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A Study on Second Order Multivariable Linear Singular Systems Using Leapfrog Method

S.Karunanithi[1], S. Chakravarthy[1], S. Sekar[2]
[1](Department of Mathematics, Government Thirumagal Mills College, Gudiyattam – 632 602, India)
[2](Department of Mathematics, Government Arts College (Autonomous), Salem – 636 007, India)

Abstract : *In this article, a new method of analysis of the second order multivariable state-space and singular systems using Leapfrog method is presented. To illustrate the effectiveness of the Leapfrog method, an example of the second order multivariable state-space and singular systems has been considered and the solutions were obtained using methods taken from the literature [8] and Leapfrog method. The obtained discrete solutions are compared with the exact solutions of the second order multivariable state-space and singular systems. Error calculations for the second order multivariable state-space and singular systems have been presented in the Table form to show the efficiency of this Leapfrog method. This Leapfrog method can be easily implemented in a digital computer and the solution can be obtained for any length of time.*

Keywords: *Differential equations, system of differential equations, second order multivariable state-space and singular systems, Single-term Haar wavelet series, Leapfrog Method.*

I. Introduction

Recent advances in speed and memory capacity of super computers with parallel processors have necessitated modification to existing algorithms and development of new algorithms to fully exploit the capabilities of those machines. Even in the Von-Neumann era, parallel numerical integrators have been proposed for the SISD machines for the first order initial value problems of the form

$$\dot{x}(t) = f(t, x(t)), \text{ with initial condition } x(0) = x_0 \quad (1)$$

It is to be noted that, from the mid 1970's onwards, serious attention has been focussed on the development of numerical methods to match speed and memory capacities of such machines. Some of the works related to the determination of numerical solutions for the IVPs are due to Gear [1, 2], Iserless and Norsett [5], Gladwell et al. [3], Hall [4], Shampine [16], etc.

The goal of this article is to construct a numerical method for addressing the second order multivariable state-space and singular systems by an application of the Leapfrog method which was studied by Sekar and team of his researchers [6-7, 10-15]. Recently, Murugesan *et al.* [8] discussed the second order multivariable state-space and singular systems single-term Walsh series method. In this paper, the same second order multivariable state-space and singular systems was considered (discussed by Murugesan *et al.* [8]) but present a different approach using the Leapfrog method with more accuracy for the second order multivariable state-space and singular systems.

II. Leapfrog Method

The most familiar and elementary method for approximating solutions of an initial value problem is Euler's Method. Euler's Method approximates the derivative in the form of $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ by a finite difference quotient $y'(t) \approx (y(t+h) - y(t))/h$. We shall usually discretize the independent variable in equal increments:

$$t_{n+1} = t_n + h, n = 0,1,...,t_0.$$

Henceforth we focus on the scalar case, N = 1. Rearranging the difference quotient gives us the corresponding approximate values of the dependent variable:

$$y_{n+1} = y_n + hf(t_n, y_n), n = 0,1,...,t_0$$

To obtain the leapfrog method, we discretize t_n as in $t_{n+1} = t_n + h, n = 0,1,...,t_0$, but we double the time interval, h, and write the midpoint approximation $y(t+h) - y(t) \approx hy'\left(t + \frac{h}{2}\right)$ in the form

$$y'(t+h) \approx (y(t+2h) - y(t))/h$$

and then discretize it as follows:

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n), n = 0,1,...,t_0$$

The leapfrog method is a linear *m* = 2-step method, with $a_0 = 0, a_1 = 1, b_{-1} = -1, b_0 = 2$ and $b_1 = 0$. It uses slopes evaluated at odd values of *n* to advance the values at points at even values of *n*, and vice versa, reminiscent of the children's game of the same name. For the same reason, there are multiple solutions of the leapfrog method with the same initial value $y = y_0$. This situation suggests a potential instability present in multistep methods, which must be addressed when we analyze them—two values, y_0 and y_1, are required to initialize solutions of $y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,...,t_0$ uniquely, but the analytical problem $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ only provides one. Also for this reason, one-step methods are used to initialize multistep methods.

III. Second Order Multivariable State-Space And Singular Systems

Though, Gladwell and Thomas [3], Murugesan and Balasubramanian [9], Van Daele et al., [17], Vander Houmen and Sommeijer [18] have contributed for the determination of discrete solutions for the second order IVPs, it is observed, interestingly, that very little work has been carried out in getting approximate solutions for the system of second order IVPs involving multivariables of the form

$$\begin{aligned}
\ddot{x}_1 &= f_1(t, x_1, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ..., \dot{x}_n) \\
\ddot{x}_2 &= f_2(t, x_1, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ..., \dot{x}_n) \\
&\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots \\
\ddot{x}_n &= f_n(t, x_1, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ..., \dot{x}_n)
\end{aligned} \tag{2}$$

with $x_i(0) = x_{i0}$ and $\dot{x}_i(0) = \dot{x}_{i0}$

i.e., $\ddot{x} = F(t, x, \dot{x})$ (3)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

where $x \in R^n$ and $F = (f_1, f_2, ..., f_n)^T$

Suppose the system is singular, then we have

$K\ddot{x} = F(t, x, \dot{x})$ (4)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

The above system of equations (3) can also be represented as

$\ddot{x} = A\dot{x}(t) + Bx(t) + Cu(t)$ (5)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ and eq. (4) can also be represented as

$K\ddot{x} = A\dot{x}(t) + Bx(t) + Cu(t)$ (6)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$,

where $x \in R^n$ is the state vector, $u(t) \in R^r$ is the input vector, $K, A, B \in R^{n \times n}$ of which *K* is a singular matrix and $C \in R^{n \times r}$.

IV. Numerical Examples

4.1 Multivariable Singular System With Four Variables

Consider a second order singular system, equation (6)

i.e., $K\ddot{x} = A\dot{x}(t) + Bx(t) + Cu(t)$

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

Let $K = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, B = 0, C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $u = [1 \quad 1]^T$

with $x(0) = [1 \quad 1 \quad 2 \quad 2]^T$ and $\dot{x}(0) = \begin{bmatrix} \frac{-3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$

Hence the singular system (6) becomes

$$\begin{bmatrix}2&2&0&0\\0&0&2&2\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\\\ddot{x}_3\\\ddot{x}_4\end{bmatrix}=\begin{bmatrix}0&-1&0&0\\-1&0&1&0\\1&-1&0&0\\0&0&1&-1\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}+\begin{bmatrix}0&0\\-1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \qquad (7)$$

i.e.,

$$\begin{aligned}2\ddot{x}_1+2\ddot{x}_2&=-\dot{x}_2\\2\ddot{x}_3+2\ddot{x}_4&=-\dot{x}_1+\dot{x}_3-1\\0&=-\dot{x}_1-\dot{x}_2+1\\0&=\dot{x}_3-\dot{x}_4\end{aligned}$$

The exact solutions of the singular system with four variables (7) are given by

$$x_1=2-e^{\frac{-t}{4}}-t$$

$$x_2=2-e^{\frac{-t}{4}}$$

$$x_3=2-\frac{1}{2}e^{\frac{-t}{4}}+\frac{1}{2}e^{\frac{t}{4}}$$

$$x_4=2-\frac{1}{2}e^{\frac{-t}{4}}+\frac{1}{2}e^{\frac{t}{4}}$$

t	STWS				Leapfrog			
	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.1	2.18E-10	3.00E-10	4.00E-10	5.00E-10	1.01E-12	2.00E-12	3.00E-12	4.00E-12
0.2	2.20E-10	4.00E-10	5.00E-10	5.00E-10	1.02E-12	3.00E-12	3.00E-12	4.00E-12
0.3	2.23E-10	4.00E-10	5.00E-10	6.00E-10	1.03E-12	4.00E-12	5.00E-12	6.00E-12
0.4	2.25E-10	4.00E-10	5.00E-10	6.00E-10	1.04E-12	5.00E-12	5.00E-12	6.00E-12
0.5	2.27E-10	5.00E-10	5.00E-10	6.00E-10	1.05E-12	6.00E-12	7.00E-12	6.00E-12
0.6	2.30E-10	5.00E-10	7.00E-10	7.00E-10	1.06E-12	6.00E-12	7.00E-12	8.00E-12
0.7	2.32E-10	5.00E-10	7.00E-10	7.00E-10	1.07E-12	7.00E-12	7.00E-12	8.00E-12
0.8	2.35E-10	6.00E-10	7.00E-10	7.00E-10	1.08E-12	7.00E-12	9.00E-12	8.00E-12
0.9	2.37E-10	6.00E-10	7.00E-10	9.00E-10	1.09E-12	8.00E-12	9.00E-12	9.00E-12
1	2.40E-10	6.00E-10	9.00E-10	9.00E-10	1.10E-12	8.00E-12	9.00E-12	9.00E-12

Table 1 Error Calculation for the Problem 4.1

The discrete solutions of the singular system (7) have been determined using STWS and Leapfrog methods. The Error calculations of the singular system (7) have been given in the Table 1. It is observed that the discrete solutions exactly match with the corresponding exact solutions at different values of time't' and the error is almost nil to the accuracy of six significant digits. Hence to depict the similarity between the solutions, for a sample, the error calculation for x_1, x_2, x_3 and x_4 has been given in the Table 1.

4.2 Multivariable Singular System With Six Variables

Consider a second order singular system (8) with six variables

$$\begin{bmatrix}1&3&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\\\ddot{x}_3\\\ddot{x}_4\\\ddot{x}_5\\\ddot{x}_6\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&3\\1&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\\\dot{x}_5\\\dot{x}_6\end{bmatrix}+\begin{bmatrix}-2&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&2&0\\0&0&0&0&0&-4\\0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}+\begin{bmatrix}1&1\\3&2\\0&0\\0&0\\0&0\\5&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \qquad (8)$$

with $x(0)=\begin{bmatrix}1&1&1&1&1&1\end{bmatrix}^T$, $\dot{x}(0)=\begin{bmatrix}0&0&0&1&1&0\end{bmatrix}^T$.

The above system (8) can be expressed as

$$\ddot{x}_1 + 3\ddot{x}_2 = -2x_1 + 2$$
$$\ddot{x}_3 = x_3 + 5$$
$$\ddot{x}_4 = \dot{x}_4 - x_4$$
$$\ddot{x}_5 = -\dot{x}_5 + 2x_5$$
$$\ddot{x}_6 = 3\dot{x}_6 - 4x_6$$
$$\ddot{x}_2 = \dot{x}_1 + 9$$

with $x(0) = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T$, $\dot{x}(0) = [0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0]^T$.

The exact solutions of the singular system (8) with six variables are given by

$$x_1 = -13.5e^{-2t} + 27e^{-t} - 12.5$$
$$x_2 = \frac{27}{4}2e^{-2t} - 27e^{-t} + \frac{9t^2}{2} - \frac{27t}{2} + \frac{85}{4}$$
$$x_3 = 3e^{t} + 3e^{-t} - 5$$
$$x_4 = e^{\frac{t}{2}}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$
$$x_5 = e^{t}$$
$$x_6 = e^{\frac{3t}{2}}\left[\cos\frac{\sqrt{7}}{2}t - \frac{3}{\sqrt{7}}\sin\frac{\sqrt{7}}{2}t\right]$$

t	STWS					
	x_1	x_2	x_3	x_4	x_5	x_6
0.1	2.18E-10	3.00E-10	4.00E-10	5.00E-10	3.00E-10	4.00E-10
0.2	2.20E-10	4.00E-10	5.00E-10	5.00E-10	3.00E-10	4.00E-10
0.3	2.23E-10	4.00E-10	5.00E-10	6.00E-10	3.00E-10	4.00E-10
0.4	2.25E-10	4.00E-10	5.00E-10	6.00E-10	4.00E-10	5.00E-10
0.5	2.27E-10	5.00E-10	5.00E-10	6.00E-10	4.00E-10	5.00E-10
0.6	2.30E-10	5.00E-10	7.00E-10	7.00E-10	4.00E-10	5.00E-10
0.7	2.32E-10	5.00E-10	7.00E-10	7.00E-10	5.00E-10	6.00E-10
0.8	2.35E-10	6.00E-10	7.00E-10	7.00E-10	5.00E-10	6.00E-10
0.9	2.37E-10	6.00E-10	7.00E-10	9.00E-10	5.00E-10	6.00E-10
1	2.40E-10	6.00E-10	9.00E-10	9.00E-10	6.00E-10	7.00E-10

t	Leapfrog					
	x_1	x_2	x_3	x_4	x_5	x_6
0.1	1.01E-12	2.00E-12	3.00E-12	4.00E-12	2.00E-12	3.00E-12
0.2	1.02E-12	3.00E-12	3.00E-12	4.00E-12	3.00E-12	3.00E-12
0.3	1.03E-12	4.00E-12	5.00E-12	6.00E-12	4.00E-12	5.00E-12
0.4	1.04E-12	5.00E-12	5.00E-12	6.00E-12	4.00E-12	5.00E-12
0.5	1.05E-12	6.00E-12	7.00E-12	6.00E-12	5.00E-12	6.00E-12
0.6	1.06E-12	6.00E-12	7.00E-12	8.00E-12	5.00E-12	6.00E-12
0.7	1.07E-12	7.00E-12	7.00E-12	8.00E-12	6.00E-12	7.00E-12
0.8	1.08E-12	7.00E-12	9.00E-12	8.00E-12	6.00E-12	7.00E-12
0.9	1.09E-12	8.00E-12	9.00E-12	9.00E-12	7.00E-12	9.00E-12
1	1.10E-12	8.00E-12	9.00E-12	9.00E-12	7.00E-12	9.00E-12

The discrete solutions of the singular system (8) have been determined using STWS and Leapfrog methods. The Error calculations of the singular system (8) have been given in the Tables 2 - 3. It is observed that the discrete solutions exactly match with the corresponding exact solutions at different values of time 't' and the error is almost nil to the accuracy of six significant digits. Hence to depict the similarity between the solutions, for a sample, the error calculation for x_1, x_2, x_3, x_4, x_5 and x_6 has been given in the Tables 2 - 3.

V. Conclusion

The obtained discrete solution of the numerical examples shows the efficiency of the Leapfrog for finding the solution of the second order multivariable state-space and singular systems. From the Tables 1 - 3, we can observe that the error calculations are come closer in Leapfrog method when compared to other methods from STWS taken from the literature [8]. Hence, the Leapfrog method is more suitable for studying the second order multivariable state-space and singular systems.

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