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Harmonic Index of Some Class of Trees with an Algorithm

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Abstract: *The harmonic index H(G) of a graph G is defined as the sum of weights* $\frac{2}{d_G(u)+d_G(v)}$ *of all edges uv of G, where* $d_G(u)$ *is the degree of a vertex u in G. In this paper we obtained the harmonic index of some class of trees and further developed an algorithmic technique to find harmonic index of any graph.*
Keywords: *Algorithm, Degree of a vertex, Harmonic Index, Tree.*

I. Introduction

Let G be an undirected graph without loops and multiple edges with n vertices. Let $V(G) = \{v_1, v_2, \ldots , v_n\}$ be the vertex set of G. The harmonic index $H(G)$ of a graph G is defined as the sum of weights $\frac{2}{d_G(u)+d_G(v)}$ of all edges uv of G, where $d_G(u)$ is the degree of a vertex u in G. It has been found that the harmonic index correlates well with the Randic index[6]. Favaron et al. [3] considered the relationship between the harmonic index and the eigenvalues of graphs. Deng et al. [2] studied the relationship between the harmonic index and chromatic number of a graph. Shwetha Shetty et al. [10] obtained the bounds for the harmonic index of graph operations like join, corona product, Cartesian product, composition and symmetric difference. The expressions for the harmonic index and Randic index of the generalized transformation graphs G^{xy} and for their complement graphs are obtained [9]. The adjacency matrix of a graph G is the $n\times n$ matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise.

II. Results

Proposition 2.1: If T_1 is a tree with n vertices and m edges as shown in *Fig.1*, then harmonic index of T_1 is

$$H(T_1)= \frac{2x}{x+2}+\frac{2(n-x-2)}{n-x}+\frac{2}{n}$$

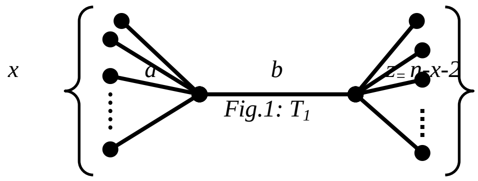

Fig.1: T_1

Proof: Without loss of generality consider the vertices a, b as shown in Fig. *1*, where $d_G(a)=x+1$, $d_G(b)=n-x-1$. Partition $E(T_1)$ into 3 sets E_1, E_2 and E_3 such that $E_1=\{uv / d_G(u)=1 \text{ and } d_G(v)=x+1\}$, $E_2=\{uv / d_G(u)=1 \text{ and } d_G(v)=z+1\}$, $E_3=\{ab\}$. It is easy to see that $|E_1|=x$, $|E_2|=n-x-2$, $|E_3|=1$ and $|E_1| + |E_2| + |E_3| = m$.

$$H(T_1) = \sum_{uv\in E(T_1)} \frac{2}{d_G(u)+d_G(v)}$$

$$= \sum_{uv\in E_1(T_1)} \frac{2}{d_G(u)+d_G(v)} + \sum_{uv\in E_2(T_1)} \frac{2}{d_G(u)+d_G(v)} + \sum_{uv\in E_3(T_1)} \frac{2}{d_G(u)+d_G(v)}$$

$$= \sum_{uv\in E_1(T_1)} \frac{2}{(x+1)+1} + \sum_{uv\in E_2(T_1)} \frac{2}{((n-x-2+1)+1)} + \sum_{uv\in E_3(T_1)} \frac{2}{(x+1)+(n-x-2+1)}$$

$$= x\frac{2}{x+2}+(n-x-2)\frac{2}{n-x}+\frac{2}{n}$$

$$H(T_1) = \frac{2x}{x+2} + \frac{2(n-x-2)}{n-x} + \frac{2}{n}$$

Proposition 2.2: If T_i is a graph with n vertices as shown in *Fig.i*, then harmonic index of T_i is as follows, where $1<i<5$.

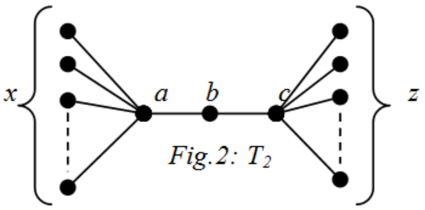

Fig.2: T_2

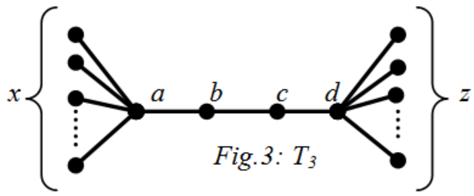

Fig.3: T_3

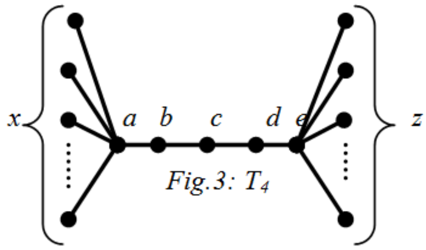

Fig.3: T_4

(i) $H(T_2) = \dfrac{2x}{x+2} + \dfrac{2(n-x-3)}{n-x-1} + \dfrac{2}{x+3} + \dfrac{2}{n-x}.$

(ii) $H(T_3) = \dfrac{1}{2} + \dfrac{2x}{x+2} + \dfrac{2(n-x-4)}{n-x-2} + \dfrac{2}{x+3} + \dfrac{2}{n-x-1}.$

(iii) $H(T_4) = 1 + \dfrac{2x}{x+2} + \dfrac{2(n-x-5)}{n-x-3} + \dfrac{2}{x+3} + \dfrac{2}{n-x-2}.$

The proof of Proposition 2.2 is analogous to the proof of the Proposition 2.1.

Proposition 2.3: If T_5 is a graph with n vertices and m edges as shown in *Fig.5*, then harmonic index of T_5 is

$$H(T_5) = \frac{2x}{x+2} + \frac{2y}{y+3} + \frac{2(n-x-y-3)}{n-x-y-1} + \frac{2}{x+y+3} + \frac{2}{n-x}.$$

\

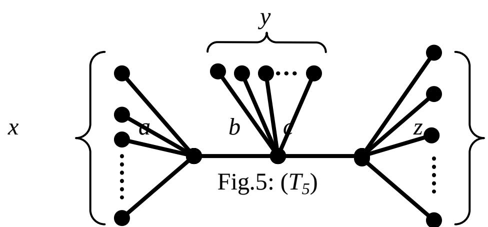

Fig.5: (T_5)

Proof: Without loss of generality consider the vertices a,b,c as shown in *Fig. 5*, where $d_G(a)=x_5+1$, $d_G(b)=y_5+2$, $d_G(c)=z_5+1$. Partition $E(T_5)$ into 5 sets E_1, E_2, E_3, E_4, and E_5 such that $E_1=\{uv / d_G(u)=1 \text{ and } d_G(v)=x_5+1\}$, $E_2=\{uv / d_G(u)=1 \text{ and } d_G(v)=y_5+2\}$, $E_3=\{uv / d\ d_G(u)=1 \text{ and } d_G(v)=z_5+1\}$, $E_4=\{ab\}$, $E_5=\{bc\}$. It is easy to see that $|E_1|=x$, $|E_2|=y$, $|E_3|=z$, $|E_4|=|E_5|=1$ and $|E_1| + |E_2| + |E_3| + |E_4| + |E_5| = m$.

$$H(T_5) = \sum_{uv\in E(T_5)} \frac{2}{d_G(u)+d_G(v)}$$

$$= \sum_{uv\in E_1(T_5)} \frac{2}{d_G(u)+d_G(v)} + \sum_{uv\in E_2(T_5)} \frac{2}{d_G(u)+d_G(v)} + \sum_{uv\in E_3(T_5)} \frac{2}{d_G(u)+d_G(v)}$$

$$+\sum_{uv\in E_4(T_5)}\frac{2}{d_G(u)+d_G(v)}+\sum_{uv\in E_5(T_5)}\frac{2}{d_G(u)+d_G(v)}$$

$$=\sum_{uv\in E_1(T_5)}\frac{2}{(x+1)+1}+\sum_{uv\in E_2(T_5)}\frac{2}{(y+2)+1}+\sum_{uv\in E_3(T_5)}\frac{2}{(z+1)+1}$$

$$+\sum_{uv\in E_4(T_5)}\frac{2}{(x+1)+(y+2)}+\sum_{uv\in E_5(T_5)}\frac{2}{(y+2)+(z+1)}$$

$$= x\frac{2}{x+2}+y\frac{2}{y+3}+z\frac{2}{z+2}+\frac{2}{x+y+3}+\frac{2}{y+z+3}$$

Here we have $n=x+y+z+3$, by replacing z value in above equation we have

$$H(T_5)=\frac{2x}{x+2}+\frac{2y}{y+3}+\frac{2(n-x-y-3)}{n-x-y-1}+\frac{2}{x+y+3}+\frac{2}{n-x}$$

Proposition 2.4: If T_i is a graph with n vertices and m edges as shown in *Fig. i*, then harmonic index of T_i is as follows, where $5<i\leq 8$..

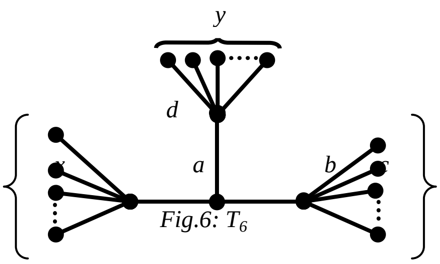

Fig.6: T_6

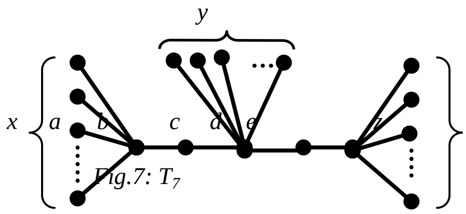

Fig.7: T_7

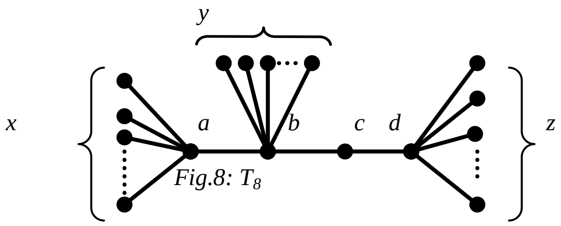

Fig.8: T_8

(i) $H(T_6)=\frac{2x}{x+2}+\frac{2y}{y+2}+\frac{2(n-x-y-4)}{n-x-y-2}+\frac{2}{n-x-y}+\frac{2}{x+4}+\frac{2}{y+4}.$

(ii) $H(T_7)=\frac{2x}{x+2}+\frac{2y}{y+3}+\frac{2(n-x-y-5)}{n-x-y-3}+\frac{2}{n-x-y-2}+\frac{2}{x+3}+\frac{4}{y+4}.$

(iii) $H(T_8)=\frac{2x}{x+2}+\frac{2y}{y+3}+\frac{2(n-x-y-4)}{n-x-y-2}+\frac{2}{n-x-y-1}+\frac{2}{x+y+3}+\frac{2}{y+4}.$

The proof of Proposition 2.4 is analogous to the proof of the Proposition 2.3.

Proposition 2.5: Let T_1 and T_j are the graphs as shown in the *Fig. 1* and *Fig. j* with n_1,n_2 vertices respectively. If each pendent vertex of T_1 is attached to the any one pendent vertex of T_j as shown in *Fig.9*, where $j=1,2,3\ldots,8$ then harmonic index of $T_1 \leftrightarrow T_j$

When $j=1$ is,

$$H(T_1\leftrightarrow T_1)=\left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1}+(n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+z+2}+\frac{2z}{z+2}\right]_{T_1}.$$

When $j=2$ is,

$H(T_1 \leftrightarrow T_2)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+3}+\frac{2}{z+3}+\frac{2z}{z+2}\right]_{T_2}.$

When $j=3$ is,

$H(T_1 \leftrightarrow T_3)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+3}+\frac{1}{2}+\frac{2}{z+3}+\frac{2z}{z+2}\right]_{T_3}.$

When $j=4$ is,

$H(T_1 \leftrightarrow T_4)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[1+\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+3}+\frac{2}{z+3}+\frac{2z}{z+2}\right]_{T_4}.$

When $j=5$ is,

$H(T_1 \leftrightarrow T_5)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+y+3}+\frac{2}{y+z+3}+\frac{2y}{y+3}+\frac{2z}{z+2}\right]_{T_5}.$

When $j=6$ is,

$H(T_1 \leftrightarrow T_6)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+4}+\frac{2}{z+4}+\frac{2}{y+4}+\frac{2y}{y+2}+\frac{2z}{z+2}\right]_{T_6}.$

When $j=7$ is,

$H(T_1 \leftrightarrow T_7)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+3}+\frac{4}{y+4}+\frac{2}{z+3}+\frac{2y}{y+3}+\frac{2z}{z+2}\right]_{T_7}.$

When $j=8$ is,

$H(T_1 \leftrightarrow T_7)= \left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1} + (n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+y+3}+\frac{2}{y+4}+\frac{2}{z+3}+\frac{2y}{y+3}+\frac{2z}{z+2}\right]_{T_8}.$

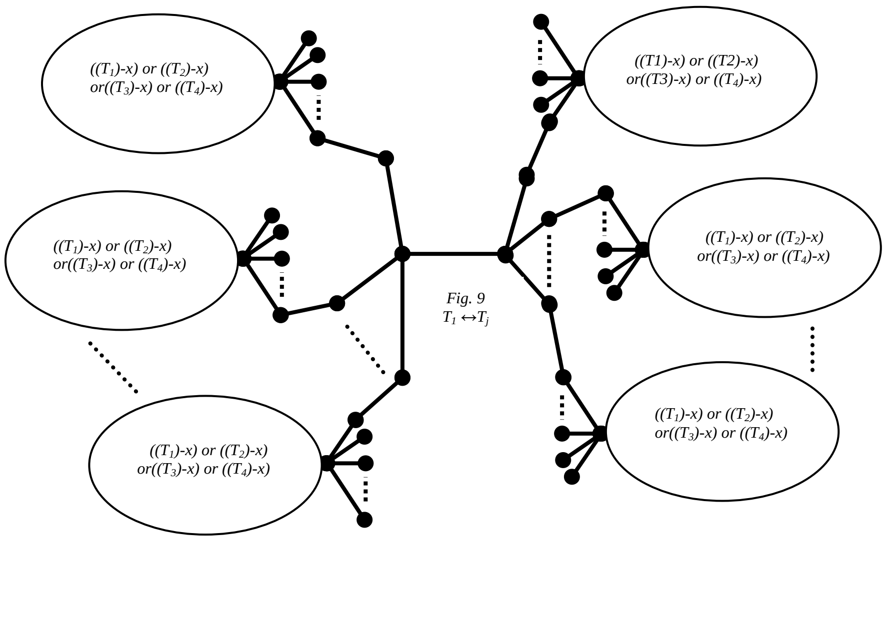

Fig. 9
$T_1 \leftrightarrow T_j$

Proof: Consider the vertices a, b as shown in *Fig. 1*, where $d_G(a)=x+1$, $d_G(b)=n-x-1=z+1$. Partition $E(T_1 \leftrightarrow T_1)$ into $E_1=\{uv / d_G(u)=2 \text{ and } d_G(v)=x+1\}$, $E_2=\{uv / d_G(u)=2 \text{ and } d_G(v)=z+1\}$, $E_3=\{ab\}$, $E_4=\{uv / d_G(u)=2 \text{ and } d_G(v)=2\}$, $E_5=\{uv / d_G(u)=2 \text{ and } d_G(v)=x+1\}$, $E_6=\{uv / d_G(u)=1 \text{ and } d_G(v)=x+1\}$, $E_7=\{uv / d_G(u)=1 \text{ and } d_G(v)=z+1\}$. It can be varify that $|E_1|=x$, $|E_2|=n-x-2=z$, $|E_3|=1$, $|E_4|=n-2$, $|E_5|=1$, $|E_6|=x-1$, $|E_7|=z$ and we attached the (n-2) copies of T_1 to a each pendent of vertex of T_1, therefore cardinality of E_5, E_6, E_7 is multiplied by $n-2$ times.

$$H(T_5) = \sum_{uv\in E(T_1 \leftrightarrow T_1)} \frac{2}{d_G(u)+d_G(v)}$$

$$= \sum_{uv\in E_1(T_1 \leftrightarrow T_1)} \frac{2}{d_G(u)+d_G(v)} + \sum_{uv\in E_2(T_1 \leftrightarrow T_1)} \frac{2}{d_G(u)+d_G(v)} + \sum_{uv\in E_3(T_1 \leftrightarrow T_1)} \frac{2}{d_G(u)+d_G(v)}$$

$$+\sum_{uv\in E_4(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{d_G(u)+d_G(v)}+\sum_{uv\in E_5(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{d_G(u)+d_G(v)}+\sum_{uv\in E_6(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{d_G(u)+d_G(v)}$$

$$+\sum_{uv\in E_7(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{d_G(u)+d_G(v)}$$

$$=\sum_{uv\in E_1(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{2+x+1}+\sum_{uv\in E_2(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{2+z+1}+\sum_{uv\in E_3(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{x+1+z+1}$$

$$+\sum_{uv\in E_4(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{2+2}+\sum_{uv\in E_5(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{2+x+1}+\sum_{uv\in E_6(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{1+x+1}+\sum_{uv\in E_7(\mathrm{T}_1\leftrightarrow \mathrm{T}_1)}\frac{2}{1+z+1}$$

Here x,z which belongs to E_1, E_2, E_3 are from the graph T_1 and x,z which belongs to $E5$, E_6, E_7 are from the graph $\mathrm{T}_{j.}$

$$=\left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1}+\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+z+2}+\frac{2z}{z+2}\right]_{T_1}.$$

Hence we attached (n-2) copies, therefore

$$H(T_1\leftrightarrow T_1)=\left[\frac{2}{x+z+2}+\frac{2x}{x+3}+\frac{2z}{z+3}+\frac{2(n-2)}{4}\right]_{T_1}+(n-2)_{T_1}\left[\frac{2}{x+3}+\frac{2(x-1)}{x+2}+\frac{2}{x+z+2}+\frac{2z}{z+2}\right]_{T_1}.$$

The proof when j=2, 3…,8 follows the proof of j=1. Hence we can have the results.

III. Algorithm

Step 1: START

Step 2: Declare: a[25][25], d[25], m as integers
sum1, s[25], sum, ts=0 as floating points
.

Step 3: Read m, a[i][j].

Step 4: Compute : Degree of each vertex of given graph
for i to m
d[i] ⟵ 0
for j to m
d[i] ⟵ d[i]+a[i][j]
Display: Degree d[i] of vertex i

Step 5: Check the condition, if a[i][j]=1 is true
Display: Vertex i is adjacent to vertex j
sum ⟵ d[i]+d[j]

Step 6: Display the sum of adjacent vertices degree
ts ⟵ ts+(2/sum)

Step 7: Display the sum Harmonic Index by dividing total sum ts by 2.

Step 8: STOP

Illustration:

G:

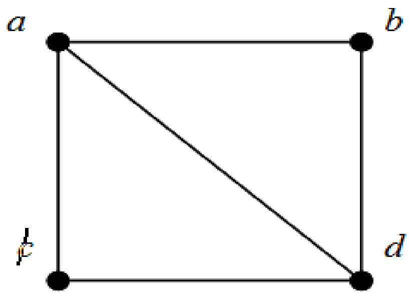

Fig. 10

We represent the graph G by adjacency matrix,(using for loop) ie.,

$$A(G) = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

In this matrix a,b,c,d represents the vertex of graph G and each elements of matrix $A(G)$ represents the adjacency of corresponding vertex in graph G, addition of each row elements gives the degree of corresponding vertex in G, ie., adding all the elements of a first row of $A(G)$ matrix gives the degree of vertex '*a*' in graph G. Using this we calculate each vertex degree and store it in $d[i]$ by using for loop. The outer loop iterates i times and the inner loop iterates j times, the statements inside the inner loop will be executed a total of $i*j$ times. It's because inner loop will iterate j times for each of the i iterations of the outer loop. This means the outer and inner loop are dependent on the problem size ie., here we considered size is n, the statement in the whole loop will be executed $O(m^2)$ times. In the loop int $i=0$, this will be executed only once. The time is actually calculated to $i=0$ and not the declaration, $i<m$ this will be executed $m+1$ times. $i++$ will be executed m times, $a[i][j]=1$, This will be executed m times (in worst case scenario).

As Harmonic Index definition we only sum the degree of vertices which are adjacent, by $A(G)$ matrix we check the adjacency of one vertex to another by using if condition, then we sum the degree of those adjacency vertices using $d[i]$, (This loop follows same procedure as explained for above loop so this also executed O(m^2) times), then we store resulting sum in one variable say ts, finally we obtain harmonic index dividing ts by 2.

IV. Conclusion

The results gives explicit formula for harmonic Index of certain class of trees and further an algorithm with the help of adjacency matrix given to compute the harmonic index of any graph.

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