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A study on Interior Domination in Graphs

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Abstract : *The concept of interior set of vertices of a graph G has applications in locating dominating set of a graph. A dominating set D is said to be an interior dominating set if every vertex v of D is an interior vertex of G. The minimum of the cardinality of the interior dominating sets of G is called as interior domination number of G. The relation between the domination number and the interior domination number has been found. Some bounds for interior domination number have been found and its exact values for some particular classes of graphs are determined.*

***Keywords**: Graphs, complete, Wheel, Star graphs, Dominating set, Interior vertices*

I. Introduction

The various domination parameters have been introduced and used in many applications in graphs by taking different sets with each of which has some specified property. These concepts are helpful to find centrally located sets to cover the entire graph in which they are defined. Let G be a finite, simple, undirected (p, q) graph with vertex set $V(G)$ and edge set $E(G)$. We refer the basic definitions and theorems used in the book [4].

Let G be a connected graph and u be a vertex of G. The *eccentricity* $e(v)$ of v is the distance to a vertex farthest from v. Thus $e(v) = \max\{ d(u, v): u\in V\}$. The radius $r(G)$ is the minimum eccentricity the vertices, whereas the diameter *diam* (G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. The vertex v is a central vertex if $e(v) = r(G)$. The centre $C(G)$ is the set of all *central vertices*. The *central subgraph* $<C(G)>$ of a graph G is the subgraph induced by the centre. The vertex v is a *peripheral vertex* if $e(v) = d(G)$. The *periphery* $P(G)$ is the set of all peripheral vertices.

For a vertex v, each vertex at a distance $e(v)$from v is an *eccentric vertex*. *Eccentric set* of a vertex v is defined as $E(v) = \{ u\in V /d(u, v) = e(v)\}$. The *open neighbourhood* $N(v)$ of a vertex $\boldsymbol{v}$ is the set of all vertices adjacent to v in G. $N[v]$ is called the closed neighbourhood of v.

A set $D\subseteq V(G)$ is a *dominating* set of G, if every vertex in $V – D$ is adjacent to some vertex in D. The dominating set D is a minimal dominating set if no proper subset D' of D is a dominating set. The minimal dominating set with minimum cardinality is known as a minimum dominating set. The cardinality of minimum dominating set is known as the *domination number* and is denoted by $\gamma(G)$.

Let x and z be two distinct vertices in G. A vertex y distinct from x and z is said to *lie between* x and z if $d(x, z) = d(x, y) + d(y, z)$. A vertex v is an *interior vertex* of G if for every vertex u distinct from v, there exists a vertex w such that v lies between u and w. A vertex v is a *boundary vertex* of u if $d(u, w) \leq d(u, v)$ for all $w\in N(v)$. A vertex u has more than one boundary vertex at different distance levels. In this paper, we initiate the study of new domination.

II. Interior Dominating Sets

Definition 2.1

A set $D \subseteq V(G)$ is an interior dominating set if D is a dominating set of G and every vertex $v\in D$ is an interior vertex of G. Any end vertex will not be a member in interior set of a graph. In a tree every vertex which is not an end vertex is an interior vertex.

Definition 2.2

The interior domination number $\gamma_{Id}(G)$ of a graph G is defined as the cardinality of the minimum interior dominating set.

Example 2.3

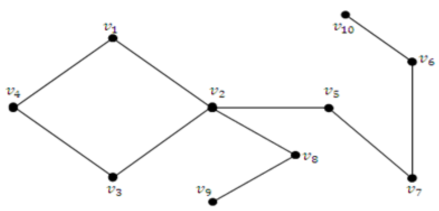

Figure 1: A graph G

$D = \{v_2, v_4, v_6, v_9\}$, $D_1 = \{v_2, v_3, v_7, v_8, v_9\}$, $D_2 = \{v_2, v_4, v_6, v_8\}$. D is a dominating set of G but it is not an interior dominating set of G. D_1 is an interior dominating set of G but not minimum. D_2 is the minimum interior dominating set of G. Hence $\gamma_{Id}(G) = 4$.

Theorem 2.4 A dominating set $D \subseteq V(G)$ is an interior dominating set if and only if for all $v \in D$, $|N(v)| \geq 2$ and for all $x \in N(v)$ there exist $y \in N(v)$ such that $d(x, y) = d(x, v) + d(v, y)$

Proof: Let D be an interior dominating set of G. Let v be an arbitrary vertex in D. If x is the only neighbourhood of v then for any $y \in V - N(v)$, $d(x, y) \neq d(x, v) + d(v, y)$. Hence v is not an interior vertex of G. Since $v \in D$ is an arbitrary vertex, no vertex in D is an interior vertex of G. Then D is not an interior dominating set of G which is a contradiction. Conversely, suppose that D is a dominating set of G. Then for all $v \in D$, $|N(v)| \geq 2$ and for all $x \in N(v)$ there exist $y \in N(v)$ such that $d(x, y) = d(x, v) + d(v, y)$. Clearly v is an interior vertex of G. Hence D is an interior dominating set of G. ■

Result 2.5 A dominating set D is an interior dominating set of G if and only if it is not a boundary dominating set of G.

Theorem 2.6 For any path of order $n \geq 3$, $\gamma_{Id}(P_n) = \left\lceil \frac{n}{3} \right\rceil$

Proof: Let $v_1, v_2, \ldots\ldots v_n$ represents a path P_n. Let $D = \{v_2, v_5, v_8 \ldots\ldots v_{n-1}\}$ be the minimum dominating set of P_n. Since no end vertex is an interior vertex and in D the dominating vertices start from v_2 and ends in v_{n-1}. No vertex in D is an end vertex of P_n. Hence D is the minimum interior dominating set and $|D| = \left\lceil \frac{n}{3} \right\rceil$. Therefore $\gamma_{Id}(P_n) = \left\lceil \frac{n}{3} \right\rceil$. ■

Theorem 2.7 For any cycle of order $n \geq 4$, $\gamma_{Id}(C_n) = \left\lceil \frac{n}{3} \right\rceil$ or $\left\lceil \frac{n}{3} \right\rceil + 1$.

Proof: Let G be a cycle $v_1, v_2, \ldots\ldots v_n, v_1$. Since C_n is a cycle, any vertex of D is an interior vertex of G.

Case 1: If $n = 3k$, where $k = 2,3,\ldots\ldots$, then $\gamma_{Id}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Case 2: If $n = 3k - 1$, where $k = 2, 3, \ldots\ldots$ Consider $D = \{v_1, v_4, v_7, \ldots\ldots v_k v_{k+3}, \ldots\ldots v_{3k-1}\}$ be a minimum dominating set of G. Since C_n is a cycle, any vertex in D is an interior vertex of G. Hence D is an interior dominating set of G and also D is a minimum interior dominating set of G.

Sub case 1: If n is odd, then $|D| = \left\lceil \frac{n}{3} \right\rceil$.

Sub case 2: If n is even, then $|D| = \left\lceil \frac{n}{3} \right\rceil + 1$.

Case 3: If $n = 3k - 2$, where $k = 2, 3, \ldots\ldots$ then it is similar to the above case.

Hence $\gamma_{Id}(C_n) = \left\lceil \frac{n}{3} \right\rceil$ or $\left\lceil \frac{n}{3} \right\rceil + 1$. ■

Theorem 2.8

(i) $\gamma_{Id}(K_{m,n}) = 2$, (ii) $\gamma_{Id}(W_n) = 1$ and (iii). $\gamma_{Id}(K_{1,n}) = 1$

Proof:

(i) When $G = K_{m,n}$, $V(G) = V_1 \cup V_2$, $|V_1| = m$ and $|V_2| = n$ such that each element of V_1 is adjacent to every vertex of V_2 and vice versa. Let $D = \{u, v\}, u \in V_1, v \in V_2$. The vertex u dominates all the vertices of V_2 and it is the interior dominating vertex. Similarly v dominates all the vertices of V_1 and it is the interior dominating vertex. Therefore D is a minimum interior dominating set and hence $\gamma_{Id}(K_{m,n}) = 2$.

(ii) When $G = W_n$,. Let $D = \{u\}$ where u is the central vertex of G and also the interior vertex of G. The central vertex dominates every vertex of G. Then D is a minimum interior dominating set of G. Thus $\gamma_{Id}(W_n) = 1$.

(iii) When $G = k_{1,n}$. Let $D = \{u\}$ where u is the central vertex of G and also the interior vertex of G. The central vertex dominates every vertex of G. Then D is a minimum interior dominating set of G. Thus $\gamma_{Id}(K_{1,n}) = 1$. ■

Note that for any complete graph K_n, there is no interior dominating set, since there is no interior vertex.

We observe the identities:

$$(i).\ \gamma_{bd}(K_{m,n}) = \gamma_{ed}(K_{m,n}) = \gamma_{Id}(K_{m,n})$$

Theorem 2.9 For any connected graph G with $n \geq 4$, $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{Id} \leq 2e - n + 1$

Proof: Let $x \in G$ be a vertex of degree Δ. Therefore x dominates itself and other Δ of neighbouring vertices. Therefore x dominates $\Delta + 1$ vertices of G and $\gamma_{Id}(G) \geq \left\lceil \frac{n}{\Delta+1} \right\rceil$. If G is a regular graph, then every vertex has degree Δ. Hence the cardinality of every dominating set is greater than $\left\lceil \frac{n}{\Delta+1} \right\rceil$. For an arbitrary graph, the cardinality of every dominating set is also greater than $\left\lceil \frac{n}{\Delta+1} \right\rceil$. The value of $\frac{n}{\Delta+1}$ in this case is greater than the value in the first case. Hence in all cases number of minimum dominating set is greater than $\left\lceil \frac{n}{\Delta+1} \right\rceil$.
For any graph, $\gamma_{Id}(G) \leq n - 1 = 2(n-1) - (n-1) \leq 2e - n + 1$. Therefore $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{Id} \leq 2e - n + 1$. ■

Theorem 2.10 Let G be a graph. If G is of radius 2 with a unique central vertex u, then $\gamma_{Id}(G) \leq n - deg(u)$
Proof: Let G be a graph with rad (G) =2. Let u be a unique central vertex, then u has an eccentric vertex v with distance 2. Hence $N[u] \neq G$ and $v \in G - N[u]$, hence there are $n - deg(u)$ - 1 vertices in $G - N[u]$. Therefore every dominating set has a cardinality $n - deg(u)$. ■

Theorem 2.11 For a connected graph G, $\gamma_{Id}(G) + \Delta(G) = n$ if $G \cong K_{1,n-1}$ or W_n.
Proof: If $G \cong K_{1,n-1}$ or W_n then $\gamma_{Id}(G) = 1$ and $\Delta(G) = n - 1$. Hence $\gamma_{Id}(G) + \Delta(G) = n$. ■
Note 2.12 Let T be tree of order n with n_1 pendant vertices. Then $\gamma_{Id}(T) < \gamma(T) + n_1$

Theorem 2.13 For a tree T, $\gamma_{Id}(T) \leq n - \Delta(T)$.
Proof: Let G be tree which is not isomorphic to $K_{1,n-1}$. Let D be a γ set of T. Let v be a vertex of maximum degree $\Delta(G)$. Then v dominates $N[v]$ and the vertices in $V - N[v]$ dominate themselves. Hence $V - N(v)$ is a dominating set of cardinality $n - \Delta(T)$ and so $\gamma_{Id}(T) \leq n - \Delta(T)$. ■

Theorem 2.14 For any triangle free graph connected graph G with $n \geq 4$ the Interior domination number lies between $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{Id}(G) \leq n - \Delta(G)$.
Proof: Let D be a γ set of G. First we consider the lower bound each vertex can dominate at most itself and $\Delta(G)$ other vertices. Hence $\gamma_{Id}(G) \geq \left\lceil \frac{n}{\Delta+1} \right\rceil$. For the upper bound, let v be a vertex of maximum degree $\Delta(G)$, then v dominates $N[v]$ and the vertices in $V - N[v]$ dominates themselves. Hence $V - N(v)$ is a dominating set of cardinality $n - \Delta(G)$ and so $\gamma_{Id}(G) \leq n - \Delta(G)$. Hence $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{Id} \leq n - \Delta(G)$. ■

Theorem 2.15 $\gamma(G) \leq \gamma_{Id}(G)$
Proof: Let D be a minimum dominating set. Then $\gamma_{Id}(G) = |D|$. Let D_{Id} be a minimum dominating set in which all vertices are interior vertices of G. If the members of D are interior vertices of G, then $|D| = |D_{Id}|$. Otherwise $|D| < |D_{Id}|$. Hence $|D| \leq |D_{Id}|$ and so $\gamma(G) \leq \gamma_{Id}(G)$. ■

Theorem 2.16 For any connected graph with $n \geq 5$ vertices, $\gamma_{Id}(G) + \kappa(G) \leq 2n - 3$.
Proof: Let G be a connected graph. For every graph G of order n, $0 \leq \kappa(G) \leq n - 1$ by [4] and also the upper bound follows from the definition of interior domination, that is $\gamma_{Id}(G) \leq n - 2$.
Hence $\gamma_{Id}(G) + \kappa(G) \leq n - 1 + n - 2$. Therefore, $\gamma_{Id}(G) + \kappa(G) \leq 2n - 3$. ■

Theorem 2.17 For any connected graph G with $n \geq 5$ vertices, $\gamma_{Id}(G) + \chi(G) \leq 2n - 2$.
Proof: Let G be a connected graph with $n \geq 5$vertices. Suppose G is a complete graph we need n colours for colouring so $\chi(G) = n$. For other graphs $\chi(G) < n$. In general for any connected graph G, $\chi(G) \leq n$. From the above theorem we have $\gamma_{Id}(G) \leq n - 2$Hence $\gamma_{Id}(G) + \chi(G) \leq n + n - 2 \leq 2n - 2$. ■

Theorem 2.18 For any connected graph G with $n \geq 5$, $\gamma_{Id}(G) + \Delta(G) \leq 2n - 3$
Proof: Let G be a connected graph with $n \geq 5$, vertices. For any connected graph G, $\Delta(G) \leq n - 1$ .From the above theorem we have $\gamma_{Id}(G) \leq n - 2$. Hence $\gamma_{Id}(G) + \Delta(G) \leq 2n - 3$. ■

Theorem 2.19 If T is a tree with l leaves, then $\gamma_{Id}(T) \geq \frac{n-l+2}{3}$
Proof: Delavina [4] proved that $\gamma(T) \geq \frac{n-l+2}{3}$ for a tree T with n vertices and leaves. But $\gamma_{Id}(G) \geq \gamma(G)$, therefore $\gamma_{Id}(T) \geq \frac{n-l+2}{3}$. ■

Theorem 2.20 For any connected graph G with x cut vertices $\gamma_{Id}(G) \geq \frac{x+2}{3}$.

Proof: Delavina [4] proved that $\gamma(G) \geq \frac{x+2}{3}$ for x cut vertices. Since $\gamma_{Id}(G) \geq \gamma(G)$, we have $\gamma_{Id}(G) \geq \frac{x+2}{3}$. ■

Next we study the interior domination properties on some special graphs.

III. Exact value for some special graphs

Example 3.1: If G is a caterpillar such that each non pendent vertex is of degree three then $\gamma_{Id}(G) = (\frac{n}{2}) - 1$

Since degree of each non pendent vertex to three then G is of the following form

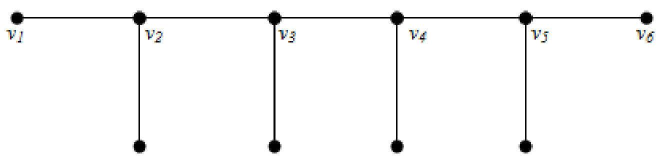

Figure 2: **A Caterpillar**

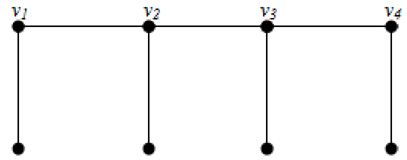

Figure 3: **A Hoffman tree**

Every non-pendent vertex set form an interior dominating set D and it is also a minimum set.

Hence $\gamma_{Id}(G) = \frac{n}{2} - 1$. If G is a Hoffman tree, then $\gamma_{Id}(G) = n$.

Example 3.2

The *Mobius-Kantor* graph is a symmetric bipartite cubic graph with 16 vertices and 24 edges as shown in the figure:

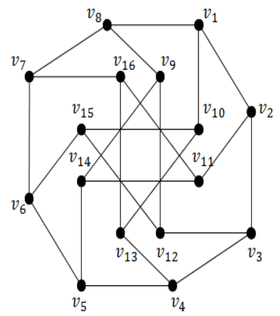

Figure 4: The Mobius-Kantor graph

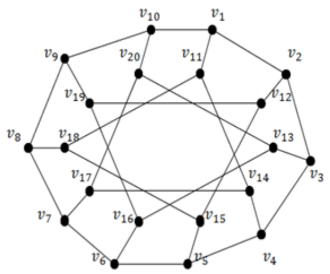

Figure 5: The *Desargues graph*

For the Mobius-Kantor graph G, $\gamma_{Id}(G) = 8$. Here the following set S is an interior dominating set of G.

Here $S = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$ ■

Example 3.3 The *Desargues graph* is a distance-transitive cubic graph with 20 vertices and 30 edges as shown in the figure For any Desargues graph G, $\gamma_{Id}(G) = 10$. Here the following set S is the interior dominating set.

Here $S = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18},, v_{19}, v_{20}\}$ ■

Example 3.4 The Chvatal graph is an undirected graph with 12 vertices and 24 edges as shown in the figure:

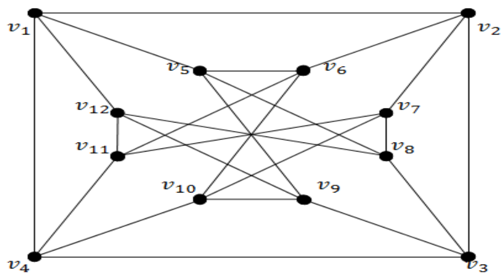

Figure 6: The Chvatal graph

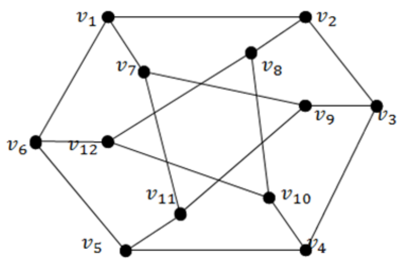

Figure 7: The Durer graph

For the Chvatal graph G, $\gamma_{Id}(G) = 4$. Here $S = \{v_1, v_2, v_3, v_4\}$ is an interior dominating set. ■

Example3.5 The Durer graph is an undirected cubic graph with 12 *v*ertices and 18 edges as shown in the figure. For the Durer graph G, $\gamma_{Id}(G) = 6$. Here $S = \{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is an interior dominating set.

IV. Conclusion

Many researchers are concentrating the dominating concept in graphs. In this paper we have defined the interior domination in graphs for different types of graphs and investigated interior domination number for them. Many results have been found and compared for some graphs.

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References

[1] Bhanumathi.M and Kavitha.M, BD-Domination in Graphs, Intern. J. Fuzzy Mathematical Archive, Vol. 7, No. 1, 75-80, 2015
[2] Buckley.F, Harary.F, Distance in graphs, Addison-Wesley Publishing Company (1990)
[3] Cockayne.E.J., Hedetniemi.S.T, Towards a theory of domination in graphs. Networks, 7:247-261.1997
[4] Ermelinda Delavina, Ryan pepper, and Bill Waller ,Lower bounds for the domination number, Discussiones Mathematicae Graph Theory, January (2010).
[5] Gary Chartand, Ping Zhang, Introduction to Graph theory, McGraw Hill Education (India) Private Limited , New Delhi (2006)
[6] Harary.F, Graph theory, Addison-Wesley Publishing Company Reading, Mass (1972).
[7] Janakiraman T.N., Bhanumathi.M and Muthammai.S, Eccentric Domination in Graphs, International Journal of Engineering
[8] Science, Advanced Computing and Bio-Technology Vol. 1, No. 2, April-June 2010, pp. 55-70.
[9] Teresa W.Haynes,Stephen T.Hedetniemi, Peter J.Slater, Fundamentals of Domination in graphs, Marcel Dekker, Inc(1998).