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Existence of Solutions of Quasi-Linear Elliptic Systems with Singular Coefficients

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Abstract : *Dedicated to research conditions of existence of solution quasi-linear differential equations with measurable coefficients, in other words, we study limitations imposed on the nonlinearity of equation or system and under the condition that system will have a unique solution and that solution belong to certain class of functions. We study weak solvability of quasi-linear differential equations of elliptic type with smooth and measurable coefficients, research methods based on the theory of monotone weakly compact operators using the method of differential forms. We introduced a new class of operators* $\Lambda_{\lambda}^{p}: W_{1}^{p}\left(R^{l}, d^{l}x\right) \rightarrow W_{-1}^{p}\left(R^{l}, d^{l}x\right)$ *associated with a given differential equation and properties of these operators are investigated by using the forms that are generated by left side of equations. So, we proved existence of solution for quasi-linear differential equations with measurable coefficients by a new, more weak, conditions on coefficients.*
Keyword :*quasi-linear differential equation, elliptical equation , monotone weakly compact operators, weak solution,*

I. Introduction

The work is dedicated to studying of conditions of the smoothness of weak solutions of quasi-linear systems of differential equations with partial derivatives in spaces $W_{1}^{p}(R^{l}, d^{l}x)$ with measurable coefficients, and establishing additional conditions on the coefficients of a system in which solutions of the system will be belong to space $W_{2}^{p}(R^{l}, d^{l}x)$. Research conducted using the Galerkin method and method of forms of nonlinear monotone operators. Conditions on the coefficients specified during the study of the properties of non-linear operators generated by the form that generated by the left parts of given quasi-linear differential equations.

The results obtained are new even in case, when the system is one equation and is very relevant in a linear case. We study the conditions of existence of the solution of quasi-linear system (7), that set limits on non linearity in which the system will have interchanges and study uniqueness and smoothness of the solution in a certain class of functions.

Research methods problem of existence of solution of systems are similar to those used to consider one equation (in our case $N = \mathbf{1}$) but when we are studying the system we are considering vector space, ie all definitions, theorems and Lemma should be reformulated in new terms, often non-trivial way, the notion of conjugate element takes a different meaning, even the concept of a generalized solution can be formulated differently. We study quasi-linear systems or equations ($N = \mathbf{1}$), the conditions of $l > \mathbf{2}$ are important concerning the respective systems (or equations).

The most important first results on the solvability of a linear equation with arbitrary elliptical matrix were obtained brilliant American mathematician John Neshom in 1958 and En. George that solved so-called 19-th Hilbert's problem, in their works were received priori estimates which allowed to formulate the corresponding theorem on the existence and smoothness of solutions but the proof was conducted only in simple cases. After receiving these important results of basic theory of linear and quasi-linear partial differential elliptic equations the theory began to grow rapidly and very classic final results of this development work are set out in O.A. Ladyzhenskaya, N.N. Uraltsevoyi, V.A. Solonnikov.

In the study of such problems, the focus is concentrated on two areas: firstly it is growth conditions, and secondly the conditions of singularity. Consider the example of those conditions in this case:

$$\left|b\left(x,u,\nabla u\right)\right| \le \mu_{1}(x)\left|\nabla u\right|^{\alpha} + \mu_{2}(x)\left|u\right|^{\delta} + \mu_{3}(x),$$

$$\left|b\left(x,u,\nabla u\right) - b\left(x,v,\nabla v\right)\right| \le \mu_{4}(x)\left|\nabla\left(u-v\right)\right|^{\varphi} + \mu_{5}(x)\left|u-v\right|^{\psi}$$

conditions in such form first started to consider M.M. Kuharchuk, in fact, they are almost indistinguishable from classic record. Real number $\alpha,\delta,\varphi,\psi$ determine the degree of growth of the function that determines the nonlinearity is subject to these numbers and the implied power of the means by growth in our case they are identically equal to one. All known conditions on the degree of growth similar or close to ours, that the degree of growth greater than one, then the operator generated by form will not be accretive, and uniqueness solution cannot be proved.

The singularity of coefficients are determined by the singularity of given functions μ_i.

II. Preliminaries

Definition 1. The space $L^p(R^l, d^l x)$ or $L^p(R^l)$ where ($1 \le p < \infty$) – Banach space consist of all measurable on R^l functions are integrated by Lebesgue with degree p.

Definition 2. Sobolev space $W_m^p(R^l, d^l x)$ (або $W_m^p(R^l)$ or W_m^p) is Banach space which is consist of all the elements of space $L^p(R^l)$ generalized derivatives which are exist to the n-th order inclusive, and are integrated with p degree.

Sobolev space $W_m^p(R^l, d^l x)$ (or $W_m^p(R^l)$ or W_m^p) are Banach spaces formed by all elements of the space $L^p(R^l)$ generalized derivatives which are to m-th order inclusive and are integrated with degree p. It is a separable space. $W_{m,0}^p(R^l, d^l x)$ - set of elements $W_m^p(R^l, d^l x)$, its support is compact.

Linear operations determined a formula similar to the case of Lebesgue space, norm introduced by the following equation:

$$\| f \|_{W_m^p} = \left(\| f \|_{L^p}^p + \sum_{1 \le |s| \le m} \left\| D^s f \right\|^p \right)^{\frac{1}{p}}, \qquad (1)$$

or p – degree of norm

$$\| f \|_{W_m^p}^p = \sum_{|s| \le m} \left\| D^s f \right\|^p . \qquad (2)$$

For Sobolev space Relliha –Kondrashova there is theorem: if $k > m$ i $1 \le p < q < \infty$, $(k-m)p < l$ and the equality $\frac{1}{q} = \frac{1}{p} - \frac{k-m}{l}$,then embedment spaces $W_k^p(R^l) \subset W_m^q(R^l)$ is continuous.

There are many versions of theorems of functional embedment almost all related to the establishment of specific functional estimations.

Conjugated to space $W_m^p(R^l, d^l x)$ is space $W_{-m}^q(R^l, d^l x)$, which by definition can be entered as a space of linear functionals on the linear space $W_m^p(R^l, d^l x)$. The conjugated element has the form

$$f^{cn} = -\sum_{r=1,..,l} \frac{\partial}{\partial x_r} \left(\frac{\partial f}{\partial x_r} \left| \frac{\partial f}{\partial x_r} \right|^{p-2} \right).$$

Since main part of this work is devoted to researching of nonlinear systems of partial differential equations so unknown is vector-function, which means that $u = (u_1(x), ..., u_N(x)), x \in R^l$ it is an ordered set of N elements of functional set, for example $u_i \in W_m^p(R^l, d^l x), i = 1, ..., N$.

We use the following system of notation throughout this paper:

$$\|u\|_{L^p(R^l)} = \left\langle \sum_{i=1,...,N} |u_i|^p \right\rangle^{\frac{1}{p}} = \left(\sum_{i=1,...,N} \left\langle |u_i|^p \right\rangle \right)^{\frac{1}{p}}, \quad (3)$$

$$\langle u, v \rangle = \sum_{i=1,...,N} \langle u_i, v_i \rangle \; \forall u \in L^p(R^l) \forall v \in L^q(R^l). \qquad (4)$$

where the element $u \in L^p(R^l)$ refers to an ordered set of length N, each of which belongs scalar space $L^p(R^l)$, a v by analogy belongs to the dual space $L^q(R^l)$ (each vector component belongs $L^q(R^l)$).

There is equality:

$$\|u\|_{L^p(R^l)}^{p-1} = \left\langle \sum_{i=1,\dots,N} |u_i|^p \right\rangle^{\frac{p-1}{p}} = \left\langle \sum_{i=1,\dots,N} \left(|u_i|^{\frac{p}{q}} \right)^q \right\rangle^{\frac{1}{q}} = \left\| u^{p-1} \right\|_{L^q(R^l)}. \quad (5)$$

Just introduced norm in vector space $W_m^p(R^l, d^l x)$:

$$\|u\|_{W_m^p} = \left(\sum_{i=1,\dots,N} \left(\|u_i\|_{L^p}^p + \sum_{1\le|s|\le m} \left\| D^s u_i \right\|^p \right) \right)^{\frac{1}{p}}, \quad (6)$$

ie belonging to vector function of a functional space means that each component of the vector function belongs to this space.

III. Problem Statement

Consider all over Euclidean space R^l the system:

$$\lambda u^k - \frac{\partial}{\partial x_i}\left(a_{ij}(x,u^k) \frac{\partial}{\partial x_j} u^k \right) + b^k(x,u^k,\nabla u^k) = f^k, \quad k = 1,\dots,N \quad (7)$$

Where is the unknown vector function $u^k(x) = (u^1,\dots.,u^N), \lambda > 0$ is real number and $f(x) = f^k = (f^1,\dots,f^N)$ is known vector function.

$b(x,u,\nabla u) = b^k(x,u^k,\nabla u^k)$ is vector-function length N three variables: the vector dimension l ,vector dimension N , matrix dimension $l \times N$.

Measurable matrix $a_{ij}(x,u)$ dimension $l \times l$ satisfies ellipticity condition: $\exists \nu: \quad 0 < \nu < \infty$ and using the following inequality $\nu \mathrm{I} \le a(x,u),$ for almost all $x \in R^l,$ scilicet

$$\nu \sum_{i=1}^{l} \xi_i^2 \le \sum_{ij=1,\dots,l} a_{ij}(x,u)\xi_i\xi_j \quad \forall \xi \in R^l \quad (8)$$

Definition 3. *We call generalized (weak) solution in* $W_1^p(R^l, d^l x)$ *an element* $u(x)$ *which satisfies the integral identity:*

$$\lambda\langle u,v\rangle + \left\langle \sum_{i,j=1,\dots N} a_{ij} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} v \right\rangle + \langle b,v\rangle = \langle f,v\rangle, \quad (9)$$

for any element $v \in W_{1,0}^q(R^l, d^l x)$. Equation (9) is a scalar integral identity. Based on this definition by left parts of the system we build the form $h_\lambda^p : W_1^p \times W_1^q \to R$:

$$h_\lambda^p(u,v) \equiv \lambda\langle u,v\rangle + \langle \nabla v \circ a \circ \nabla u\rangle + \langle b(x,u,\nabla u), v\rangle, \quad (10)$$

which will assume correctly defined (conditions on the coefficients specify below) for all items $u \in W_1^p(R^l, d^l x), v \in W_1^q(R^l, d^l x)$.

Thus, based on the subset of differential equations we builded scalar form (form which is defined on function vector spaces $W_1^p(R^l, d^l x) \times W_1^q(R^l, d^l x)$ dimension N and

$$h_\lambda^p : \left(\overset{N}{\underset{1}{\times}} W_1^p(R^l, d^l x) \right) \times \left(\overset{N}{\underset{1}{\times}} W_1^q(R^l, d^l x) \right) \to R.$$

Also there are other weak solution concept, for example, its own system solution (7) defined as a function $u(x)$, if this function identically satisfies the system of equations (7).

The main object of this study is the existence of the solution of such systems, ie establishing affiliation generalized solution specific functional set provided that equation coefficients belonging to certain classes and functional spaces.

IV. The Conditions On The Coefficients Of The System To Guarantee The Existence Of The Solution

Definition 4. Let $\varphi \in L^2_{loc}(R^l, d^l x)$, then define the function $n(\varphi;\lambda)$ the formula

$$n(\varphi;\lambda) = ess\sup_{x\in R^l}\int_0^\infty e^{-\lambda t}(e^{t\Delta}|\varphi|^2(x))^{\frac{1}{2}}\frac{dt}{\sqrt{t}};$$

this function sometimes called normal Nash functions φ .

Similarly, for functions $\varphi \in L^1_{loc}(R^l, d^l x)$ introduce function $k_l(\varphi;\lambda)$ i $k_{l+1}(\varphi;\lambda)$ by definition:

$$k_l(\varphi;\lambda) = ess\sup_{x\in R^l}\int_0^\infty e^{-\lambda t}e^{t\Delta}|\varphi|(x)dt,$$

$$k_{l+1}(\varphi;\lambda) = ess\sup_{x\in R^l}\int_0^\infty e^{-\lambda t}e^{t\Delta}|\varphi|(x)\frac{dt}{\sqrt{t}}.$$

Function φ belongs to Nash class N_2 if and only if $\lim_{\lambda\to\infty} n(\varphi;\lambda) = 0.$

Similarly, function φ belongs Kato classes K_l i K_{l+1} , if and only if $\lim_{\lambda\to\infty} k_l(\varphi;\lambda) = 0$ i $\lim_{\lambda\to\infty} k_{l+1}(\varphi;\lambda) = 0$, respectively.

Definition 5. We introduce class of form bounded functions $PK_\beta(A)$ is given by formula:

$$PK_\beta(A) = \left\{ f \in L^1_{loc}(R^{l+1}) : \left|\left\langle f\,|h|^2\right\rangle\right| \le \beta\left\langle A^{\frac{1}{2}}h, A^{\frac{1}{2}}h\right\rangle + c(\beta)\,\|h\|_2^2 \right\},$$

where $h \in D(A^{\frac{1}{2}})$, $\beta > 0$, $\quad c(\beta) \in R^1$.

1. $b(x,y,z)$ is measurable vector function and its arguments $b \in L^1_{loc}(R^l)$;

2. Vector function $b(x,y,z)$ almost everywhere satisfies the inequality:

$$\left|b(x,u,\nabla u)\right| \le \mu_1(x)\left|\nabla u\right| + \mu_2(x)|u| + \mu_3(x). \quad (11)$$

We introduce a class of functions:

$$PK_\beta(A) = \left\{ f \in L^1_{loc}(R^l, d^l x) : \left|\left\langle h f h\right\rangle\right| \le \beta\left\langle \nabla h \circ a \circ \nabla h\right\rangle + c(\beta)\,\|h\|_2^2 \right\},$$

where $\beta > 0$, $\quad c(\beta) \in R^1$. In the condition (11) functions $\mu_1^2 \in PK_\beta(A)$, $\mu_2 \in PK_\beta(A)$, function $\mu_3 \in L^p(R^l)$.

3. The growth vector function $b(x,y,z)$ almost everywhere satisfies the condition:

$$\left|b(x,u,\nabla u) - b(x,v,\nabla v)\right| \le \mu_4(x)\left|\nabla(u-v)\right| + \mu_5(x)|u-v|, \quad (12)$$

where $\mu_4^2 \in PK_\beta(A)$, $\mu_5 \in PK_\beta(A)$.

Remark. The above conditions on the nonlinearity is weaker than previously seen by the system in the scientific literature. These conditions, such as type functions satisfying Coulomb potential.

V. Existence Of The Solution Of The System (7)

The proof of the existence of the solution of the system (7) is carried out under the scheme: the given system shape and explore its properties, proving that this form is associated with non-linear operator and studying the properties associated operator by means of generating forms; applying Galerkin method can show that a given system has a solution in the Sobolev space. To prove necessity of existence of solution of the system (7) we need some a priori estimates which are theorems about properties of solutions under certain conditions on the function that forming this system; assuming smooth of "coefficients" and get the smoothness of solutions that actually assert that solution is exist. The estimates of solutions is the key point to proving theorem of existence, in case such estimates are known; we can use different methods of proving the solvability of the system.

Lemma*(the acute angle.) Let the field* $S_R = \left(\bar{C} : |\bar{C}| = R\right)$, *where* $R > 0$ *– some appropriately chosen number, given continuous mapping* $\vec{B} : R^n \to R^n$ *for which executed counterpart conditions of acute angle, ie,* $\left\langle \bar{B}\left(\bar{C}\right), \bar{C}^* \right\rangle \geq 0$. *Then there is at least one such point* $\vec{C} : \ |\vec{C}| \leq R$, *що* $\vec{B}\left(\vec{C}\right) = 0$.

Suppose that the coefficients limited sufficiently smooth function, proving the existence of the solution will produce using Galerkin method.

Let $\{v_i\}$ i $\{v_i^*\}$ are smooth bases of spaces $W_1^p(R^l, d^l x)$, $\_W_1^q(R^l, d^l x)$, in accordance, and let $[v_1, ...v_k]$ islinear shell basic elements $\langle u_k, u_k^* \rangle = \| u_k \|_p^p$. Granted, by definition, $u_k = \sum_{i=1}^k c_i v_i$, $u_k^* = \sum_{i=1}^k c_i^* v_i^*$. To find the sequence of Galerkin up a system of equations:

$$\left\langle A^p(u_k) - f, v_i^* \right\rangle = 0, i = 1, ..., k,$$

then show that this system has a solution in linear shell first n base elements $\{v_i\}$. Indeed, the system determines a continuous mapping sphere in Euclidean space, and therefore there is an analogue of Lemma acute angle, ie, reflection $\vec{B}\left(\vec{C}\right) : B_i\left(\vec{C}\right) = \left\langle A^p(u_k) - f, v_i^* \right\rangle$, due coercitive of operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$, satisfies the conditions of Lemma analog acute angle:

$$\left\langle \vec{B}\left(\vec{C}\right), \vec{C}^* \right\rangle = \left\langle A^p\left(\sum_{i=1}^k c_i v_i\right) - f, \sum_{i=1}^k c_i^* v_i^* \right\rangle = \left\langle A^p(u_k) - f, u_k |u_k|^{p-2} \right\rangle \geq$$

$$\geq \left(\frac{h_\lambda^p\left(u_k, u_k |u_k|^{p-2}\right)}{\| u_k |u_k|^{p-2} \|_{W_1^q}} - \| f \|_{W_{-1}^p} \right) \| u_k |u_k|^{p-2} \|_{W_1^q} \geq 0.$$

Because of $A^p : W_1^p \to W_{-1}^p$ is continuous mapping on finite-dimensional subspace space $W_1^p\left(R^l, d^l x\right)$, then due analogue of Lemma acute angle for sufficiently large values of $R > 0$ there exists an element $\vec{C}, \ |\vec{C}| = R$, that $\vec{B}\left(\vec{C}\right) = 0$.

That is way above specified construction sequence $\{u_k(x)\}$, elements which are the solutions of systems of equations $\left\langle A^p(u_k) - f, v_i^* \right\rangle = 0.$ We show that this sequence converges to the solution of the system.

Using coerciveness of operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$, we obtain inequality $\| A^p(u_k) \|_{W_{-1}^p} \leq \| f \|_{L^p}$.

Theorem 1.*The system (7) under condition (11, 12) has unique solution in the space* W_1^p.

Proof. We estimate form (10), which made up the system (7) for the conjugate element $u|u|^{p-2} = \left(u_1 |u_1|^{p-2}, ..., u_N |u_N|^{p-2}\right)$:

$$\left| h_\lambda^p(u, u|u|^{p-2}) \right| =$$

$$= \left| \lambda \left\langle u, u|u|^{p-2} \right\rangle + \left\langle \nabla\left(u|u|^{p-2}\right) \circ a \circ \nabla u \right\rangle + \left\langle b(x, u, \nabla u), u|u|^{p-2} \right\rangle \right| \leq$$

$$\le \lambda \|w\|^2 + \frac{4(p-1)}{p^2}\langle \nabla w \circ a \circ \nabla w\rangle +$$

$$+\left\langle \mu_1(x)|\nabla u| + \mu_2(x)|u| + \mu_3(x), |u|^{p-1}\right\rangle \le$$

$$\le \lambda \|w\|^2 + \frac{4(p-1)}{p^2}\langle \nabla w \circ a \circ \nabla w\rangle + \frac{2}{p}\langle \mu_1 |\nabla w|, |w|\rangle +$$

$$+\beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2 + \|\mu_3\|\|u\|^{p-1},$$

where by the definition used vector function $w = u|u|^{\frac{p-2}{2}}$, according matrix $\nabla w = \frac{p}{2}|u|^{\frac{p-2}{2}}\nabla u$ we have the estimation:

$$\left\langle \mu_1 |\nabla u|, |u|^{p-1}\right\rangle = \left\langle \mu_1 |u|^{\frac{p-2}{2}}|\nabla u|, |u|^{\frac{p}{2}}\right\rangle \le \frac{2}{p}\langle \mu_1 |\nabla w|, |w|\rangle,$$

$$\left\langle \mu_2(x), w^2\right\rangle \le \beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2,$$

$$\left\langle \mu_3(x), |u|^{p-1}\right\rangle \le \|\mu_3\|\left\||u|^{p-1}\right\| = \|\mu_3\|\|u\|^{p-1},$$

the defenition is used $|u|^{p-1} = \left(\sum_{i=1,..,N}|u_i|^p\right)^{\frac{p-1}{p}}$ i then

$$|u||u|^{p-1} = \left(\sum_{i=1,..,N}|u_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1,..,N}|u_i|^p\right)^{\frac{p-1}{p}} =$$

$$= \left(\sum_{i=1,..,N}|u_i|^p\right) = |u|^p.$$

By Holder and Young

$$\frac{2}{p}\langle \mu_1 |\nabla w|, |w|\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\|,$$

$$\|\mu_1 w\| = \left\langle (\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le \left(\beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}},$$

so

$$\frac{2}{p}\langle \mu_1 |\nabla w|, |w|\rangle \le \frac{2}{p}\|\mu_1 w\|\|\nabla w\| = \frac{2}{p}\|\nabla w\|\left\langle (\mu_1 w)^2\right\rangle^{\frac{1}{2}} \le$$

$$\le \frac{2}{p}\|\nabla w\|\left(\beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2\right)^{\frac{1}{2}} \le$$

$$\le \frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2 + \varepsilon^2\left(\beta\langle \nabla w \circ a \circ \nabla w\rangle + c(\beta)\|w\|^2\right)\right).$$

Then we get estimate

$$\left|h_\lambda^p(u, u|u|^{p-2})\right| \le \lambda\|w\|^2 + \frac{4(p-1)}{p^2}\langle \nabla w \circ a \circ \nabla w\rangle +$$

$$+\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2+\varepsilon^2\left(\beta\langle\nabla w\circ a\circ\nabla w\rangle+c(\beta)\|w\|^2\right)\right)+$$

$$+\beta\langle\nabla w\circ a\circ\nabla w\rangle+c(\beta)\|w\|^2+\frac{\sigma^p}{p}\|\mu_3\|^p+\frac{1}{\sigma^q q}\|u\|^p,$$

finally

$$\left|h_\lambda^p(u,u|u|^{p-2})\right|\le\left(\lambda+\left(\frac{\varepsilon^2}{p}+1\right)c(\beta)+\frac{1}{\sigma^q q}\right)\|w\|^2+$$

$$+\left(\frac{4(p-1)}{p^2}+\frac{\beta\varepsilon^2}{p}+\beta\right)\langle\nabla w\circ a\circ\nabla w\rangle+\frac{1}{p}\frac{1}{\varepsilon^2}\|\nabla w\|^2+\frac{\sigma^p}{p}\|\mu_3\|^p.$$

Remark. We show that $\|w\|^2_{L^2(R^l)}=\|u\|^p_{L^p(R^l)}$, indeed

$$\|w\|^2_{L^2(R^l)}=\left\langle\sum_{i=1,..,N}w_i^2\right\rangle=\left\langle\sum_{i=1,..,N}u_i|u_i|^{\frac{p-2}{2}},u_i|u_i|^{\frac{p-2}{2}}\right\rangle=$$

$$=\|u\|^p_{L^p(R^l)}.$$

Let us analyze the nature of the numerical coefficients that are included in the estimation of form, coefficient β depends on the data of (smoothness coefficient system (μ_i)); coefficient $c(\beta)$ depends on β; coefficients ε i σ – arbitrary positive; coefficients ε selected based on the matrix a constants ellipticity, coefficient σ shifts affect the range of the value, it is less substantial.

So for every fixed vector $u\in W_1^p$ form $h_\lambda^p(u,v)$ is a continuous linear (in $v\in W_1^q$) functional of W_1^q, and therefore each $u\in W_1^p$ is associated with an element conjugate to W_1^q space W_{-1}^p, so function exists that $A^p:W_1^p\to W_{-1}^p$. Operator $A^p:W_1^p\to W_{-1}^p$ takes the following action: $h_\lambda^p(u,v)=\langle A^p(u),v\rangle$. We will prove the existence of solution of the system (7) and its uniqueness in several stages. First suppose that the vector functions that form the equation quite smooth and prove the existence of solution Galerkin method with a special basis of uniqueness of the solution is the result of strict accretive operator generated by form, which is composed from system and can be proved by contradiction. Then suppose that coefficients are measurable and by it cutting and smoothing we reduce the problem to the previous case. Next step is removal of the conditions of cutting and smoothing.

Theorem 2. Generalized solutions of equations (7) with conditions (11) uniformly limited in W_1^p.

Proof. We form an integral identity:

$$\lambda\langle u_k,\xi\rangle+\langle d\xi\circ a\circ du_k\rangle+\langle b(x,u_k,\nabla u_k),\xi\rangle\equiv\langle f,\xi\rangle,$$

We put $\xi=u_k|u_k|^{p-2}$, obtain:

$$\lambda\langle u_k,u_k|u_k|^{p-2}\rangle+\frac{4(p-1)}{p^2}\left\langle d\left(u_k|u_k|^{\frac{p-2}{2}}\right)\circ a\circ d\left(u_k|u_k|^{\frac{p-2}{2}}\right)\right\rangle+$$

$$+\left\langle b,u_k|u_k|^{p-2}\right\rangle\equiv\left\langle f,u_k|u_k|^{p-2}\right\rangle.$$

With conditions (11), using Young's and Holder inequalitys , we find:

$$|\langle b,u_k|u_k|^{p-2}\rangle|\le$$

$$\le\frac{1}{p}\left(\frac{1}{\varepsilon^2}\|\nabla w\|^2+\varepsilon^2\left(\beta\langle\nabla w\circ a\circ\nabla w\rangle+c(\beta)\|w\|^2\right)\right)+$$

$$+\beta\langle\nabla w\circ a\circ\nabla w\rangle+c(\beta)\|w\|^2+\frac{\sigma^p}{p}\|\mu_3\|^p+\frac{1}{\sigma^q q}\|u_k\|^p,$$

Next, we get:

$$|\langle f,u_k|u_k|^{p-2}\rangle|\leq\|f\|_p\|u_k|u_k|^{p-2}\|_q\leq\|f\|_p\|u_k\|^{p-1}.$$

Then, using arguments similar to the previous one, we obtain

$$\|u_k\|+\|\nabla u_k\|\leq c(\lambda,p,l,\lambda_0,N)\|f\|.$$

So, since $\|u_k\|_{W_1^p}<C$, where constant has depends on function coefficient (structure of equations), then because of weak compactness of space $W_1^p\left(R^l,d^l x\right)$ we find that there exists a subsequence $\left(u_{k'}(x)\right)$, that is a property: $u_{k'}\xrightarrow[W_1^p]{}u_0$ weak i $A^p(u_{n'})\xrightarrow[W_1^p]{}y$ weak.

Show that $y=A^p(u_0)=f$. It follows that the mapping $A^p:W_1^p\left(R^l,d^l x\right)\to W_{-1}^p\left(R^l,d^l x\right)$ is reflection "on". We form the integral identity:

$$\left\langle A^p(u_{k'}),v_i^*\right\rangle=\left\langle f,v_i^*\right\rangle,i=1,...,k'$$

and go to the limit $k'\to+\infty$. Then we obtain:

$$\lim_{n\to\infty}A^p(u_{k'})=y=f,$$

the limit in $W_{-1}^p\left(R^l,d^l x\right)$ space.

Using the accretiveness of operator $A^p(\cdot):W_1^p(R^1,d^l x)\to W_{-1}^p(R^1,d^l x)$ in space $L^p(R^l,d^l x)$, we have:

$$\lim_{k'\to\infty}\left\langle A^p(u_{k'})-A^p(v),(u_{k'}-v)\left|u_{k'}-v\right|^{p-2}\right\rangle=$$

$$=\left\langle y-A^p(v),(u_0-v)\left|u_0-v\right|^{p-2}\right\rangle\geq 0.$$

We put $v=u_0-tz,t>0,\quad z\in W_1^p\left(R^l,d^l x\right)$ and reducing both sides of the resulting inequality in t^{p-1}, obtain $\left\langle y-A^p(u_0-tz),z\left|z\right|^{p-2}\right\rangle\geq 0.$

With semi-continuity of operator $A^p:W_1^p\to W_{-1}^p,$ given the arbitrary element $z\in W_1^p\left(R^l,d^l x\right)$, obtain $y=A^p(u_0)=f$, ie for given initial data constructed sequence $\{u_{k'}\}$ and proved its convergence to the element $u_0\in W_1^p\left(R^l,d^l x\right)$, therefore element $u_0\in W_1^p\left(R^l,d^l x\right)$ will be solutions of the conditions mentioned above.

The uniqueness of this solution follows from the properties of accretiveness of operator $A^p(\cdot)$. Indeed, let u_0, u_0' are two such solutions.Then, just equality:

$$\left\langle A^p(u_0),w\right\rangle=f,\quad\left\langle A^p(u_0'),w\right\rangle=f\quad\forall w\in W_1^q(R^l,d^l x),$$

that $\left\langle A^p(u_0)-A^p(u_0'),w\right\rangle=0$.

Let $w=(u_0-u_0')|u_0-u_0'|^{p-2}$, so:

$$0=\left\langle A^p(u_0)-A^p(u_0'),(u_0-u_0')|u_0-u_0'|^{p-2}\right\rangle\geq$$

$$\geq\lambda\|u_0-u_0'\|_p^p+(p-1)\left\langle\nabla(u_0-u_0')\circ a\circ\nabla(u_0-u_0'),|u_0-u_0'|^{p-2}\right\rangle-$$

$$-\left\langle \mu_4(x)\left|\nabla\left(u_0-u_0'\right)\right|+\mu_5(x)\left|u_0-u_0'\right|,\left(u_0-u_0'\right)\left|u_0-u_0'\right|^{p-2}\right\rangle \geq$$

$$\geq\left(\lambda-\frac{\varepsilon^2 c(\beta)}{p}-c(\beta)\right)\|w\|^2+$$

$$+\left(\frac{4(p-1)}{p^2}-\beta\frac{\varepsilon^2}{p}-\frac{1}{p\varepsilon^2\nu}-\beta\right)\langle\nabla w\circ a\circ\nabla w\rangle\geq 0,$$

that due to strict accretiveness of operator A^p equivalent equality $u_0=u_0'$, that solutions of equations coincide.

VI. Research of the case measurable coefficients

Let the functions that form the quasi-linear system (7) is measurable and satisfy the above conditions for the growth and power of the singularities.

We introduce the notation $u(x)$. Нехай $u(x)$ is Lebesgue measurable vector function at R^l linear dimension N. We denote by u_ϑ and s_ϑ cutting and support, respectively $i=1,\dots,N$

$$u_\vartheta=\begin{cases}u_i-\vartheta_i, u_i>\vartheta_i,\\ 0;|u_i|\leq\vartheta_i,\\ u_i+\vartheta_i, u_i<-\vartheta_i,\end{cases} \qquad (13)$$

$$s_\vartheta(u)=\left\{x\in R^l:|u_i(x)|>\vartheta_i,\vartheta_i>0\right\}.$$

Let $a^m(x)$, $f^m(x)$, $b^m(x,y,z)$ are cutting for argument x functions $a(x)$, $f(x)$, $x\in R^l$, $b(x,y,z),(x,y,z)\in\{R^l\times R\times R^l\}$, respectively.

Consider the "smooth" approximation of functions $a^m(x),;b^m(x,y,z)$, $f^m(x)$ for argument x:

$$a_i^{n,m}(x)=\int_{R^l}\rho_n(x-t)\,a_i^m(t)dt=\rho_n * a_i^m, \qquad (14)$$

where $\rho_n(t)$ is smooth integral approximation 1 in R^l.

Research will be conducted under such a scheme, system of equations is approximated equations with smooth coefficients, then this system is approximating forms that give rise to operators approximating systems of equations and we study the properties of the operators that have to exist for solving quasilinear elliptic system conditions which were discussed above. At the final stage being established existence of approximate solution of equations and the transition to the limit. We go to the border is as follows initially removed smoothing, that go to the limit with $n\to\infty$ then remove the cut, that go to the limit with $m\to\infty$(sequence of limits is important).

The system of equations:

$$\lambda u-d\circ a^{m,n}\circ du+b^{m,n}(x,u,\nabla u)=f^{m,n},\lambda>0, \qquad (15)$$

and form

$$h_\lambda^{p,mn}(u,v)=\lambda\langle u,v\rangle+\langle dv\circ a^{m,n}\circ du\rangle+\langle b^{m,n},v\rangle. \qquad (16)$$

For every fixed vector $u\in W_1^p$ the form $h_\lambda^{p,mn}(u,v)$ is a continuous linear (по $v\in W_1^q$) functional of W_1^q, and therefore each $u\in W_1^p$ is associated with an element conjugate to W_1^q space W_{-1}^p, there is mapping $A^{p,mn}:W_1^p\to W_{-1}^p$ generated by this form.

Theorem 3. *The system (14) under condition (11, 11 a) has unique solution in* W_1^p.

Proof. This result proved above for $\forall n\in N$ і $\forall m\in R^+$.

Assume that the solutions of $u^{m,n}(x)$ equally limited, ie, $\| u^{m,n}(x)\|_{W_1^p} < c$. Because coerciveness of operator $A^p : W_1^p\left(R^l, d^l x\right) \to W_{-1}^p\left(R^l, d^l x\right)$ we obtain inequality $\| A^p(u^{m,n})\|_{W_{-1}^p} \le \| f \|_{W_{-1}^p}$. Then, according to the properties of weak compact space W_1^p there exists a sequence $n' = (m', n')$, that

$$u^{n'} \xrightarrow[W_1^p]{\omega} u_0, A^p(u^{n'}) \to y, n' = (m', n') \to \infty.$$

Now we have enough to show that $A^p(u_0) = f$. Indeed, the move to the limit in the identity $\left\langle A^p(u_{n'}), v\right\rangle = \left\langle f^{n'}, v\right\rangle$. From the properties and conditions of approximation (11, 11a), we get $w \xrightarrow[n'\to\infty]{} A^p(u^{n'}) = y = f$.

On the other hand:

$$\langle A^p(u^{n'}) - A^p(v), (u^{n'} - v) | u^{n'} - v |^{p-2}\rangle \ge 0,$$

resulting by accretiveness of operator $A^p : W_1^p \to W_{-1}^p$ в L^p.

Let $v \equiv u_0 - tz, \quad z \in W_1^p$, and reducing both sides of the resulting inequality in t^{p-1}, we obtain the inequality:

$$\langle y - A^p(u_0 - tz), z | z |^{p-2}\rangle \ge 0.$$

As operator $A^p : W_1^p \to W_{-1}^p$ is semi-continuous, go in it to the limit $t \to 0$, finally get $y = A^p(u_0) = f$, ie the system of (11) has a unique solution in space W_1^p.

To complete the proof we must still get priori estimate $\| u^{m,n} \| < c$.

Theorem 4. *Generalized solutions of system (14) with the conditions (11) uniformly limited in* W_1^p.

Proof. We form the integral identity:

$$\lambda\langle u^{m,n}, \xi\rangle + \langle d\xi \circ a^{m,n} \circ du^{m,n}\rangle + \langle b^{m,n}(x, u^{m,n}, \nabla u^{m,n}), \xi\rangle \equiv \langle f^{m,n}, \xi\rangle,$$

Let $\xi = u^{m,n} | u^{m,n} |^{p-2}$, we obtain:

$$\lambda\langle u^{m,n}, u^{m,n} | u^{m,n} |^{p-2}\rangle +$$

$$+\frac{4(p-1)}{p^2}\left\langle d\left(u^{m,n} | u^{m,n} |^{\frac{p-2}{2}}\right) \circ a^{m,n} \circ d\left(u^{m,n} | u^{m,n} |^{\frac{p-2}{2}}\right)\right\rangle +$$

$$+\left\langle b^{m,n}, u^{m,n} | u^{m,n} |^{p-2}\right\rangle \equiv \left\langle f^{m,n}, u^{m,n} | u^{m,n} |^{p-2}\right\rangle.$$

With conditions (15), using Holder and Young's inequality, we find:

$$| \langle b^{m,n}, u^{m,n} | u^{m,n} |^{p-2}\rangle | \le$$

$$\le \int_{R^l}\int_{R^l} | b^{m,n}(t) | \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2}\, dtdx \le$$

$$\le \int_{R^l}\int_{R^l} (\mu_1^m(t) \nabla u^{m,n}(x) +$$

$$+\mu_2^m(t) u^{m,n}(x) + \mu_3^m(t)) \rho_n(x-t) \times\times u^{m,n}(x) | u^{m,n}(x) |^{p-2}\, dtdx \le$$

$$\le \int_{R^l}\int_{R^l} (\mu_1^m(t) | \nabla u^{m,n}(x) | \rho_n(x-t) u^{m,n}(x) | u^{m,n}(x) |^{p-2}\, dtdx +$$

$$+\int_{R^l}\int_{R^l}(\mu_2^m(t)\,|\,u^{m,n}(x)\,|\,\rho_n(x-t)u^{m,n}(x)\,|\,u^{m,n}(x)\,|^{p-2}\,dtdx+$$

$$+\int_{R^l}\int_{R^l}(\mu_3^m(t)\rho_n(x-t)u^{m,n}(x)\,|\,u^{m,n}(x)\,|^{p-2}\,dtdx\le$$

$$\le\frac{2}{p}\int_{R^l}\int_{R^l}(\mu_1^m(t)\,|\,\nabla W(x)\,|\,\rho_n(x-t)W(x)dtdx+$$

$$+\int_{R^l}\int_{R^l}(\mu_2^m(t)W^2(x)\rho(x-t)dtdx+$$

$$+\int_{R^l}\int_{R^l}(\mu_3^m(t)\rho_n(x-t)u^{m,n}(x)\,|\,u^{m,n}(x)\,|^{p-2}\,dtdx,$$

estimate can be written:

$$\int_{R^l}\int_{R^l}(\mu_2^m(t)W^2(x)\rho(x-t))dtdx=$$

$$\int_{R^l}\left(\int_{R^l}\mu_2^m(t)\left|\rho_n(x-t)\right|\,|\,W(x)\,|^2\,dx\right)=$$

$$=\int_{R^l}\left(\int_{R^l}\mu_2^m(t)dt\right)\left|\rho_n(x)\right|W^2(x-t)dx=$$

$$=\int_{R^l}\left|\rho_n(x)\right|\int_{R^l}(\mu_2^m(t)W^2(x-t)dt)dx\le\int_{R^l}\left|\rho_n(x)\right|\_(\beta\,\|\,\nabla W(x)\,\|_2^2+$$

$$+c(\beta)\,\|\,\nabla W(x)\,\|_2^2)dx=\beta\,\|\,\nabla W\,\|_2^2+c(\beta)\,\|\,\nabla W\,\|_2^2,$$

we estimate integrals:

$$\int_{R^l}\int_{R^l}\mu_1^m(t)\,|\,\nabla W(x)\,|\,\rho_n(x-t)W(x)dtdx\le$$

$$\le\left(\int_{R^l}\int_{R^l}\rho_n(x-t)\,|\,\nabla W(x)\,|^2\,dtdx\right)^{\frac{1}{2}}*$$

$$*\left(\int_{R^l}\int_{R^l}\left|\rho_n(x-t)\right|(\mu_1^m(t)\,|\,W(x)\,|)^2\,dtdx\right)^{\frac{1}{2}},$$

where $W=u^{m,n}\,|\,u^{m,n}\,|^{\frac{p-2}{2}}$.

Then again used Young inequality and form-boundedness μ_1^2 . We have:

$$\int_{R^l}\left(\int_{R^l}\rho_n(x-t)W^2(x)dx(\mu_1^m(t))^2\,dt\right)=$$

$$=\int_{R^l}\left(\int_{R^l}(\mu_1^m(t))^2\,dt\right)\rho_n(x)\,|\,W(x-t)\,|^2\,dx=$$

$$=\int_{R^l}\rho_n(x)\int_{R^l}(\left(\mu_1^m(t)\right)^2\,|\,W(x-t)\,|^2\,dt)dx\le$$

$$\le\int_{R^l}\rho_n(x)\_(\beta\,\|\,\nabla W(x-t)\,\|_2^2+$$

$$+c(\beta)\|W(x)\|_2^2)dx = \beta\|\nabla W\|_2^2 + c(\beta)\|W\|_2^2 .$$

Then

$$\int_{R^l}\int_{R^l}(\mu_3^m(t)\rho_n(x-t)u^{m,n}(x)|u^{m,n}(x)|^{p-2}\,dtdx \le$$

$$\le \|\mu_3\|_p \left\|u^{m,n}(x)|u^{m,n}(x)|^{p-2}\right\|_q = \|\mu_3\|_p \left\|u^{m,n}(x)\right\|^{p-1}$$

Using the properties of approximation of function f we obtain:

$$|\langle f^{m,n}, u^{m,n}|u^{m,n}|^{p-2}\rangle| \le \| f^{m,n}\|_p \| u^{m,n}|u^{m,n}|^{p-2}\|_q \le$$

$$\le \| f^n\|_p \| u^{m,n}\|^{p-1} \le \| f\|_p \| u^{m,n}\|_p^{p-1} .$$

Then, using Young's inequality, for sufficiently large m, n and considerations such as the previous one, obtain $\| u^{m,n}\| + \|\nabla u^{m,n}\| \le c(\lambda, p, l, \lambda_0, N)\| f\|_p$.

VII. Conclusions

We proved existence generalized (weak) solution of quasi-linear elliptic systems in whole Euclidean space R^l on conditions of $\frac{\partial}{\partial x_i}a_{ij} \in L^\infty(R^l, dx^l)$ and almost everywhere satisfies ellipticity condition: $\exists\, \nu : 0 < \nu < \infty$ and executed the following inequality $\nu \mathrm{I} \le a(x,u),$ almost all $x \in R^l,$ so $\nu\sum_{i=1}^{l}\xi_i^2 \le \sum_{ij=1,\dots,l} a_{ij}(x,u)\xi_i\xi_j\ \forall \xi \in R^l$, and also satisfies conditions (11) and (12) with some form-bounded functions $\mu_i^2 \in PK_\beta(A)$. As a generalization of the results of this study may improve the smoothness of solutions with increases smoothness of the initial data problem.

References

[1]. Вишик М.Й. О сильно эллиптических системах дифференциальных уравнений / М.Й. Вишик // Матем. сб. - 1951. - Т. 29(71), №3. - С. 615-676.

[2]. Kato T. Perturbation theory for linear operators / T. Kato. – Berlin-Heidelberg-New York: Springer-Verlag, 1980. – 578 p.

[3]. Kato T. Non-linear semigroups and evolution equations / T. Kato // J. Math. Soc. Japan. – 1967. – V. 19. – P. 508 – 520.

[4]. Komura Y. Differentiability of nonlinear semigroups / Y. Komura // J. Math. Soc. Japan. –1969. – V. 21. – P. 375–402.

[5]. Komura Y. Nonlinear semi-groups in Hilbert space / Y. Komura // J. Math. Soc. Japan. – 1967. – V. 19. – P. 493 – 507.

[6]. Konishy Y. Cosine functions of operators in locally convex spaces / Y.Konishy // J. Fac. Sci. Univ. Tokyo. – 1972. – V. 18. – P. 443 – 463.

[7]. Ладыженская О.А., Уральцева Н.Н. Линейные и квазилинейные уравнения эллиптического типа / О.А. Ладыженская, Н.Н.Уральцева. - М.: Наука, 1973. - 579 с.

[8]. Лионс Ж.-Л. Некоторые методы решения нелинейных краевых задач / Ж.-Л.Лионс. - М: Мир, 1972. - 587 с.

[9]. Яременко М.І. Квазілінійні рівняння та нелінійні напівгрупи стиску / М.І. Яременко. - К.: НТУУ "КПІ", 2013. - 201 с.

[10]. Miyadera I. On perturbation theory for semi-groups of operators / I. Miyadera // Tohoku Math. J. – 1966. – V. 18. – P. 299 – 310.

[11]. Nagy B. On cosine operator functions in Banach spaces / B. Nagy // Acta Science Math. – 1974. – V. 36. – P. 281 – 289.