}{(N-1)\,!\,q\,!} = |\mathbb{N}_q|\,, \tag{10}$$

the totality of distinct numbers belonging to and having got prime, with multiplicity allowed, factors. The totality of numbers belonging to turns out to be:

$$\begin{aligned} Q^{\mathbb{N}} &= 1 + \sum_{q=1}^{\infty} Q_q^{\mathbb{N}} \\ &= 1 + \sum_{q=1}^{\infty} \lim_{N\to\infty} C_{N+q-1,\,q} \\ &= 1 + \sum_{q=1}^{\infty} \lim_{N\to\infty} \frac{(N+q-1)\,!}{(N-1)\,!\,q\,!} \\ &= 1 + \sum_{q=1}^{\infty} |\mathbb{N}_q| = |\mathbb{N}| = \aleph_0 . \end{aligned} \tag{11}$$

We will be interested in the infinite sum:

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda(n) &= \lambda(1) + \sum_{n=2}^{\infty} \lambda(n) \\ &= 1 + \sum_{n=2}^{\infty} \lambda(n) \\ &\therefore \end{aligned}$$

$$\sum_{n=1}^{\infty} \lambda(n) = 1 + \sum_{k=1}^{\infty} Q_{2k}^{\mathbb{N}} - \sum_{k=1}^{\infty} Q_{2k-1}^{\mathbb{N}}, \tag{12}$$

where:

$$\begin{aligned} Q_{2k}^{\mathbb{N}} &= \lim_{N\to\infty} C_{N+2k-1,\,2k} \\ &= \lim_{N\to\infty} \frac{(N+2k-1)\,!}{(N-1)\,!\,(2k)\,!}, \end{aligned} \tag{13}$$

And:

$$\begin{aligned} Q_{2k-1}^{\mathbb{N}} &= \lim_{N\to\infty} C_{N+(2k-1)-1,\,2k-1} \\ &= \lim_{N\to\infty} \frac{(N+2k-2)\,!}{(N-1)\,!\,(2k-1)\,!}, \end{aligned} \tag{14}$$

. Considering the partial difference:

$$D_k^N = Q_{2k}^N - Q_{2k-1}^N, \tag{15}$$

Which, by virtue of (9), leads to:

$$\begin{aligned}
D_k^N &= C_{N+2k-1,\,2k} - C_{N+(2k-1)-1,\,2k-1} = \frac{(N+2k-1)\,!}{(N-1)\,!\,(2k)\,!} - \frac{(N+2k-2)\,!}{(N-1)\,!\,(2k-1)\,!} \\
&= \frac{(N+2k-1)\,! - 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N+2k-1)\,(N+2k-2)\,! - 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N+2k-1-2k)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N-1)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\
&= \frac{(N-1)\,(N+2k-2)\,!}{(N-1)\,(N-2)\,!\,(2k)\,!} \\
&= \frac{(N+2k-2)\,!}{(N-2)\,!\,(2k)\,!} = \frac{[(N-2)+2k]\,!}{(N-2)\,!\,(2k)\,!} \\
&= \frac{\overbrace{[(N-2)+2k]\,[(N-2)+2k-1]\times\cdots\times[(N-2)+1]}^{2k\ factors}\,(N-2)\,!}{(N-2)\,!\,(2k)\,!} \\
&= \frac{\overbrace{[(N-2)+2k]\,[(N-2)+2k-1]\times\cdots\times[(N-2)+1]}^{2k\ factors}}{(2k)\,!} \\
&= \frac{1}{(2k)\,!}\prod_{j=1}^{2k}[(N-2)+j]\,,
\end{aligned}$$

therefore:

$$D_k^N > d_k^N, \tag{16}$$

where:

$$d_k^N = \frac{(N-2)^{2k}}{(2k)\,!} \tag{17}$$

Hence, from (15), (16) and (17):

$$\begin{aligned}
\sum_{k=1}^{\infty} D_k^N &= \sum_{k=1}^{\infty} Q_{2k}^N - \sum_{k=1}^{\infty} Q_{2k-1}^N \\
&= \sum_{k=1}^{\infty} \frac{1}{(2k)\,!}\prod_{j=1}^{2k}[(N-2)+j] \\
&> \sum_{k=1}^{\infty} d_k^N = \sum_{k=1}^{\infty} \frac{(N-2)^{2k}}{(2k)\,!} = \cosh(N-2) - 1.
\end{aligned} \tag{18}$$

Now, consider the set , hence exhausts any prime factorization. This is so, since a given prime factorization represents one and only one element in and a given element of has got one and only one prime factorization. Hence, all the factorizations exhaust . is the entire set of possible prime factorizations and the entire set of possible prime factorizations is . The arithmetic progression:

$$a_n = 1 + (n-1)1 = n \Leftrightarrow a_n = n, \qquad (19)$$

With, shows that the number of possible factorizations grows as grows. Hence, since this is essentially the set , the quantity of possible factorizations exhaustively grows [a given finite set of factorizations will have a factorization, say , representing a greatest factorized number in this set and being one-to-one to an for this set, with the remaining factorizations ()] as . Hence, as discussed in the appendix (see, also, (11) and its subsequent result giving emphasis on the cardinal context of):

$$n \to 1 + \sum_{k=1}^{\infty} \lim_{N\to\infty} Q_{2k}^{N} + \sum_{k=1}^{\infty} \lim_{N\to\infty} Q_{2k-1}^{N}, \qquad (20)$$

viz.:

$$n \to 1 + \sum_{k=1}^{\infty} \left(Q_{2k}^{\mathbb{N}} + Q_{2k-1}^{\mathbb{N}}\right) = \infty \qquad (21)$$

By virtue of (21):

$$n^{\frac{1}{2}+\epsilon} \to \left[1 + \sum_{k=1}^{\infty} \left(Q_{2k}^{\mathbb{N}} + Q_{2k-1}^{\mathbb{N}}\right)\right]^{\frac{1}{2}+\epsilon} = \infty \qquad (22)$$

Considering the partial sum:

$$s_k^N = Q_{2k}^N + Q_{2k-1}^N, \qquad (23)$$

which, by virtue of the Eq. (9), leads to:

$$\begin{aligned} s_k^N &= C_{N+2k-1,\,2k} + C_{N+(2k-1)-1,\,2k-1} = \frac{(N+2k-1)\,!}{(N-1)\,!\,(2k)\,!} + \frac{(N+2k-2)\,!}{(N-1)\,!\,(2k-1)\,!} \\ &= \frac{(N+2k-1)\,! + 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\ &= \frac{(N+2k-1)\,(N+2k-2)\,! + 2k(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\ &= \frac{(N+2k-1+2k)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} = \frac{(N+4k-1)\,(N+2k-2)\,!}{(N-1)\,!\,(2k)\,!} \\ &= \frac{(N+4k-1)\,[(N-1)+(2k-1)]\,!}{(N-1)\,!\,(2k)\,!} \end{aligned}$$

$$= \frac{(N+4k-1)\overbrace{[(N-1)+(2k-1)]\{(N-1)+[(2k-1)-1]\}\times\cdots\times[(N-1)+1]}^{2k-1\ factors}}{(N-1)!\,(2k)!}\cdot$$

$$\times(N-1)!$$

$$= \frac{(N+4k-1)\overbrace{[(N-1)+(2k-1)]\{(N-1)+[(2k-1)-1]\}\times\cdots\times[(N-1)+1]}^{2k-1\ factors}}{(2k)!}$$

therefore:

$$s_k^N < S_k^N, \tag{24}$$

where:

$$S_k^N = \frac{N+4k}{(2k)!}(N+4k)^{2k-1} = \frac{(N+4k)^{2k}}{(2k)!}, \tag{25}$$

remembering: $k \in \mathbb{N}$. Furthermore:

$$S_k^N < \Xi_k^N, \tag{26}$$

where:

$$\Xi_k^N = \frac{(N+N/\alpha)^{2k}}{(2k)!} = \left(1+\frac{1}{\alpha}\right)^{2k}\frac{N^{2k}}{(2k)!}, \tag{27}$$

$\forall$ *fixed* $\alpha \in \mathbb{R}_+^*$. The (26) follows from the proposition:

$$4\alpha k < N,\ \forall\,\alpha \in \mathbb{R}_+^* \tag{28}$$

Since is to exhaustively cover the set , this latter being the entire set of prime numbers, viz., in:

$$\mathbb{P}_N = \{p_1, p_2, p_3, \cdots, p_N\} = \{2, 3, 5, \cdots, p_N\}, \tag{29}$$

has been taken, defined, the set of prime numbers, as we have defined from the beginning of this paper. Now, to conversely suppose the condition stated by (28), i.e.:

$$\exists\ \alpha \in \mathbb{R}_+^* \mid\ 4\alpha k \geq N, \tag{30}$$

one turns out to be, by hypothesis, considering an implied superior quota for the existence of prime numbers. Suppose (30) is correct. Hence, there exists, by virtue of (30), an such that the successive values of to exhaustively cover the set of prime numbers never exceed . Putting such , with the obeyer number of(30), :
a fortiori:

$$4\alpha_0 k = [4\alpha_0 k] + \{4\alpha_0 k\}, \tag{31}$$

with and being, respectively, the integer and the fractionary parts of , one is led to a superior quota:

$$N = [4\alpha_0 k], \tag{32}$$

since . By virtue of this superior quota for the quantity of prime numbers, there exist only finitely many primes . Hence, let , and consider . Since is a product of primes, it turns out to have a common prime divisor with , implying this common prime divisor divides : an absurd! Henceforth, (30) is an absurd, the proposition given by (28) is correct fixed and, by virtue of (23), (24), (26) and (27), one turns out to be led to:

$$s_k^N = Q_{2k}^N + Q_{2k-1}^N < \Xi_k^N = \frac{1}{(2k)\,!}\left[\left(1+\frac{1}{\alpha}\right)N\right]^{2k} \tag{33}$$

Back to the interest carried from (20), (21) and (22), now, we consider the consequence to the sum:

$$\sum_{k=1}^{\infty} s_k^N = \sum_{k=1}^{\infty} Q_{2k}^N + \sum_{k=1}^{\infty} Q_{2k-1}^N, \tag{34}$$

i.e.:

$$\begin{aligned}\sum_{k=1}^{\infty} s_k^N &= \sum_{k=1}^{\infty} Q_{2k}^N + \sum_{k=1}^{\infty} Q_{2k-1}^N\\ &< \sum_{k=1}^{\infty} \Xi_k^N = \sum_{k=1}^{\infty} \frac{1}{(2k)\,!}\left[\left(1+\frac{1}{\alpha}\right)N\right]^{2k} = \cosh\left[\left(1+\frac{1}{\alpha}\right)N\right]-1\end{aligned}$$

therefore:

$$\sum_{k=1}^{\infty} s_k^N = \sum_{k=1}^{\infty} Q_{2k}^N + \sum_{k=1}^{\infty} Q_{2k-1}^N < \sum_{k=1}^{\infty} \Xi_k^N = \cosh\left[\left(1+\frac{1}{\alpha}\right)N\right]-1 \tag{35}$$

From the reasonings we have carried throughout the march that has been led from that (12), accomplished from its previous combinatorics, to the (35), one reaches, *a fortiori*, the following implied consequence:

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &= \lim_{n\to\infty}\frac{1+\sum_{l=2}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}}\\ &> \lim_{N\to\infty}\frac{1+[\cosh(N-2)-1]}{\left\{1+\cosh\left[\left(1+\frac{1}{\alpha}\right)N\right]-1\right\}^{\frac{1}{2}+\epsilon}} \Rightarrow\\ \lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &> \lim_{N\to\infty}\frac{\cosh(N-2)}{\left\{\cosh\left[\left(1+\frac{1}{\alpha}\right)N\right]\right\}^{\frac{1}{2}+\epsilon}}\\ &\therefore\\ \lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}} &> \frac{2^{\epsilon}}{\sqrt{2}}\lim_{N\to\infty}\frac{e^{(N-2)}+e^{(2-N)}}{\left[e^{(1+1/\alpha)N}+e^{-(1+1/\alpha)N}\right]^{\frac{1}{2}+\epsilon}}\end{aligned}$$

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}}>\frac{2^{\epsilon}}{e^{2}\sqrt{2}}\lim_{N\to\infty}\frac{e^{N}\left(1+e^{-2(N-2)}\right)}{\left[e^{(1+1/\alpha)N}\left(1+e^{-2(1+1/\alpha)N}\right)\right]^{\frac{1}{2}+\epsilon}},$$

therefore:

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}}>\frac{2^{\epsilon}}{e^{2}\sqrt{2}}\lim_{N\to\infty}\frac{1+e^{-2(N-2)}}{\left(1+e^{-2(1+1/\alpha)N}\right)^{\frac{1}{2}+\epsilon}}e^{N[1-(\epsilon+1/2)(1+1/\alpha)]} \tag{36}$$

Since the Riemann Hypothesis [3] is true *if and only if* [4,5]:

$$\lim_{n\to\infty}\frac{\sum_{l=1}^{n}\lambda(l)}{n^{\frac{1}{2}+\epsilon}}=0,\ \forall\ \textit{fixed}\ \epsilon>0, \tag{37}$$

it follows that the convergence of the right-hand side of (36), $\forall$ *fixed* $\epsilon>0$, is a necessary condition for the validity of the Riemann Hypothesis. It follows, then, that:

- *If there exists some $\epsilon>0$ such that the right-hand side of (36) diverges, then, the Riemann Hypothesis turns out to be false.*

Supposing the Riemann Hypothesis is true, the condition:

$$(\epsilon+1/2)(1+1/\alpha)>1, \tag{39}$$

$\forall$ *fixed* $\epsilon>0$. But this latter condition is an absurd, since, for $\epsilon=1/8$ and $\alpha=2$, e.g.:

$$(\epsilon+1/2)(1+1/\alpha)|_{(\alpha,\epsilon)=(2,1/8)}=\frac{15}{16}<1, \tag{40}$$

implying the Riemann hypothesis turns out to be false.

II. Appendix

The (20) must be correct, otherwise the Fundamental Theorem of Arithmetic turns out to be contradicted, as one grasps. This theorem asserts the factorization is unique, except for the order of the prime factors being multiplied to generate a natural number. Hence, the ∞, the upper sum limit, is a symbol for the natural numerical limit of $\aleph_0$. From this, follows the bijection between the natural numbers and their [unique] factorizations, with the degeneracy of a given factorization removed from the convention we have adopted in this paper, completely exhausting the members of both the sets, $\mathbb{N}$ and [their unique] factorizations, one-to-one. In fact, one can define the following function with domain $\mathbb{N}$:

$$\begin{aligned} f(n) &= 1=f^{-1}(n),\ \text{for } n=1; \\ f(n) &= \text{Product of the prime factors of } n \\ &= n \\ &= f^{-1}(n),\ \text{for } n\in\mathbb{N} \text{ and } n\neq 1, \end{aligned} \tag{41}$$

which is obviously invertible, with image $\mathbb{N}$, having got, hence, a one-to-one correspondence, a bijection, where $f^{-1}(n)$ is the inverse function of $f(n)$. The cardinality of $\mathbb{N}$ (remember our definition of $\mathbb{N}$, at the beginning of this paper, the positive integers) is $\aleph_0$. By (41), it is crystalline the set of factorizations is the very same $\mathbb{N}$, hence with the same cardinality, $\aleph_0$. The sequence $(1,\ 2,\ 3,\ 4,\ \cdots,\ n,\ \cdots)$ tends to the infinity (∞) that is

Symbolized by the cardinal . Hence, the cardinality of the factorizations must be , otherwise there would exist some number without factorization and vice-versa, contradicting the Fundamental Theorem of Arithmetic. Since is denumerable, hence countable, the set of factorizations is [denumerable, hence countable] too. Counting, must tend to as given by (20), since tends to the infinity that is symbolized by the cardinal , asseverating, as well as the infinity quantity of all the factorizations, (20).

III. Conclusion

We have concluded the Riemann Hypothesis turned out to be false.

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