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Sums Of Squares Of Fractal Sequences:Farey Sequence & Negative Jacobsthal Sequence

A.Gnanam[1], B.Anitha[2]
[1]*(Assistant Professor,Department of Mathematics, Government Arts College,Trichy-22,India)*
[2]*(Research Scholar,Department of Mathematics, Government Arts College,Trichy-22,India)*

Abstract : *Negative Jacobsthal numbers are fractal sequence of numbers. We find sums of squares of negative jacobsthal sequence of numbers. We also find sum of squares of Farey sequence of numbers.*
Keywords: *Sums of squares,Jacobsthal sequence,Jacobsthal Lucas Sequence,Farey sequence*

I. Introduction

The Jacobsthal and Jacobsthal –Lucas sequences J_n and j_n are defined by the recurrence relations

$J_0 = 0, J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$.

$j_0 = 2, j_1 = 1,\ j_n = j_{n-1} + 2j_{n-2}$ for $n \geq 2$.

The first ten terms of the sequence J_n are 0,1,1,3,5,11,21,43,85 ,171 and 341. It is given by the formula $J_n = \frac{(2)^n-(-1)^n}{3}$. The first ten terms of the sequence j_n are 2,1,5,7,17,31,65,127,257,511 and 1025. It is given by the formula $j_n = (2)^n + (-1)^n$. Negative Jacobsthal numbers are fractal sequence of numbers. Here we find the sums of squares of Negative Jacobsthal and Negative Jacobsthal Lucas sequence of numbers.

A Farey sequence of order N is a set of irreducible fractions between 0 and 1 arranged in increasing order, the denominators of which do not exceed N.In this paper we establish a formula to find the sum of squares of Farey sequence of numbers.

Sums of Squares of Jacobsthal and Jacobthal-Lucas Sequences

Theorem

$$J_{-n}^2 + j_{-n}^2 = \frac{10}{9}\, j_{-2n} + \frac{1}{3}.\, 2^{-n+2}(-1)^{-n}$$

Proof

By the formula $J_{-n} = \frac{2^{-n}-(-1)^{-n}}{3}$; $j_{-n} = 2^{-n} + (-1)^{-n}$

we have

$$J_{-n}^2 + j_{-n}^2 = \left[\frac{2^{-n} - (-1)^{-n}}{3}\right]^2 + [2^{-n} + (-1)^{-n}]^2$$

Squaring and adding the above formula

$$= \frac{1}{9}2^{-2n} + 2^{-2n} + \frac{1}{9}(-1)^{-2n} + (-1)^{-2n} - \frac{2}{3}2^{-n}(-1)^{-n} + 2.2^{-n}(-1)^{-n}$$

$$= \frac{10}{9}2^{-2n} + \frac{10}{9}(-1)^{-2n} + 2.2^{-n}(-1)^{-n}\left(1 - \frac{1}{3}\right)$$

$$= \frac{10}{9}(2^{-2n} + (-1)^{-2n}) + \frac{2^2}{3}.\, 2^{-n}(-1)^{-n}$$

$$J_{-n}^2 + j_{-n}^2 = \frac{10}{9}\, j_{-2n} + \frac{1}{3}.\, 2^{-n+2}(-1)^{-n}$$

Remark

When n is odd

$$J_{-n}^2 + j_{-n}^2 = \frac{10}{9}\, j_{-2n} - \frac{1}{3}.\, 2^{-n+2}$$

When n is even

$$J_{-n}^2 + j_{-n}^2 = \frac{10}{9}\, j_{-2n} + \frac{1}{3}.\, 2^{-n+2}$$

Summation of the Jacobsthal-Lucas sequence

Theorem

$$\sum_{n=1}^{k} j_{-n}^2 = \frac{1}{3} + k + 2\sum_{n=1}^{k} \frac{(-1)^{-n}}{\sum_{i=1}^{n} \binom{n}{i}}$$

Proof

By the formula $j_{-n} = 2^{-n} + (-1)^{-n}$

We have

$$\sum_{n=1}^{k} j_{-n}^2 = \sum_{n=1}^{k} (2^{-n} + (-1)^{-n})^2$$

$$= \sum_{n=1}^{k} 2^{-2n} + \sum_{n=1}^{k} (-1)^{-2n} + 2\sum_{n=1}^{k} 2^{-n}\,(-1)^{-n}$$

$$= \left\{1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots - 1\right\} + \{1 + 1 + 1 + \cdots.\} + 2\left\{-\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots\right\}$$

Using the geometric series

$$= \left(\frac{1}{1 - \frac{1}{2^2}} - 1\right) + k + 2\sum_{n=1}^{k} \frac{(-1)^{-n}}{2^n}$$

$$= \left(\frac{4}{3} - 1\right) + k + 2\sum_{n=1}^{k} \frac{(-1)^{-n}}{2^n}$$

$$\sum_{n=1}^{k} j_{-n}^2 = \frac{1}{3} + k + 2\sum_{n=1}^{k} \frac{(-1)^{-n}}{\sum_{i=1}^{n} \binom{n}{i}}$$

Theorem

Sums of squares of Farey fractions of order N is given by

$$S_N^2 = \sum_{\substack{k=2 \\ (x,k)=1 \\ x<k}}^{N} \left(\frac{x}{k}\right)^2$$

Proof

In [1] it is proved that

$$S_N = S_{N-1} + \frac{\varphi(N)}{2}$$

Where S_N denotes the sum of Farey fractions of order N.

The corresponding formula for summation of squares of Farey sequence can be taken as

$$S_N^2 = S_{N-1}^2 + \sum_{(x,N)=1} \left(\frac{x}{N}\right)^2$$

Where S_N^2 denotes the sum of squares of Farey fractions of order N.

Now

$$S_N^2 = S_{N-2}^2 + \sum_{(x,N-1)=1} \left(\frac{x}{N-1}\right)^2 + \sum_{(x,N)=1} \left(\frac{x}{N}\right)^2$$

$$S_N^2 = S_{N-3}^2 + \sum_{(x,N-2)=1} \left(\frac{x}{N-2}\right)^2 + \sum_{(x,N-1)=1} \left(\frac{x}{N-1}\right)^2 + \sum_{(x,N)=1} \left(\frac{x}{N}\right)^2$$

$$S_N^2 = S_{N-3}^2 + \sum_{(x,N-2)=1} \left(\frac{x}{N-2}\right)^2 + \sum_{(x,N-1)=1} \left(\frac{x}{N-1}\right)^2 + \sum_{(x,N)=1} \left(\frac{x}{N}\right)^2 + \cdots$$

Continuing we get

$$S_N^2 = \sum_{\substack{k=2 \\ (x,k)=1 \\ x<k}}^{N} \left(\frac{x}{k}\right)^2$$

Illustration

$$S_5^2 = S_4^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$$

Proof

$$S_5^2 = \sum_{\substack{k=2 \\ (x,k)=1 \\ x<k}}^{5} \left(\frac{x}{k}\right)^2$$

$$S_5^2 = \left(\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2\right) + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$$

$$S_5^2 = S_4^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$$

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