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Estimation of a Mixture of Two Weibull Distributions under Generalized Order Statistics

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Abstract:*This paper deals with the estimation of the parameters, reliability and hazard rate functions of the mixture of two Weibull distributions (MTWD), with a common shape parameter, based on the generalized order statistics (GOS). The maximum likelihood and Bayes methods of estimation are used for this purpose with standard errors and credible intervals. The Markov Chain Monte Carlo (MCMC) method is used for obtaining Bayes estimates under the squared error loss function. Our results are specialized to progressive Type II censoring and Type II censoring. Comparisons are made between Bayesian and maximum likelihood estimates and between the two censoring types, progressive Type II censoring and Type II censoring. A real data set is used for illustration purpose.*

Keywords: *Mixture of two Weibull distributions, generalized order statistics (GOS), Maximum likelihood estimation, Bayesian estimation, Markov Chain Monte Carlo*

I. Introduction

The finite mixture of distributions have multiple uses in a various scientific fields such as physics, biology, medicine and industrial engineering, among others. In fact, in life testing, each failure occurs may not has one type of failure, it could be categorized to more than one type. Moreover, the failure time populations could be heterogeneous since it could be consisting of weak components corresponding to short lives and strong components corresponding to long lives. Mixtures of Weibull distributions are widely used to model lifetime data. That is because of the flexibility of the Weibull distribution in modeling both increasing and decreasing failure rates. This flexibility mainly depends on the shape parameter; which led us, when estimating the distribution parameters, to let the value of this parameter to be known and controlled by the researcher carrying out the application. Where, it could be selected according to the type of failure rate that fits the data. For the increasing failure rate data, the shape parameter can be set to a value greater than one. While, for the decreasing failure rate data, the shape parameter can be set to a valueless than one; but for the constant failure rate data, the value of the parameter will be equal to one. This study is concerned with studying the finite mixture of two Weibull distributions, denoted by MTWD, as a lifetime model with two unknown scale parameters and a common known shape parameter.

Some of the most important references that discussed different types of mixtures of distributions are the monographs by [1], [2], [3] and [4]. [5]considered Bayesian estimation of the mixing parameter, mean and reliability function of a mixture of two exponential lifetime distributions based on right censored samples.[6] studied and compared classical and Bayesian estimates of the parameters of a finite mixture of two Gomperetz lifetime models based on simulated data sets with Type I and Type II censoring. [7]obtained Bayesian predictive density of order statistics based on finite mixture models. Based on Type I censored samples from a finite mixture of two truncated Type I generalized logistic components,[8] computed Bayes estimates of parameters, reliability and hazard rate functions.[9]considered estimation for the parameters of mixed exponential distribution based on record statistics.[10]considered Bayes inference under a finite mixture of two compound Gompertz components model. [11]considered estimation forthe parameters of mixture of two component exponentiated gamma distribution. [12]applied the order statisticson a mixture model of exponentiated Rayleigh and exponentiated exponential distributions. [13]estimated the parameters of a two-parameter weighted Lindley distribution based on hybrid censoring.[14]introduced the finite mixture of two exponentiatedKumaraswamy distributions.

Mixtures of Weibull distributions are widely used to model lifetime data and they have been considered extensively by many authors, [15] estimated the parameters of a mixture Weibull distribution using MLE and Bayes estimation under Type I censoring. [16]studied the classification of the aging properties of generalized mixtures of two or three Weibull distributions in terms of the mixing weights, scale parameters and a common shape parameter. [17]estimated the parameters from the mixture of two Weibull distributions under the informative and non-informative priors they also determined, the Bayes predictive intervals. A mixture of two and three Weibull distributions were used to analyze the data of failure times [18]. [19]proposed a mixture Weibull proportional hazards model to predict the failure of a mechanical system with multiple failure modes.

II. Model Description

The considered MTWD, which is produced from mixing two Weibull distributions, the first with two parameters, the scale parameter $\alpha_1 > 0$, and the shape parameter, $\beta > 0$ and the second also has the scale parameter $\alpha_2 > 0$, and the shape parameter, $\beta > 0$.That is the scale parameters of the MTWD are α_1 and α_2 and the common shape parameteris β, which is assumed as a known parameter, its value could be selected (greater, less or equals 1) according to the type of the failure rate that fits the data set under the study.

The probability density function(PDF) of MTWDis given as

$$f(t) = pf_1(t) + (1-p)f_2(t) \ , \qquad t > 0 \tag{1}$$

where $0 \le p \le 1$,and the PDF of the j^{th} component, $j = 1,2$, is

$$f_j(t) = \frac{\beta t^{\beta-1}}{\alpha_j{}^{\beta}} exp\left[-\left(\frac{t}{\alpha_j}\right)^{\beta}\right], \qquad t > 0. \tag{2}$$

The cumulative distribution function(CDF), reliability function(RF) and hazard function (HF) of the MTWD are,respectively, given by

$$F(t) = p\left\{1 - exp\left[-\left(\frac{t}{\alpha_1}\right)^{\beta}\right]\right\} + (1-p)\left\{1 - exp\left[-\left(\frac{t}{\alpha_2}\right)^{\beta}\right]\right\}, \tag{3}$$

$$R(t) = p\left\{exp\left[-\left(\frac{t}{\alpha_1}\right)^{\beta}\right]\right\} + (1-p)\left\{exp\left[-\left(\frac{t}{\alpha_2}\right)^{\beta}\right]\right\}. \tag{4}$$

$$H(t) = \left(\frac{1}{1+g(t)}\right)H_1(t) + \left(1 - \frac{1}{1+g(t)}\right)H_2(t), \tag{5}$$

where,

$$g(t) = \frac{(1-p)R_2(t)}{pR_1(t)},$$

$$R_j(t) = exp\left[-\left(\frac{t}{\alpha_j}\right)\right], \ \ H_j(t) = \frac{\beta t^{\beta-1}}{\alpha_j{}^{\beta}}, \qquad j = 1,2.$$

The mean, variance, skewness and kurtosis of the MTWDare computed for different values of the parameters and are given in Table (1).

Table (1): Mean, variance, skewness and kurtosis of MTWD

$\underline{\theta} = (p,\beta,\alpha_1,\alpha_2)$	Mean	variance	skewness	kurtosis
$\underline{\theta} = (0.3,1,0.5,1.5)$	1.2000	1.8600	2.3558	11.1779
$\underline{\theta} = (0.3,2,0.5,1.5)$	1.0634	0.5190	0.8872	3.3878
$\underline{\theta} = (0.3,3.5,0.5,1.5)$	1.0796	0.3037	0.1918	2.0173
$\underline{\theta} = (0.1,3,1,2)$	1.6966	0.4614	0.2132	2.5954
$\underline{\theta} = (0.5,3,1,2)$	1.3394	0.4626	0.7546	3.0271
$\underline{\theta} = (0.8,3,1,2)$	1.0715	0.2961	1.3503	5.3735
$\underline{\theta} = (0.6,3,1.5,2.8)$	1.8038	0.7959	0.8862	3.5089
$\underline{\theta} = (0.6,3,2,2.8)$	2.0717	0.7056	0.5122	3.2050
$\underline{\theta} = (0.6,3,3.5,2.8)$	2.8753	1.1982	0.3016	2.8712
$\underline{\theta} = (0.7,2,1.5,1.1)$	1.2229	0.4422	0.7458	3.5087
$\underline{\theta} = (0.7,2,1.5,2.8)$	1.6749	1.1214	1.3140	5.3765
$\underline{\theta} = (0.7,2,1.5,3.4)$	1.8344	1.6776	1.5685	6.0598

Table (1) shows that, when the parameters (p,α_1,α_2) are fixed, then the relationship between the shape parameter $\beta$and each of the skewness and kurtosis is inverse. However, when $(\beta,\alpha_1,\alpha_2)$ are fixed we have positive relationship between the mixing proportion parameter $p$and each of the skewness and kurtosis. With the fixed(p,β,α_2) we have negative relationship between scale parameter $\alpha_1$and each of the skewness and kurtosis. With the fixed (β,p,α_1) we have positive relationship between scale parameter $\alpha_2$and skewness and kurtosis.

III. Maximum Likelihood Estimation

Let $T_{1;n,\tilde{m},k}, T_{2;n,\tilde{m},k}, \ldots, T_{n;n,\tilde{m},k}$, $k > 0$, $\tilde{m} = (m_1, \ldots, m_{n-1}) \in \Re^{n-1}, m_1, \ldots, m_{n-1} \in \Re$, are n GOS drawn from the MTWD. The likelihood function (LF) is given in [20], for $-\infty < t_1, \ldots, t_n < \infty$, by

$$L(\theta|t) = k \prod_{i=1}^{n-1} \gamma_i \left\{ \prod_{i=1}^{n-1} [R(t_i)]^{m_i} f(t_i) \right\} [R(t_n)]^{k-1} f(t_n), \tag{6}$$

where, $t = (t_1, \ldots, t_n), \theta \epsilon \Theta$ is parameter space, and

$$\gamma_i = k + n - i + M_i > 0, M_i = \sum_{v=1}^{n-1} m_v.$$

Substituting (1) and (4), in (6), then the likelihood function takes the form of

$$L(\theta|t) \propto [pR_1(t_n) + (1-p)R_2(t_n)]^{k-1} \prod_{i=1}^{n-1} [pR_1(t_i) + p_2 R_2(t_i)]^{m_i} \prod_{i=1}^{n} [(1-p)f_1(t_i) + p_2 f_2(t_i)]. \tag{7}$$

Taking the logarithm of (7), to obtain

$$\begin{aligned} \ell(\theta) = ln\, L(\theta|t) &\propto (k-1)\, ln[pR_1(t_n) + (1-p)R_2(t_n)] \\ &+ \sum_{i=1}^{n-1} m_i\, ln[pR_1(t_i) + (1-p)R_2(t_i)] \\ &+ \sum_{i=1}^{n} ln[pf_1(t_i) + (1-p)f_2(t_i)]\,. \end{aligned} \tag{8}$$

Differentiating (8) with respect to the parameters p and α_j , $j = 1,2.$ and equating to zero gives the following likelihood equations

$$\left.\begin{aligned} \frac{\partial \ell}{\partial p} &= (k-1)\vartheta^*(t_n) + \sum_{i=1}^{n-1} m_i \vartheta^*(t_i) + \sum_{i=1}^{n} \vartheta(t_i) = 0 \\ \frac{\partial \ell}{\partial \alpha_1} &= p\left\{(k-1)\psi^*{}_1(t_n) + \sum_{i=1}^{n} \xi_1(t_i)\,\psi_1(t_i) + \sum_{i=1}^{n-1} m_i \psi^*{}_1(t_i)\right\} = 0 \\ \frac{\partial \ell}{\partial \alpha_2} &= (1-p)\left\{(k-1)\psi^*{}_2(t_n) + \sum_{i=1}^{n} \xi_2(t_i)\,\psi_2(t_i) + \sum_{i=1}^{n-1} m_i \psi^*{}_2(t_i)\right\} = 0 \end{aligned}\right\}. \tag{9}$$

Where, for *j = 1, 2*

$$\left.\begin{aligned} &\vartheta(t_i) = \frac{f_1(t_i) - f_2(t_i)}{pf_1(t_i) + (1-p)f_2(t_i)}\ , \quad \vartheta^*(t_i) = \frac{R_1(t_i) - R_2(t_i)}{pR_1(t_i) + (1-p)R_2(t_i)} \\ &\psi_j(t_i) = \frac{f_j(t_i)}{pf_1(t_i) + (1-p)f_2(t_i)}\ , \quad \psi^*{}_j(t_i) = \frac{R_j(t_i)\beta t_i{}^{\beta} \alpha_j{}^{-\beta-1}}{pR_1(t_i) + (1-p)R_2(t_i)} \\ &\xi_j(t_i) = \left[\frac{-\beta}{\alpha_j} + \beta t_i{}^{\beta} \alpha_j{}^{-\beta-1}\right] \end{aligned}\right\}$$

The solution of the three nonlinear likelihood equations in (9) using numerical method, yields the maximum likelihood (ML) estimates $\hat{p}, \hat{\alpha}_1$ and $\hat{\alpha}_2$. The ML estimates of the $R(t)$ and the $H(t)$ are given, respectively, by (4) and (5) after replacing $\underline{\theta} = (p, \alpha_1, \alpha_2)$ by their corresponding ML estimates.

IV. Bayesian Estimation

This section deals with Bayesian using conjugate and non-informative priors.

4.1 Bayesian estimation using conjugate prior

Let p, α_1 and α_2 are independent random variables such that p follows $Beta(b_1, b_2)$ and for $j = 1, 2, \alpha_j$ to follow an inverted gamma prior distribution with PDFs, respectively, given by

$$\pi(p) = \frac{1}{\beta(b_1, b_2)} (p)^{b_1 - 1}(1-p)^{b_2 - 1}\ , \qquad 0 \le p \le 1, \qquad b_1, b_2 > 0.$$

$$\pi(\alpha_j) = \frac{{\gamma_j}^{\tau_j}}{\Gamma\tau_j}\alpha_j^{-\tau_j-1}e^{-\frac{\gamma_j}{\alpha_j}}, \quad \alpha_j > 0,$$

where $j = 1, 2$ $\alpha_j > 0$, $(b_j, \tau_j, \gamma_j) > 0$.

A joint prior density function of $\underline{\theta} = (p, \alpha_1, \alpha_2)$ is then given by

$$\pi(\underline{\theta}) = \pi(p)\pi(\alpha_1)\pi(\alpha_2).$$

$$\pi(\underline{\theta}) \propto (p)^{b_1-1}(1-p)^{b_2-1}\prod_{j=1}^{2}\alpha_j^{-\tau_j-1}e^{-\sum_{j=1}^{2}\frac{\gamma_j}{\alpha_j}}. \qquad (10)$$

It follows, from (10) and (7), that the joint posterior density function is given by

$$\pi^*(\underline{\theta}|\underline{t}) = A_1^{-1}p^{b_1-1}(1-p)^{b_2-1}\prod_{j=1}^{2}\alpha_j^{-\tau_j-1}e^{-\sum_{j=1}^{2}\frac{\gamma_j}{\alpha_j}}[pR_1(t_n) + (1-p)R_2(t_n)]^{k-1}$$
$$\prod_{i=1}^{n-1}[pR_1(t_i) + (1-p)R_2(t_i)]^{m_i}\prod_{i=1}^{n}[pf_1(t_i) + (1-p)f_2(t_i)], \qquad (11)$$

where,

$$A_1^{-1} = \int_\theta L(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}.$$

Under the squared error loss function (SE), the Bayes estimator of a function, say $\varphi \equiv \varphi(p, \alpha_1, \alpha_2)$, is given by

$$\hat{\varphi}_{Bs} = E(\varphi|\underline{t}) = \int_{\underline{\theta}} \varphi\pi^*(\underline{\theta}|\underline{t})d\theta, \qquad (12)$$

where the integral is taken over the three dimensional space. To compute the integral we propose to consider MCMC methods.

The conditional posterior distribution of the parameters p, α_1 and α_2 using conjugate prior can be computed and written, respectively, by

$$\pi^*(p|\alpha_1, \alpha_2, \underline{t}) \propto (p)^{b_1-1}(1-p)^{b_2-1}[pR_1(t_n) + (1-p)R_2(t_n)]^{k-1}$$
$$\prod_{i=1}^{n-1}[pR_1(t_i) + (1-p)R_2(t_i)]^{m_i}\prod_{i=1}^{n}[pf_1(t_i) + (1-p)f_2(t_i)].$$

$$\pi^*(\alpha_i|p, \alpha_j, \underline{t}) \propto \alpha_i^{-\theta_1-1}e^{-\frac{\gamma_1}{\alpha_i}}[pR_1(t_n) + (1-p)R_2(t_n)]^{k-1}$$
$$\prod_{i=1}^{n-1}[pR_1(t_i) + (1-p)R_2(t_i)]^{m_i}\prod_{i=1}^{n}[pf_1(t_i) + (1-p)f_2(t_i)],$$

where, $i \neq j$ (i and $j = 1,2$).

4.2 Bayesian estimation using non-informative prior

Assuming that all of the parameters consisting θ are positive and independent, and that we are ignoring the prior information about θ so that we set improper non-informative prior to $\alpha_j, j = 1, 2$, and p as follows

$$\pi(p) \propto 1, \quad \pi(\alpha_1) \propto \frac{1}{\alpha_1}, \quad \pi(\alpha_2) \propto \frac{1}{\alpha_2}.$$

Hence, the joint prior density function of $\underline{\theta} = (p, \alpha_1, \alpha_2)$ is then given by

$\pi(\underline{\theta}) = \pi_1(p)\pi_2(\alpha_1)\pi_3(\alpha_2)$.

$$\pi(\underline{\theta}) \propto \prod_{j=1}^{2}(\alpha_j)^{-1}. \qquad (13)$$

For $j = 1,2, \alpha_j > 0$, it follows, from (13) and (7), that the joint posterior density function is given by

$$\pi^*(\underline{\theta}|\underline{t}) = A_2^{-1}\prod_{j=1}^{2}\alpha_j^{-1}[pR_1(t_n) + (1-p)R_2(t_n)]^{k-1}$$
$$\prod_{i=1}^{n-1}[pR_1(t_i) + (1-p)R_2(t_i)]^{m_i}\prod_{i=1}^{n}[pf_1(t_i) + (1-p)f_2(t_i)], \qquad (14)$$

where,

$$A_2^{-1} = \int_\theta L(\underline{\theta}|\underline{t})\pi(\underline{\theta})d\underline{\theta}.$$

Under the squared error loss function (SE), the Bayes estimator of a function, say $\varphi \equiv \varphi(p, \alpha_1, \alpha_2)$, is given in (12) where the integral is taken over the three dimensional space. To compute the integral we propose to consider MCMC methods.

The conditional posterior distribution of the parameters p, α_1 and α_2 using conjugate prior can be computed and written, respectively, by

$$\pi^*(p|\alpha_1, \alpha_2, \underline{t}) \propto [pR_1(t_n) + (1-p)R_2(t_n)]^{k-1} \prod_{i=1}^{n-1} [pR_1(t_i) + (1-p)R_2(t_i)]^{m_i} \prod_{i=1}^{n} [pf_1(t_i) + (1-p)f_2(t_i)]$$

$$\pi^*(\alpha_i|p, \alpha_j, \underline{t}) \propto \alpha_i^{-1} [pR_1(t_n) + (1-p)R_2(t_n)]^{k-1} \prod_{i=1}^{n-1} [pR_1(t_i) + (1-p)R_2(t_i)]^{m_i} \prod_{i=1}^{n} [pf_1(t_i) + (1-p)f_2(t_i)],$$

where, $i \neq j (i \text{and} j = 1,2)$.

V. Numerical Computations

In this section, the computations regarding the comparisons are performed and the MTWD with a common parameter $\beta = 1.5$ is considered. A comparison between the ML and Bayesian estimates, under the squared error loss function (SE), is made by using Monte Carlo simulation study (based on progressive Type II censoring and Type II censoring) which is considered in subsection 5.1 and a real data set is considered in subsection 5.2.

5.1 Mont Carlo simulation

(1) Based on progressive Type II censored samples

The estimation of the parameters (p, α_1, α_2) using ML and Bayesian techniques on cases when parameter β is known have been obtained, based of progressive Type II censored sample which is special case of GOS that can be derived from it, by choosing

$$\left.\begin{aligned} m_i &= R_i \,; i = 1,2, \dots, m-1 \\ \gamma_i &= R_r + 1 \end{aligned}\right\}. \qquad (15)$$

Therefore, the ML and Bayes estimates of the unknown parameters can be obtainedby substituting (15) in (6),and then the likelihood reduces to the following

$$L(\theta|t) \propto \prod_{i=1}^{m} [pR_1(t_i) + (1-p)R_2(t_i)]^{R_i} [pf_1(t_i) + (1-p)f_2(t_i)].$$

We compare the performance of the ML and Bayes estimates with their standard deviation (SD), bias, mean squared errors(MSE) and the width of 95 % asymptotic confidence intervals (CI) and the width of 95% highest posterior density (HPD) intervals. The simulation is performed for different choices of the sample sizes n and censored sample sizes$m$. The procedure is repeated for 40,000 times. We computed the estimates, their SD, bias, MSE, width of asymptotic CI and the HPD width for the ML method and Bayesian method (based on the MCMC technique). The results are reported in Tables (2) and (3). The HPD confidence widths are smaller than the widths of asymptotic CI. It seems plausible to say that the performances of the ML estimates and Bayesian estimates are very comparable in most results. The SD is computed by take the square root of the variance; however the bias is computed by take the absolute value of the difference between true value ofθ and its estimate$\hat{\theta}$ and MSE is computed by add the variance to the bias squared

It is clear from Tables (2) and (3) that the computed estimates of the parameters p , α_1 and α_2 are better when the sample size increases. Since, the MSE and bias are small. One can see that the estimates of the complete samples are better than the censored ones. In addition comparing the censored samples for the considered sample sizes, the ML and Bayesian estimates have smaller bias, MSE and interval width when the censoring is large. That is the estimates are better than the corresponding estimates when the censoring is large. In general, the computed bias and MSE of Bayesian estimates are smaller than there corresponding for ML estimates.The estimates of the reliability and hazard rate functions are found after substituting the parameters estimates in their functions and fixed time.

Table (2): Classical and Bayesian estimates of the parameters, RF and HF at $t_0 = 0.5$ with their SD, bias and MSE for progressive Type II censored samples from the MTWD with parameters $p = 0.5$, $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$

n	*m*	$\underline{\theta}$	ML (SD) bias (MSE)	Bayesian	
				CP (SD) bias (MSE)	NP (SD) bias (MSE)
30	30	p	0.3967 (0.1448) 0.1033 (0.0316)	0.5002 (0.0009) 0.0002 (0.0000008)	0.5022 (0.0022) 0.0022 (0.0000096)
		α_1	0.0720 (0.0284) 0.028 (0.0015)	0.0958 (0.0019) 0.0042 (0.000021)	0.0996 (0.0009) 0.0004 (0.00000097)
		α_2	0.5263 (0.1254) 0.1737 (0.0458)	0.6990 (0.0009) 0.001 (0.00000181)	0.6988 (0.0006) 0.0012 (0.0000018)
		$R(t_0)$	0.1451	0.1711	0.1704
		$H(t_0)$	2.6195	1.7162	1.7182
	25	p	0.3812 (0.1114) 0.1188 (0.0265)	0.4964(0.0016) 0.0036 (0.0000155)	0.4975 (0.0014) 0.0025 (0.0000082)
		α_1	0.1080 (0.0323) 0.008 (0.0011)	0.1002 (0.0022) 0.0002 (0.0000048)	0.10001 (0.0010) 0.00001 (0.000001)
		α_2	0.9984 (0.2383) 0.2984 (0.1458)	0.6991 (0.0015) 0.0009 (0.000003)	0.7003 (0.0010) 0.0003 (0.0000010)
		$R(t_0)$	0.2923236	0.1725109	0.1724
		$H(t_0)$	1.012017	1.717216	1.7125
50	50	p	0.5192 (0.0971) 0.0192 (0.0097)	0.4990 (0.0007) 0.001 (0.0000014)	0.4995 (0.0011) 0.0005 (0.00000146)
		α_1	0.0660 (0.0151) 0.034 (0.00138)	0.1042 (0.0026) 0.0042 (0.0000244)	0.1014 (0.0010) 0.0014 (0.0000029)
		α_2	0.5486 (0.1086) 0.1514 (0.03471)	0.7017 (0.0011) 0.0017 (0.0000041)	0.6983 (0.0008) 0.0017 (0.0000035)
		$R(t_0)$	0.1225	0.1724	0.1712
		$H(t_0)$	2.4609	1.7090	1.7205
	25	p	0.4148 (0.0752) 0.0851 (0.0129)	0.5004 (0.0010) 0.0004 (1.3e-06)	0.5011 (0.0008) 0.0011 (2.1e-06)
		α_1	0.1569 (0.0413) 0.0569 (0.0049)	0.0964 (0.0016) 0.0035 (1.56e-05)	0.0974 (0.0010) 0.0025 (7.5e-06)
		α_2	3.8578 (1.3268) 3.1578 (11.7326)	0.6967 (0.0026) 0.0032 (1.7e-05)	0.6969 (0.0018) 0.0030 (1.2e-05)
		$R(t_0)$	0.4852	0.1704	0.1702
		$H(t_0)$	0.1905	1.7245	1.7242
100	100	p	0.4506 (0.0994) 0.0494 (0.01232)	0.4998 (0.0014) 0.0002 (0.000002)	0.4989 (0.0009) 0.0011 (0.0000020)
		α_1	0.0823 (0.0188) 0.0177 (0.00066)	0.1016 (0.0013) 0.0016 (0.0000042)	0.1021 (0.0017) 0.0021 (0.0000073)
		α_2	0.4530 (0.0689) 0.2470 (0.06576)	0.7017 (0.0012) 0.0017 (0.0000043)	0.6970 (0.0011) 0.0030 (0.0000102)
		$R(t_0)$	0.1049	0.1720	0.1711
		$H(t_0)$	3.2800	1.7081	1.7255
	25	p	0.1726 (0.0344) 0.3273 (0.1083)	0.4977 (0.0012) 0.0022 (6.3e-06)	0.5042 (0.0026) 0.0042 (2.5e-05)
		α_1	0.1486 (0.0368) 0.0486 (0.0037)	0.0968 (0.0011) 0.0031 (1.1e-05)	0.0983 (0.0009) 0.0016 (3.7e-06)
		α_2	3.3370 (0.5472) 2.6370 (7.2533)	0.6981 (0.0019) 0.0018 (7.01e-06)	0.6996 (0.0021) 0.0003 (4.5e-06)
		$R(t_0)$	0.6619	0.1717	0.1699
		$H(t_0)$	0.1843	1.7196	1.7145

(CP and NP denote Bayesian estimates when conjugate and non-informative priors are considered, respectively.)

Table (3):Average 95% confidence/HPD intervals and their width under different progressive Type II censoring for $p = 0.5$, $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$

n	*m*	$\underline{\theta}$	CI [Width]	Bayes HPDC [Width]	HPDN [Width]
30	30	p	(0.1128,0.6806) [0.5678]	(0.4985,0.5017) [0.0031]	(0.4975,0.5051) [0.0075]
		α_1	(0.0162,0.1278) [0.1115]	(0.0928,0.0996) [0.0068]	(0.0979,0.1015) [0.0035]
		α_2	(0.2804,0.7721) [0.4917]	(0.6967,0.7007) [0.0039]	(0.6976,0.7002) [0.0026]
	25	p	(0.3784, 0.7959) [0.4175]	(0.4994,0.5022) [0.0028]	(0.4955,0.5005) [0.0049]
		α_1	(0.0624, 0.1608) [0.0984]	(0.0996,0.1065) [0.0068]	(0.0981,0.1022) [0.0041]
		α_2	(0.3759, 1.7711) [1.3952]	(0.6953,0.7014) [0.0060]	(0.6982,0.7020) [0.0038]
50	50	p	(0.3287,0.7096) [0.3809]	(0.4975,0.5003) [0.0028]	(0.4974,0.5014) [0.0039]
		α_1	(0.0364,0.0956) [0.0591]	(0.1001,0.1087) [0.0085]	(0.0997,0.10330) [0.0035]
		α_2	(0.3357,0.76152) [0.4257]	(0.6996,0.7035) [0.0039]	(0.6968,0.7001) [0.0032]
	25	p	(0.2673,0.5623) [0.2950]	(0.4985,0.5023) [0.0038]	(0.4994,0.5024) [0.0030]
		α_1	(0.0758,0.2379) 0.1621	(0.0941,0.1002) [0.0060]	(0.0954,0.0993) [0.0038]
		α_2	(1.2574,6.4583) [5.2009]	(0.6917,0.7004) [0.0087]	(0.6940,0.7005) [0.0064]
100	100	p	(0.2558,0.6455) [0.3897]	(0.4972,0.5020) [0.0048]	(0.4966,0.5004) [0.0037]
		α_1	(0.0453,0.1193) [0.0740]	(0.0997,0.1043) [0.0046]	(0.0990,0.1052) [0.0062]
		α_2	(0.3179,0.5881) [0.2702]	(0.6996,0.7041) [0.0045]	(0.6948,0.6991) [0.0042]
	25	p	(0.1050,0.2401) [0.1351]	(0.4953,0.5002) [0.0049]	(0.5001,0.5087) [0.0085]
		α_1	(0.0763,0.2209) [0.1445]	(0.0951,0.0992) [0.0041]	(0.0963,0.1001) [0.0037]
		α_2	(2.2644,4.4096) [2.1451]	(0.6946,0.7010) [0.0063]	(0.6951,0.7024) [0.0072]

(HPDC and HPDN denote the HPD intervals when conjugate and non-informative priors are considered respectively.)

(2) Based on Type II censored samples

The estimation of the parameters (p, α_1, α_2) using the ML and Bayesian techniques on cases when the parameter β is known have been obtained, based on Type II censored sample which is special case of GOS, that can be derived from it by choosing

$$\left.\begin{array}{c} m_i = 0 \quad ; i = 1,2, \dots, r-1 \\ k = n - r + 1 \end{array}\right\}. \tag{16}$$

By substituting (16) in (6), the likelihood equations reduced to the following

$$L(\theta|t) \propto [R(t_r)]^{n-r} \left\{\prod_{i=1}^{r} f(t_i)\right\}$$

It is clear from Tables (4) and (5) that the computed estimates of the parameters p, α_1 and α_2 are better when the sample size increases. Since, the MSE and bias are small. One can see that the estimates of the complete samples are better than the censored ones. In addition, comparing the censored samples for the considered sample sizes, the ML and Bayesian estimates have smaller bias, MSE and interval width when the censoring is large. That is the estimates are better than the corresponding estimates when the censoring is small. In general, the computed bias and MSE of Bayesian estimates are smaller than there correspondingcomputedbias and MSE for the ML estimates.

Table (4):Classical and Bayesian estimates of the parameters, RF and HF at $t_0 = 0.5$ with their SD, MSE and bias for Type II censored samples from the MTWD with parameters $p = 0.5$, $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$

n	*m*	$\underline{\theta}$	ML (SD) bias (MSE)	Bayes	
				CP (SD) bias (MSE)	NP (SD) bias (MSE)
30	30	p	0.3707 (0.1270) 0.1293 (0.03285)	0.5002 (0.0008) 0.0002 (0.0000006)	0.4987 (0.0018) 0.0013 (0.0000049)
		α_1	0.0784 (0.0257) 0.0216 (0.00113)	0.0998 (0.0013) 0.0002 (0.0000017)	0.1008 (0.0015) 0.0008 (0.0000029)
		α_2	0.6030 (0.1332) 0.0970 (0.02715)	0.7031 (0.0021) 0.0031 (0.0000140)	0.7032 (0.0022) 0.0032 (0.0000151)
		$R(t_0)$	0.1814	0.1722	0.1728
		$H(t_0)$	2.1364	1.7024	1.7024
	15	p	0.4362 (0.1747) 0.0638 (0.03459)	0.4986 (0.0007) 0.0014 (0.0000025)	0.4989 (0.0009) 0.0011 (0.0000020)
		α_1	0.1156 (0.0472) 0.0156 (0.00247)	0.0967 (0.0023) 0.0033 (0.0000162)	0.1003 (0.0013) 0.0003 (0.0000018)
		α_2	0.6745 (0.2101) 0.0225 (0.04479)	0.7009 (0.0010) 0.0009 (0.0000018)	0.6959 (0.0019) 0.0041 (0.0000204)
		$R(t_0)$	0.1861	0.1721	0.1708
		$H(t_0)$	1.8163	1.7095	1.7290
50	50	p	0.5014 (0.1590) 0.0014 (0.02528)	0.5036 (0.0017) 0.0036 (0.0000159)	0.5012 (0.0018) 0.0012 (0.0000047)
		α_1	0.1271 (0.0432) 0.0271 (0.00260)	0.0985 (0.0018) 0.0015 (0.0000055)	0.1024 (0.0014) 0.0024 (0.0000077)
		α_2	0.6479 (0.1575) 0.0521 (0.02752)	0.6982 (0.0013) 0.0018 (0.0000049)	0.7013 (0.0014) 0.0013 (0.0000037)
		$R(t_0)$	0.1580	0.1697	0.1715
		$H(t_0)$	1.9329	1.7199	1.7098
	30	p	0.4656 (0.1032) 0.0344 (0.01183)	0.5051 (0.0032) 0.0051 (0.0000363)	0.5021 (0.0012) 0.0021 (0.0000059)
		α_1	0.0929 (0.0236) 0.0071 (0.00061)	0.1003 (0.0009) 0.0003 (0.0000009)	0.1030 (0.0013) 0.0030 (0.0000107)
		α_2	0.8005 (0.2076) 0.1005 (0.05319)	0.6999 (0.0012) 0.0001 (0.000015)	0.6966 (0.0018) 0.0034 (0.0000148)
		$R(t_0)$	0.2095	0.1697	0.1700
		$H(t_0)$	1.4006	1.7143	1.7272
100	100	p	0.5075 (0.0828) 0.0075 (0.00069)	0.4966 (0.0013) 0.0034 (0.0000133)	0.4995 (0.0010) 0.0005 (0.0000013)
		α_1	0.0937 (0.0177) 0.0063 (0.000353)	0.0951 (0.0015) 0.0049 (0.0000263)	0.0996 (0.0010) 0.0004 (0.0000012)
		α_2	0.6280 (0.0945) 0.0720 (0.01411)	0.7005 (0.0013) 0.0005 (0.0000019)	0.7014 (0.0018) 0.0014 (0.0000052)
		$R(t_0)$	0.1493	0.1727	0.1720
		$H(t_0)$	2.0132	1.7104	1.7085
	50	p	0.2493 (0.0715) 0.2507 (0.06796)	0.5020 (0.0013) 0.0020 (0.0000057)	0.5017 (0.0012) 0.0017 (0.0000043)
		α_1	0.0783 (0.0225) 0.0217 (0.00098)	0.0972 (0.0014) 0.0028 (0.0000098)	0.1006 (0.0015) 0.0006 (0.0000026)
		α_2	0.6000 (0.0843) 0.1000 (0.01711)	0.70001 (0.0011) 0.00001 (0.0000012)	0.7014 (0.0013) 0.0014 (0.0000037)
		$R(t_0)$	0.2150	0.1707	0.1713
		$H(t_0)$	2.1521	1.7128	1.7091

Table (5): Average 95% confidence/HPD intervals with their width under different Type II censoring for fixed $p = 0.5$, $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$

n	*m*	$\underline{\theta}$	CI [Width]	Bayes	
				HPDC [Width]	HPDN [Width]
30	30	p	(0.1216,0.6198) [0.4981]	(0.4985,0.5016) [0.0030]	(0.4953,0.5014) [0.0060]
		α_1	(0.0280,0.1288) [0.1007]	(0.0977,0.1023) [0.0046]	(0.0985,0.1038) [0.0052]
		α_2	(0.3419,0.8641) [0.5221]	(0.6998,0.7064) [0.0066]	(0.6995,0.7064) [0.0069]
	15	p	(0.0938,0.7787) [0.6848]	(0.4969,0.5000) [0.0031]	(0.4967,0.5007) [0.0039]
		α_1	(0.0229,0.2083) [0.1853]	(0.0919,0.1005) [0.0085]	(0.0979,0.1029) [0.0050]
		α_2	(0.2626,1.0865) [0.8239]	(0.6991,0.7030) [0.0038]	(0.6928,0.6997) [0.0068]
50	50	p	(0.2398,0.7630) [0.5231]	(0.4999,0.5064) [0.0065]	(0.4984,0.5044) [0.0059]
		α_1	(0.0560,0.1982) [0.1421]	(0.0947,0.1013) [0.0065]	(0.0997,0.1047) [0.0050]
		α_2	(0.3888,0.9069) [0.5181]	(0.6956,0.7007) [0.0051]	(0.6991,0.7039) [0.0048]
	30	p	(0.2958,0.6354) [0.3395]	(0.4996,0.5101) [0.0104]	(0.4994,0.5036) [0.0042]
		α_1	(0.0541,0.1318) [0.0777]	(0.0983,0.1019) [0.0035]	(0.0999,0.1049) [0.0050]
		α_2	(0.4590,1.1420) [0.6830]	(0.6977,0.7022) [0.0044]	(0.6945,0.7012) [0.0067]
100	100	p	(0.3712,0.6437) [0.2725]	(0.4945,0.5000) [0.0054]	(0.4969,0.5013) [0.0043]
		α_1	(0.0644,0.1229) [0.0585]	(0.0927,0.0987) [0.0060]	(0.0978,0.1015) [0.0037]
		α_2	(0.4724,0.7835) [0.3111]	(0.6979,0.7026) [0.0046]	(0.6974,0.7035) [0.0060]
	50	p	(0.1317,0.3669) [0.2352]	(0.4998,0.5038) [0.0040]	(0.4996,0.5041) [0.0045]
		α_1	(0.0412,0.1154) [0.0741]	(0.0945,0.0999) [0.0054]	(0.0974,0.1030) [0.0055]
		α_2	(0.4613,0.7387) [0.2773]	(0.6972,0.7024) [0.0051]	(0.6994,0.7040) [0.0046]

5.2 Real data analysis

In this subsection, a real data set has been analyzed. This data set represents the time between failures of secondary reactor pumps. This data set has been originally discussed by [21]and [22] . The chance of the failure of the secondary reactor pump is of the increasing nature in early stage of the experiment and after that it decreases. It has been checked by [23] that flexible Weibull distribution is well fitted model to this data set. The times between failures of 23 secondary reactor pumps are as follows: 2.160, 0.150, 4.082, 0.746, 0.358, 0.199, 0.402, 0.101, 0.605, 0.954, 1.359, 0.273, 0.491, 3.465, 0.070, 6.560, 1.060, 0.062, 4.992, 0.614, 5.320, 0.347, and 1.921.

Table (6): Classical and Bayesian estimates of the parameters, RF and HF at $t_0 = 0.5$ with their SD for the real data set under progressive Type II censoring

m	$\underline{\theta}$	ML (SD)	Bayes	
			CP (SD)	NP (SD)
	p	0.5522 (0.1593)	0.5524 (0.0030)	0.5540 (0.0012)
	α_1	0.4043 (0.1427)	0.4031 (0.0018)	0.4048 (0.0010)
Complete	α_2	3.0255 (0.9763)	3.0260 (0.0018)	3.0288 (0.0033)
	$R(t_0)$	0.4358	0.4352	0.4350
	$H(t_0)$	0.5088	0.5086	0.5096
	p	0.4385 (0.1329)	0.4406 (0.0012)	0.4418 (0.0022)
	α_1	0.38647(0.1348)	0.3894 (0.0013)	0.3825 (0.0020)
18	α_2	3.9042 (1.2420)	3.9009 (0.0018)	3.9051 (0.0015)
	$R(t_0)$	0.5263	0.5257	0.5228
	$H(t_0)$	0.3921	0.3942	0.3930
	p	0.3520 (0.1367)	0.3546 (0.0014)	0.3552 (0.0018)
	α_1	0.3574 (0.1541)	0.3577 (0.0010)	0.3576 (0.0009)
15	α_2	3.3218 (1.0288)	3.3194 (0.0014)	3.3210 (0.0013)
	$R(t_0)$	0.5601	0.5584	0.5580
	$H(t_0)$	0.3668	0.3684	0.3687

Table (7):Average 95% confidence/HPD intervals with their width for the real data set under progressive Type II censoring

m	θ	CI [Width]	Bayes	
			HPDC [Width]	HPDN [Width]
	p	(0.2400, 0.8645) [0.6244]	(0.5475, 0.5575) [0.0099]	(0.5515, 0.5557) [0.0041]
Complete	α_1	(0.1246, 0.6841) [0.5595]	(0.4000, 0.4062) [0.0062]	(0.4029, 0.4067) [0.0038]
	α_2	(1.1119, 4.9390) [3.8270]	(3.0227, 3.0282) [0.0054]	(3.0245, 3.0348) [0.0102]
	p	(0.1780, 0.6990) [0.5210]	(0.4379, 0.4421) [0.0042]	(0.4382, 0.4451) [0.0069]
18	α_1	(0.1222, 0.6507) [0.5285]	(0.3863, 0.3913) [0.0050]	(0.3803, 0.3870) [0.0067]
	α_2	(1.4699, 6.3385) [4.8685]	(3.8987, 3.9055) [0.0067]	(3.9028, 3.9081) [0.0052]
	p	(0.0840, 0.6201) [0.5360]	(0.3518, 0.3568) [0.0050]	(0.3525, 0.3602) [0.0076]
15	α_1	(0.0553, 0.6595) [0.6042]	(0.3554, 0.3593) [0.0038]	(0.3559, 0.3591) [0.0031]
	α_2	(1.3053, 5.3383) [4.0330]	(3.3174, 3.3223) [0.0049]	(3.3188, 3.3237) [0.0048]

Table (8):Classical and Bayesian estimates of the parameters, RF and HRF at $t_0 = 0.5$ with their SD forthereal data set under Type II censoring

m	θ	ML(SD)	Bayes	
			CP(SD)	NP(SD)
	p	0.5522 (0.1593)	0.5546 (0.0015)	0.5573 (0.0016)
	α_1	0.4043 (0.1427)	0.4046 (0.0007)	0.4067 (0.0019)
Complete	α_2	3.0255 (0.9763)	3.0247 (0.0012)	3.0239 (0.0015)
	$R(t_0)$	0.4358	0.4344	0.4335
	$H(t_0)$	0.5088	0.5101	0.5120
	p	0.6295 (0.1530)	0.6301 (0.0008)	0.6293 (0.0009)
	α_1	0.4667 (0.1636)	0.4636 (0.0011)	0.4677 (0.0016)
20	α_2	4.5123 (2.2741)	4.5113 (0.0009)	4.5155 (0.0016)
	$R(t_0)$	0.4399	0.4382	0.4405
	$H(t_0)$	0.4942	0.4945	0.4941

Table (9):Average 95% confidence/HPD intervals with their width for real data set under Type II censoring

m	*Par*	CI [Width]	Bayes (MCMC)	
			HPDC [Width]	HPDN [Width]
	p	(0.2400, 0.8645) [0.6244]	(0.5525, 0.5575) [0.0049]	(0.5534, 0.5596) [0.0061]
Complete	α_1	(0.1246, 0.6841) [0.5595]	(0.4028, 0.4060) [0.0032]	(0.4030, 0.4092) [0.0062]
	α_2	(1.1119, 4.9390) [3.8270]	(3.0225, 3.0266) [0.0040]	(3.0209, 3.0265) [0.0055]
	p	(0.3296, 0.9295) [0.5999]	(0.6286, 0.6318) [0.0031]	(0.6277, 0.6312) [0.0035]
20	α_1	(0.1460, 0.7874) [0.6413]	(0.4614, 0.4657) [0.0043]	(0.4654, 0.4706) [0.0051]
	α_2	(0.0551, 8.9696) [8.9145]	(4.5090, 4.5128) [0.0037]	(4.5119, 4.5178) [0.0059]

VI. Conclusion

In this paper, some achievements have been done for the MTWD model, with a common shape parameter, based on GOS. The GOS approach is used for obtaining the statistical inferences of MTWD. All the results that have been obtained for GOS of MTWD can be specialized to any special cases of GOS. In this paper, the progressive Type II censored sample and Type II censored sample are used as special cases of GOS.
The ML estimates are obtainedusingthenlm (Non-Linear Minimization) routine of R 3.0.3 to compute the nonlinear ML equation. Bayesian estimates based on SE loss function are obtained by using MCMC technique under two cases, first with the conjugate prior distributions and the other case with the non-informative prior distributions. The simulation have been studied under different sizes and different censoring schemes in

progressive Type II censored samples and Type II censored samples. In many situations, Bayesian estimates are more efficient than ML estimates.

In general, Bayesian estimation is better than ML because the MSE of Bayesian is less than MSE of ML. Bayesian estimation with conjugate prior and non-informative prior have the same quality because the SD, bias, MSE and width are similar. But in some cases using conjugate prior is better than non-informative prior because the MSE in conjugate prior is less than MSE in non-informative prior and this appears in parameter p computations. For each n when m is large, the estimates have smaller MSE than their corresponding, when m is small. Usually for each m when n is small, the estimates have smaller MSE than their corresponding, when n is large. Finally, the estimation based on the progressive Type II censoring is better than the corresponding estimation based on censoringType II because the MSE in the progressive Type II censoring is less than the MSE in the censoring Type II in the same value of n and m.

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