IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. V (Mar. - Apr. 2016), PP 01-07
www.iosrjournals.org

Inequalities on Multivalent Harmonic Starlike Functions Involving Hypergeometric Functions

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Abstract: *In this paper we obtain some inequalities as sufficient conditions for the harmonic multivalent function G(z) to be in classes* $S^*H_m(\alpha)$. *Inequalities for convolution multiplier of two harmonic multivalent functions f and G are also obtained. Also it shown that these inequalities are necessary and sufficient for the function* $G_1(z)$. *Further, the necessary and sufficient conditions for the functions* $G_2(z)$ *to be in classes* $S^*H_m(\alpha)$ *are obtained*.

I. Introduction

Let SH(m), $m \geq 1$, denotes the class of all m-valent, harmonic and orientation-preserving functions in the open unit disk $\Delta = \{z : |z| < 1\}$. A function f in SH(m), $m \geq 1$ can be expressed as $f = h + \overline{g}$, where h and g are analytic functions of the form

(1.1) $$h(z) = z^m + \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1}, \ g(z) = \sum_{n=1}^{\infty} B_{n+m-1} z^{n+m-1}, \ |B_m| < 1$$

Definition 1.1[1,2]

Let $S^*H_m(\alpha), m \geq 1 \text{ and } 0 \leq \alpha < 1$ denotes the class of functions $f = h + \overline{g} \in SH(m)$ which satisfy the condition.

(1.2) $$\frac{\partial}{\partial\theta}(\arg(f(re^{i\theta}))) \geq m\alpha$$

for each $z = re^{i\theta}, \ 0 \leq \theta < 2\pi \text{ and } 0 \leq r < 1$. A function in $S^*H_m(\alpha)$ is called m-valent harmonic starlike function of order α.

The class $S^*H_m(\alpha)$ was studied by Ahuja and Jahangiri [5],[6]. In particular, they stated the following Lemma.

Lemma 1.1 [5],[6]

Let $f = h + \overline{g}$ be given by (1.1) if

(1.3) $$\sum_{n=1}^{\infty}\left[\frac{n-1+m(1-\alpha)}{m(1-\alpha)}|A_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)}|B_{n+m-1}|\right] \leq 2$$

where $A_m = 1$ and $m \geq 1, \ 0 \leq \alpha < 1$. then the harmonic function f is sense-preserving, m-valent and $f \in S^*H_m(\alpha)$

Denote by $T^*H_m(\alpha)$ is the subclasses of consisting of functions $f = h + \overline{g}$, $f \in S^*H_m(\alpha)$ so that h and g are of the form

(1.6) $$h(z) = z^m - \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1},$$

$$g(z) = \sum_{n=1}^{\infty} B_{n+m-1} z^{n+m-1}, \ A_{n+m-1} \geq 0, \ B_{n+m-1} \geq 0, \ B_m < 1$$

. Lemma 1.2 [5],[9]

Let $f = h + \overline{g}$ be given by (1.6) then $f \in T^{*}H_m(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}\{n-1+m(1-\alpha)\}A_{n+m-1} + \sum_{n=1}^{\infty}\{n-1+m(1+\alpha)\}B_{n+m-1} \le m(1-\alpha)$$

Definition:

The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ for ……,-2,-1,0 $\neq$ c,a,b $\in$ C, a set of complex numbers is defined as

$${}_2F_1(a,b;c;z) \equiv F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n.$$

where $(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \lambda(\lambda+1)\ldots.(\lambda+n-1) \text{ for } n = 1,2,3\ldots$ and $(\lambda)_0 = 1$.

$F(a,b;c;z)$ is analytic in $\Delta = \{z : |z| < 1\}$ and for Re(c-a-b)>0,

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, c \ne 0,-1,-2,\ldots.$$

In this paper a harmonic m-valent function $G(z) = \phi_1(z) + \overline{\phi_2(z)}$ is considered, where $\phi_1(z)$ and $\phi_2(z)$ are m-valent analytic functions in $\Delta = \{z : |z| < 1\}$ defined in terms of above mentioned hypergeometric function as :

(1.7) $$\phi_1(z) = \phi_1(a_1,b_1;c_1;z) = z^m F(a_1,b_1;c_1;z)$$
$$= z^m + \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^{n+m-1},$$

(1.8) $$\phi_2(z) = \phi_2(a_2,b_2;c_2;z) = z^{m-1}[F(a_2,b_2;c_2;z)-1]$$
$$= \sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^{n+m-1},\ a_2b_2 < c_2.$$

Also, consider a harmonic m-valent function $G_1(z)$ which is defined as :

$$G_1(z) = z^m\left(2 - \frac{\phi_1(z)}{z^m}\right) + \overline{\phi_2(z)} \text{ for } a_j, b_j > 0, c_j > a_j + b_j + 1, j = 1,2.$$

Further, a convolution $L_m(f,G)$ of two harmonic m-valent functions f and G is considered as follows:

$$L_m(f,G)(z) = (f \,\tilde{*}\, G)(z)$$
$$= h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)}$$
$$= P(z) + \overline{Q(z)}$$

where $P(z) = z^m + \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_{n+m-1}z^{n+m-1}$

and $Q(z) = \sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}B_{n+m-1}z^{n+m-1},\ a_2b_2|B_m| < c_2$.

Ahuja and Silverman [4] have given a nice connection between Harmonic univalent functions and hypergeometric functions and obtained some inequalities harmonic univalent functions which are sense-preserving, harmonic starlike univalent (harmonic convex univalent) in Δ. They also defined a convolution multipliers between two harmonic univalent functions. Motivated by the work of Ahuja and Silverman [4] in this chapter some inequalities as sufficient conditions for m-valent harmonic function $G(z)$ to be sense-preserving starlike and convex of positive order $\alpha(0 \le \alpha < 1)$ are obtained. It is also shown that these inequalities are necessary and sufficient for the function $G_1(z)$. Further necessary and sufficient conditions for the function $G_2(z)$ to be in class $f \in S^*H_m(\alpha)$ are obatained. Inequalities for convolution of two harmonic m-valent functions f and G are also obtained.

2: Main Results

Theorem 2.1 s

If $a_j, b_j > 0, c_j > a_j + b_j + 1$ for j=1,2 then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ where $\phi_1 \text{ and } \phi_2$ are given in (1.7) and (1.8) respectively to be sense-preserving harmonic m-valent in Δ and $G \in S^*H_m(\alpha)$ is that

$$(3.2.1)\quad F(a_1,b_1;c_1;1)\left(\frac{a_1b_1}{c_1-a_1-b_1-1}+m(1-\alpha)\right)+ F(a_2,b_2;c_2;1)\left(\frac{a_2b_2}{c_2-a_2-b_2-1}+m(1+\alpha)-1\right) \le m(3-\alpha)-1.$$

Proof

To prove that G is sense-preserving in Δ, it only needs to show that

$|\phi_1'(z)| > |\phi_2'(z)|, z \in \Delta$

By the hypothesis, it noted that

$$|\phi_1'(z)| = |mz^{m-1} + \sum_{n=2}^{\infty}(n+m-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^{n+m-2}|$$

$$= |mz^{m-1} + \sum_{n=2}^{\infty}(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^{n+m-2} + \sum_{n=2}^{\infty}m\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^{n+m-2}|$$

$$> \left[m - \sum_{n=2}^{\infty}(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - m\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\right]|z|^{m-1}$$

$$= \left[m - \frac{a_1b_1}{c_1}\sum_{n=1}^{\infty}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_{n-1}} - m\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}\right]|z|^{m-1}$$

$$= \left[2m - \frac{a_1b_1}{c_1}\frac{\Gamma(c_1+1)\Gamma(c_1-a_1-b_1-1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)} - m\frac{\Gamma(c_1)\Gamma(c_1-a_1-b_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)}\right]|z|^{m-1}$$

$$= \left[2m - \left(\frac{a_1b_1}{c_1-a_1-b_1-1}+m\right)F(a_1,b_1;c_1;1)\right]|z|^{m-1}$$

using (2.1)

$$\ge \left[\left\{\frac{a_2b_2}{c_2-a_2-b_2-1}+m(1+\alpha)-1\right\}F(a_2,b_2;c_1;1) - m\alpha\left(F(a_1,b_1;c_1;1)-1\right) - m + 1\right]|z|^{m-1}$$

$$\ge \left[\left\{\frac{a_2b_2}{c_2-a_2-b_2-1}+(m-1)\right\}F(a_2,b_2;c_1;1) - m + 1\right]|z|^{m-1}, 0 \le \alpha < 1$$

$$= \left[\frac{a_2b_2}{c_2}\frac{\Gamma(c_2+1)\Gamma(c_2-a_2-b_2-1)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)} + (m-1)\frac{\Gamma(c_2)\Gamma(c_2-a_2-b_2)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)} - m + 1\right]|z|^{m-1}$$

$$= \left[\frac{a_2b_2}{c_2}\sum_{n=1}^{\infty}\frac{(a_2+1)_{n-1}(b_2+1)_{n-1}}{(c_2+1)_{n-1}(1)_{n-1}} + (m-1)\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right]|z|^{m-1}$$

$$= \sum_{n=1}^{\infty}(n+m-1)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}|z|^{m-1}$$

$$\geq \sum_{n=1}^{\infty}(n+m-1)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}|z|^{n+m-2}$$

$$\geq \left|\sum_{n=1}^{\infty}(n+m-1)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^{n+m-2}\right| = |\phi_2'(z)|$$

so, G is sense-preserving in Δ.

To, show that G is m-valent and $G \in S^*H_m(\alpha)$, on applying Lemma 1.1 it only needs to show that

$$\sum_{n=2}^{\infty}[n-1+m(1-\alpha)]\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \tag{2.2}$$

$$+\sum_{n=1}^{\infty}[n-1+m(1+\alpha)]\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \leq m(1-\alpha)$$

The left hand side of (2.2) is equivalent to

$$\sum_{n=2}^{\infty}(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + m(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty}\frac{n(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$+\{m(1+\alpha)-1\}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$= \frac{a_1b_1}{c_1-a_1-b_1-1}F(a_1,b_1;c_1;1) + m(1-\alpha)[F(a_1,b_1;c_1;1)-1] +$$

$$+\frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1) + \{m(1+\alpha)-1\}[F(a_2,b_2;c_2;1)-1]$$

$$= F(a_1,b_1;c_1;1)\left(\frac{a_1b_1}{c_1-a_1-b_1-1} + m(1-\alpha)\right) +$$

$$+F(a_2,b_2;c_2;1)\left(\frac{a_2b_2}{c_2-a_2-b_2-1} + m(1+\alpha) - 1\right) - m(1-\alpha) - m(1+\alpha) + 1$$

The last expression is bounded by $m(1-\alpha)$ provided that (2.1) is satisfied. Therefore, $G \in S^*H_m(\alpha)$. Consequently G is sense-preserving and m-valent of order α in Δ.

For m=1 and α=0 the following corollary [4] is obtained.

Corollary 2.2 [4]

If $a_j, b_j > 0,\ c_j > a_j + b_j + 1$ for $j = 1,2$, then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ with m=1 to be harmonic univalent in Δ and $G \in S^*H$ is that

$$\left(1+\frac{a_1b_1}{c_1-a_1-b_1-1}\right)F(a_1,b_1;c_1;1) + \frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1) \leq 2.$$

Theorem 2.5

Let $a_j, b_j > 0,\ c_j > a_j + b_j + 1$ for j=1,2 and $a_2b_2 < c_2$, if

$$G_1(z) = z^m\left(2 - \frac{\phi_1(z)}{z^m}\right) + \overline{\phi_2(z)} \tag{2.5}$$

then,

$G_1 \in T^*H_m(\alpha)$ if and only if (2.1) holds.

Proof

It is observed that

$$G_1(z) = z^m - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n+m-1} + \overline{\sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} z^{n+m-1}}$$

and $T^*H_m(\alpha) \subset S^*H_m(\alpha)$. In view of Theorem 2.1, it only needs to show the necessary condition for G_1 to be in $T^*H_m(\alpha)$. If $G_1 \in T^*H_m(\alpha)$

Theorem 2.6

If $a_j, b_j > 0$ and $c_j > a_j + b_j$ for j=1,2 then a sufficient condition for a function

$$G_2(z) = m\int_0^z t^{m-1}F(a_1,b_1;c_1;t)dt + \overline{\int_0^z t^{m-1}[F(a_2,b_2;c_2;t)-1]dt}$$

to be in $S^*H^0(m)$ is that

(3.2.6) $\quad mF(a_1,b_1;c_1;1) + F(a_2,b_2;c_2;1) \le 2m+1$.

Proof

In view of Lemma 3.1.1, the function

$$G_2(z) = z^m + \sum_{n=2}^{\infty} \frac{m}{n+m-1} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n+m-1} + \overline{\sum_{n=2}^{\infty} \frac{1}{n+m-1} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} z^{n+m-1}}$$

is in $S^*H^0(m)$ if

$$\sum_{n=2}^{\infty} \frac{n+m-1}{m} \frac{m}{n+m-1} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=2}^{\infty} \frac{n+m-1}{m} \frac{1}{n+m-1} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \le 1$$

That is, if

$$\sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{1}{m}\sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \le 1.$$

which holds if (2.6) is true.

Theorem 2.6

If $a_1, b_1 > -1,\ c_1 > 0,\ a_1b_1 < 0,\ a_2 > 0,\ b_2 > 0$ and $c_j > a_j + b_j + 1,$ j=1,2 then

$$G_2(z) = m\int_0^z t^{m-1}F(a_1,b_1;c_1;t)dt + \overline{\int_0^z t^{m-1}[F(a_2,b_2;c_2;t)-1]dt}$$

is in $S^*H^0(m)$ if and only if

(2.7) $\quad mF(a_1,b_1;c_1;1) - F(a_2,b_2;c_2;1) \ge -1$.

Proof

By the hypothesis using Lemma (1.3) to

$$G_2(z) = z^m - \frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty} \frac{m}{n+m-1} \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(n-1)(c_1+1)_{n-2}(1)_{n-2}} z^{n+m-1}$$
$$+ \overline{\sum_{n=2}^{\infty} \frac{1}{n+m-1} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} z^{n+m-1}}$$

It suffices to show that

$$\frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty} \frac{n+m-1}{m} \frac{m}{n+m-1} \frac{1}{(n-1)} \frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_{n-2}}$$
$$+ \sum_{n=2}^{\infty} \frac{n+m-1}{m(n+m-1)} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \le 1$$

or,

$$\sum_{n=1}^{\infty}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_{n-1}n}+\frac{c_1}{|a_1b_1|}\frac{1}{m}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\le\frac{c_1}{|a_1b_1|}$$

But, this is equivalent to

$$\frac{c_1}{a_1b_1}\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\frac{c_1}{|a_1b_1|}\frac{1}{m}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\le\frac{c_1}{|a_1b_1|}$$

That is if (2.7) holds.

Theorem 2.7

Let $a_j, b_j > 0,\ c_j > a_j + b_j + 1,$ for j=1,2 and $a_2b_2 < c_2$. A necessary and sufficient condition for $L_m(f,G)(z) = f * (\phi_1+\overline{\phi_2}) \in T^*H_m(\alpha)$ and $G=\phi_1+\overline{\phi_2}$ given by (1.7) and (1.8) for $f \in T^*H_m(\alpha)$ is that

(2.8) $$F(a_1,b_1;c_1;1)+F(a_2,b_2;c_2;1)\le 3.$$

Proof

Let $f = h+\overline{g} \in T^*H_m(\alpha)$, where h and g are given by (1.6).Also,

$$L_m(f,G)(z) = z^m - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_{n+m-1}z^{n+m-1}+\overline{\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}B_{n+m-1}z^{n+m-1}}$$

In view of Lemma 1.3 it only needs to prove that $L_m(f,G)(z)\in T^*H_m(\alpha)$.

As an application of Lemma 1.3,

$$A_{n+m-1}\le\frac{m(1-\alpha)}{n-1+m(1-\alpha)},\ B_{n+m-1}\le\frac{m(1-\alpha)}{n-1+m(1+\alpha)}.$$

Consider

$$\sum_{n=2}^{\infty}\{n-1+m(1-\alpha)\}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}1)_{n-1}}A_{n+m-1}$$
$$+\sum_{n=1}^{\infty}\{n-1+m(1+\alpha)\}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}B_{n+m-1}$$
$$\le m(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+m(1-\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$= m(1-\alpha)F(a_1,b_1;c_1;1)+m(1-\alpha)F(a_2,b_2;c_2;1)-2m(1-\alpha).$$

The last expression is bounded above by $m(1-\alpha)$ if and only if (2.8) is satisfied. This proves the result.

Theorem 2.9

Let $a_j, b_j > 0,\ c_j > a_j + b_j + 1$ for j=1,2 then (2.) is a necessary and sufficient condition for a function

$$L_{m,c}(f,G)(z)=\frac{c+m}{z^c}\int_0^z t^{c-1}P(t)dt+\overline{\frac{c+m}{z^c}\int_0^z t^{c-1}Q(t)dt}$$

to be in $T^*H_m(\alpha)$ for $f\in T^*H_m(\alpha)$ of the form (1.6) and $G=\phi_1+\overline{\phi_2}$ given by (1.7) and (1.8).

Proof

By the hypothesis

$$L_{m,c}(f,G)(z)=z^m-\sum_{n=2}^{\infty}\frac{c+m}{n+m+c-1}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_{n+m-1}+\overline{\sum_{n=1}^{\infty}\frac{c+m}{n+m+c-1}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}B_{n+m-1}}$$ and

as $f\in T^*H_m(\alpha)$,

$$A_{n+m-1}\le\frac{m(1-\alpha)}{n-1+m(1-\alpha)}\ \text{and}\ B_{n+m-1}\le\frac{m(1-\alpha)}{n-1+m(1+\alpha)}.$$

Consider,

$$\sum_{n=2}^{\infty} \left(n-1+m(1-\alpha)\right)\frac{c+m}{n+m+c-1}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_{n+m-1} +$$

$$+\sum_{n=1}^{\infty} \left(n-1+m(1+\alpha)\right)\frac{c+m}{n+m+c-1}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}B_{n+m-1}$$

$$\leq m(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + m(1-\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$= m(1-\alpha)F(a_1,b_1;c_1;1) + m(1-\alpha)F(a_2,b_2;c_2;1) - 2m(1-\alpha).$$

The last expression is bounded above by $m(1-\alpha)$ if and only if (2.8) is satisfied. This proves the result.

$$\sum_{n=2}^{\infty}\frac{(n+m-1)\{n-1+m(1-\alpha)\}}{m}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_{n+m-1}$$

$$+\sum_{n=1}^{\infty}\frac{(n+m-1)\{n-1+m(1+\alpha)\}}{m}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}B_{n+m-1}$$

$$\leq m(1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + m(1-\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$= m(1-\alpha)F(a_1,b_1;c_1;1) + m(1-\alpha)F(a_2,b_2;c_2;1) - 2m(1-\alpha).$$

The last expression is bounded above by $m(1-\alpha)$ if and only if (2.11) is satisfied. This proves the result.

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