IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. III (Mar. - Apr. 2016), PP 37-41
www.iosrjournals.org

Series of Integers in Arithmetic Progression with a Common Property of Two Parts of Equal Sums

Soumendra Bera
Mahishadal Raj College, Vidyasagar University, India.

Abstract: *3 + 5 + 7 + 9 + 11 + 13 is a summation series of six integers in arithmetic progression. If we put the sign of equality (=) between 9 and 11 replacing the plus sign (+) then we find: 3 + 5 + 7 + 9 = 11 + 13. The paper describes the general forms of such equalities and their amazing properties.*
Keywords: *integer; summation; identity; arithmetic progression.*

I. Introduction

The preliminary summation series of the first three natural numbers is: 1 + 2 + 3 from which we get the equality: 1 + 2 = 3. Some other examples of the equalities of the particular pattern are:

$$2 + 3 + 4 + 5 + 6 + 7 = 8 + 9 + 10.$$
$$5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 21 + 23 + 25 + 27.$$
$$1 + 4 + 7 + 10 + 13 = 16 + 19.$$
$$3 + 7 + 11 + 15 + 19 + 23 + 27 = 31 + 35 + 39.$$

The common differences between the consecutive terms in the above four series are 1, 2, 3 and 4 respectively. The greatest common divisor (GCD) of any two consecutive terms of a series is 1. In each example, a term on the right is greater than any term on the left so that the number of terms on the left is higher than the number of terms on the right. In the thesis, we show some identities which can be the beautiful elementary part of number theory. Some results in general forms are shown below.

(1) $n^2 + (n^2 + 1) + \ldots + (n^2 + n) = (n^2 + n + 1) + (n^2 + n + 2) + \ldots + (n^2 + 2n)$
$= 3\,(1^2 + 2^2 + \ldots + n^2).$

(2) $(n + 1) + (n + 2) + \ldots + (5n + 2) = (5n + 3) + (5n + 4) + \ldots + (7n + 3)$
$= 3\,(2n + 1)^2.$

(3) $(n^2 - 1) + (n^2 + 1) + \ldots + (n + 1)^2 = \{(n + 1)^2 + 2\} + \{(n + 1)^2 + 4\} + \ldots + \{(n + 1)^2 + 2n\}$
$= n\,(n + 1)\,(n + 2)\,.$

(4) $(2n^2 - 2n) + (2n^2 - 2n + 1) + \ldots + 2n^2 = (2n^2 + 1) + (2n^2 + 2) + \ldots + (2n^2 + 2n - 1)$
$= n\,(2n + 1)\,(2n - 1).$

(5) $(6n^2 - 6n - 10) + (6n^2 - 6n - 9) + \ldots + (6n^2 + 6n + 7)$
$= (6n^2 + 6n + 8) + (6n^2 + 6n + 9) + \ldots + (6n^2 + 18n + 1) = 9\,(2n + 3)\,(2n + 1)\,(2n - 1).$

(6) $(2n + 1) + (2n + 3) + \ldots + (10n - 1) = (10n + 1) + (10n + 3) + \ldots + (14n - 1) = 24n^2\,.$

(7) $\left\{\frac{1}{2}n(3 - n)\right\} + \left\{\frac{1}{2}n(3 - n) + (n - 1)\right\} + \left\{\frac{1}{2}n\,(3 - n) + 2(n - 1)\right\} + \ldots + \left\{\frac{1}{2}n(3 - n) + (n - 1)^2\right\}$
$= \left\{\frac{1}{2}n\,(3 - n) + (n - 1)^2 + (n - 1)\right\} = \frac{1}{2}\,n\,(n + 1).$

(8) $\{n\,(2n + 1)\}^2 + \{n\,(2n + 1) + 1\}^2 + \{n\,(2n + 1) + 2\}^2 + \ldots + \{n\,(2n + 1) + n\}^2$
$= \{2n\,(n + 1) + 1\}^2 + \{2n\,(n + 1) + 2\}^2 + \ldots + \{n\,(2n + 3)\}^2.$

II. The Basic Identity With The Terms In Ap

Consider the inequality:

$$1 + 2 + \ldots + x < (x + 1) + (x + 2) + \ldots + (2x - p). \qquad (1)$$

x and x_1 are the numbers of terms on the left and right hand sides of (1) respectively and p is the difference between x and x_1 where x_1, $p < x$. Obviously $x_1 = x$ – p. If S and S_1 are the sums on the left and right of (1) respectively, then

$$S = \frac{1}{2}\,x\,(x + 1)\,;$$
$$S_1 = \frac{1}{2}\,(x - p)\,(3x + 1 - p)\,;$$
$$S + \frac{1}{2}\{2x^2 - 4xp + p(p - 1)\} = S_1\,. \qquad (2)$$

Let $S_1 - S$ be divided into some equal parts to find an identity of the following form:

$$(a + 1) + (a + 2) + \ldots + (a + x) = (a + x + 1) + (a + x + 2) + \ldots + (\mathrm{a} + 2x - p)\,. \qquad (3)$$

Comparing (2) and (3), we get:

$$xa - (x - p)\mathrm{a} = \frac{1}{2}\,\{2x^2 - 4xp + p(p - 1)\}$$

$$\Rightarrow\ a\ =\ \frac{1}{2p}\{2x^2-4xp+p(p-1)\}.$$

The first term on the left of (3) is:

$$T_1\ =\ a+1\ =\ \frac{1}{2p}\ \{2x^2-4xp\ +\ p(p+1)\}\ . \qquad (4)$$

Thus for $T_1\ =\ \frac{1}{2p}\{2x^2-\ 4xp\ +\ p(p+1)\}$, we have the following identity:

$$T_1+(T_1+1)+\ \ldots+(T_1+x-1)\ =\ (T_1+x)+\ (T_1+x+1)+\ \ldots+(T_1+2x-p-1). \qquad (5)$$

(5) is the basic identity to obtain the subsequent results.

III. Common Difference: *d* = 1

3.1. Relation of (5) with the Summation Series of Square Integers

We can get the integer-values of the terms of (5) for some particular values of x and p; Let x be a multiple of p such that $x = (n + 1)\, p$, $n \in \mathbb{N}$. Then

$$T_1\ =\ n^2p\ -\ \frac{p-1}{2}. \qquad (6)$$

Consequently (5) is equal to the sum:

$$S_2\ =\ \frac{x}{2}\ (2T_1+x-1).$$

Putting $T_1\ = n^2p-\ \frac{p-1}{2}$ and $x = (n + 1)\, p$, we get:

$$S_2\ =\ \frac{1}{2}\ p^2\ n\ (n+1)(2n+1)$$
$$=3p^2(1^2+\ 2^2+\ \ldots\ +\ n^2).$$

3.2. Number Next to the Last Term on the Right of (5)

The desired number is: $T_1+2x-p\ =\ (n+1)^2p\ -\ \frac{p-1}{2}$. This is again the first term on the left of (5) by the substitution of $n + 1$ for n.

Examples: Suppose $p = 3$ and $n = 2$. Then $T_1\ =\ n^2p-\ \frac{p-1}{2}$ = 11; and (5) yields:

11 + 12 + … + 19 = 20 + 21 + … + 25.

The integer next to 25 is 26. Starting with 26, (5) yields:

26 + 27 + … + 37 = 38 + 39 + … + 46.

The integer next to 46 is 47. Starting with 47, (5) yields:

47 + 48 + … + 61 = 62 + 63 + … + 73.

… …

The general case of Topic 3.2 is shown in Topic 5.

3.3. Odd Values of *p*

(i) For $x = (n + 1)\, p$ with the successive odd values of p: 1, 3, 5, …, we get:

$$n^2 + (n^2 + 1) + \ldots + (n^2 + n) = (n^2 + n + 1) + (n^2 + n + 2) + \ldots + (n^2 + 2n). \qquad (7.1)$$
$$(3n^2-1) + 3n^2 + \ldots + (3n^2 + 3n + 1) = (3n^2 + 3n + 2) + \ldots + (3n^2 + 6n + 1). \qquad (7.2)$$
$$(5n^2-2) + (5n^2-1) + \ldots + (5n^2 + 5n + 2) = (5n^2 + 5n + 3) + \ldots + (5n^2 + 10n + 2). \qquad (7.3)$$
$$(7n^2-3) + (7n^2-2) + \ldots + (7n^2 + 7n + 3) = (7n^2 + 7n + 4) + \ldots + (7n^2 + 14n + 3). \qquad (7.4)$$

… …

(ii) For $n = 1$, $p = 2a + 1$; (6) is reduced as: $T_1 = a + 1$. Then from (5), we get:

$$(a + 1) + (a + 2) + \ldots + (5a + 2) = (5a + 3) + (5a + 4) + \ldots + (7a + 3) = 3\,(2a + 1)^2. \qquad (8)$$

Examples: $1 + 2 = 3 = 3 \cdot 1^2$.

$2 + 3 + \ldots + 7 = 8 + 9 + 10 = 3 \cdot 3^2$.

$3 + 4 + \ldots + 12 = 13 + 14 + 15 + 16 + 17 = 3 \cdot 5^2$.

… …

Obviously starting with any positive integer, we can find an equality of the desired pattern.

3.4. Even Values of *p*

We can write (4) in other form:

$$T_1\ =\ \frac{x^2}{p}\ +\frac{1}{2}-\ 2p\ +\frac{p}{2}\,.$$

For an even value of p, T_1 can be an integer if $\frac{x^2}{p}$ is a mixed number whose fractional part is $\frac{1}{2}$. Then letting $x = 2^a\,(2b + 1)\,(2c + 1)$ and $p = 2^{2a+1}\,(2b + 1)$, we get:

$$\frac{x^2}{p}\ =\ \frac{(2b+1)\,(2c+1)^2}{2}.$$

The fractional part of the expression on the right is $\frac{1}{2}$.

Again,

$$x > p.$$
$$\Rightarrow 2^a\,(2b+1)\,(2c+1) > 2^{2a+1}\,(2b+1)$$
$$\Rightarrow 2c+1 > 2^{a+1}.$$
$$\Rightarrow 2c+1 = 2^{a+1} + 2n - 1,\ \ n \in \mathbb{N}.$$

Consequently,

$$x = 2^a\,(2b+1)\,(2^{a+1} + 2n - 1) \tag{9.1}$$

Putting the values of p and x in (4), we get:

$$T_1 = (2b+1)\left\{2n\left(n-1\right) - 2^{2a} + \tfrac{1}{2}\right\} + \tfrac{1}{2}. \tag{9.2}$$

The number of terms on the right of (5) is:

$$x_1 = x - p = 2^a\,(2b+1)\left(2^{a+1} + 2n - 1\right) - 2^{2a+1}\,(2b+1)$$
$$= 2^a\,(2b+1)\,(2n-1). \tag{9.3}$$

Examples: (i) For $p = 2^{2a+1}\,(2b+1) = 24$, we have:

$$a = b = 1;$$
$$x = 2^a\,(2b+1)\,(2^{a+1} + 2n - 1) = 6\,(2n+3);$$
$$x_1 = 2^a\,(2b+1)\,(2n-1) = 6\,(2n-1);$$
$$T_1 = (2b+1)\,\{2n\,(n-1) - 2^{2a} + \tfrac{1}{2}\} + \tfrac{1}{2} = 6n^2 - 6n - 10.$$

Then from (5),

$$(6n^2 - 6n - 10) + (6n^2 - 6n - 9) + \ldots + (6n^2 + 6n + 7)$$
$$= (6n^2 + 6n + 8) + (6n^2 + 6n + 9) + \ldots + (6n^2 + 18n + 1)$$
$$= 9\,(2n+3)\,(2n+1)\,(2n-1). \tag{10.1}$$

(ii) Similarly for $p = 2^5 \cdot 7$ or 224,

$$(14n^2 - 14n - 108) + (14n^2 - 14n - 107) + \ldots + (14n^2 + 42n + 87)$$
$$= (14n^2 + 42n + 88) + (14n^2 + 42n + 89) + \ldots + (14n^2 + 98n + 59).$$
$$= 98\,(2n+7)\,(2n+3)\,(2n-1). \tag{10.2}$$

… …

IV. Common Difference: *d* = *P*

Multiplying (5) throughout by p, we get the following identity:

For $T_2 = x^2 - 2xp + \frac{1}{2}\,p(p+1)$,

$$T_2 + (T_2 + p) + (T_2 + 2p) + \ldots + \{T_2 + p\,(x-1)\} = (T_2 + px) + \{T_2 + p\,(x+1)\} + \ldots + \{T_2 + p\,(2x - p - 1)\}. \tag{11}$$

4.1. *d* = *p* = 2

For $d = p = 2$, we get: $T_2 = x^2 - 4x + 3 = (x-3)\,(x-1)$.

Since $x > p$ and $p = 2$, we can write: $x = n + 2$.

Consequently,

$$T_2 = (x-3)\,(x-1) = (n+1)\,(n-1).$$

Then we get:

$$(n^2 - 1) + (n^2 + 1) + (n^2 + 3) + \ldots + (n+1)^2 = \{(n+1)^2 + 2\} + \{(n+1)^2 + 4\} + \ldots + \{(n+1)^2 + 2n\}. \tag{12}$$

(12) is comparable with (7.1). It is amazing that the left hand side of (12) starts with the product of two consecutive integers and ends with a square integer; and on the other hand starting and ending of the left hand side of (7.1) is just opposite.

(12) is equal to the sum:

$$S_3 = \frac{n+2}{2}\,\{2\,(n^2 - 1) + (n + 2 - 1)\,2\}$$
$$= n\,(n+1)\,(n+2). \tag{13}$$

The beauty of the integer-sequences in (13) is:

$$1 \cdot 2 \cdot 3 = 0 + 2 + 4 = 6;$$
$$2 \cdot 3 \cdot 4 = 3 + 5 + 7 + 9 = 11 + 13;$$
$$3 \cdot 4 \cdot 5 = 8 + 10 + 12 + 14 + 16 = 18 + 20 + 22;$$
$$4 \cdot 5 \cdot 6 = 15 + 17 + 19 + 21 + 23 + 25 = 27 + 29 + 31 + 33;$$

… …

(i) Even Values of *n*

Replacing n by $2m$ in (13), we get the sum:

$$S_4 = 4\,m\,(m+1)\,(2m+1)$$
$$= 24\,(1^2 + 2^2 + \ldots + m^2). \tag{14}$$

Putting n = 2, 4, 6, … successively in (12), we get the following results.

$$3 + 5 + 7 + 9 = 11 + 13 = 24 \cdot 1^2.$$

The odd number next to 13 is 15, starting with 15, we get:

$$15 + 17 + 19 + 21 + 23 + 25 = 27 + 29 + 31 + 33 = 24\,(1^2 + 2^2).$$

Similarly the next equality is:

$$35 + 37 + 39 + 41 + 43 + 45 + 47 + 49 = 51 + 53 + 55 + 57 + 59 + 61 = 24\,(1^2 + 2^2 + 3^2).$$

… …

(ii) Odd Values of *n*

When n is odd then 2 is the common divisor of all terms of (12). Dividing the terms by 2, we find the particular form of (5) where $d = 1$, $p = 2$, and $x = 2n + 1$ as shown

$$(2n^2 - 2n) + (2n^2 - 2n + 1) + \ldots + 2n^2 = (2n^2 + 1) + (2n^2 + 2) + \ldots + (2n^2 + 2n - 1) \qquad (15)$$
$$= n\,(2n + 1)\,(2n - 1).$$

Obviously any two consecutive terms of (15) are mutually prime to each other.

4.2. $d = P \geq 3$

If GCD(d, x) or GCD(p, x) = 1, then (11) yields the identities of which any two consecutive terms are mutually prime to each other. Some examples are shown here.

For $d = p = 3$ and $x = 4$, $\quad -2 + 1 + 4 + 7 = 10.$
For $d = p = 3$ and $x = 5$, $\quad 1 + 4 + 7 + 10 + 13 = 16 + 19.$
For $d = p = 4$ and $x = 7$, $\quad 3 + 7 + 11 + 15 + 19 + 23 + 27 = 31 + 35 + 39.$
For $d = p = 5$ and $x = 7$, $\quad -6 - 1 + 4 + 9 + 14 + 19 + 24 = 29 + 34.$
For $d = p = 5$ and $x = 8$, $\quad -1 + 4 + 9 + 14 + 19 + 24 + 29 + 34 = 39 + 44.$
For $d = p = 5$ and $x = 9$, $\quad 6 + 11 + 16 + 21 + 26 + 31 + 36 + 41 + 46 = 51 + 56 + 61 + 66.$

… …

4.3. Identity for the Triangular Numbers

Let d = p = x – 1. Then from (11), we get:

$$\left\{\tfrac{1}{2}x\,(3 - x)\right\} + \left\{\tfrac{1}{2}x\,(3 - x) + (x - 1)\right\} + \left\{\tfrac{1}{2}\,x\,(3 - x) + 2(x - 1)\right\} + \ldots + \left\{\tfrac{1}{2}x\,(3 - x) + (x - 1)^2\right\}$$
$$= \left\{\tfrac{1}{2}x\,(3 - x) + (x - 1)^2 + (x - 1)\right\} \qquad (16)$$
$$= \tfrac{1}{2}\,x\,(x + 1) = 1 + \ldots + x.$$

Examples: $1 + 2 = 3$
$0 + 2 + 4 = 6$
$-2 + 1 + 4 + 7 = 10$
$-5 - 1 + 3 + 7 + 11 = 15$

… …

V. Common Difference: $d \neq p$

Multiplying (5) throughout by d, we get the following identity:

For $T_3 = \frac{d}{2p}\{2\,x^2 - 4xp + p(p + 1)\}$,

$$T_3 + (T_3 + d) + (T_3 + 2d) + \ldots + \{T_3 + d\,(x - 1)\} = (T_3 + dx) + \{T_3 + d(x + 1)\} + \ldots + \{T_3 + d\,(2x - p - 1)\}. \qquad (17)$$

Letting $x = (n + 1)p$, we get: $T_3 = d\,(n^2p - \frac{p-1}{2})$; and then adding d with the last term on the right of (17), we get the sum:

$$S_5 = T_3 + d\,(2x - p - 1) + d$$
$$\Rightarrow S_5 = d\left\{(n + 1)^2p - \frac{p-1}{2}\right\} \qquad (18)$$

It follows that when $x = (n + 1)\,p$ then (18) is the first term on the left of (17) by the substitution of $n + 1$ for n. Topic 3.2 is the initial condition of (18) where $d = 1$.

5.1. $d = 2$ and $x = (n + 1)\,p$

For $d = 2$ and $x = (n + 1)\,p$, (11) is modified as:

$$\{p\,(2n^2 - 1) + 1\} + \{p\,(2n^2 - 1) + 3\} + \ldots + (2n^2p + 2n + p - 1)$$
$$= (2n^2p + 2np + p + 1) + (2n^2p + 2np + p + 3) + \ldots + (2n^2p + 4np + p - 1). \qquad (19)$$
$$= 6\,p^2\,(1^2 + 2^2 + \ldots + n^2).$$

For $p = 2a$ and $n = 1$, (19) is reduced as:

$$(2a + 1) + (2a + 3) + \ldots + (10a - 1) = (10a + 1) + (10a + 3) + \ldots + (14a - 1) = 24a^2. \qquad (20)$$

(20) shows the identity of desired pattern involving odd numbers only.

5.2. $d, p \geq 3$; $d \neq p$

(i) d, p and x are all odd; $d \neq p$

Let $d = (2a + 1)$; $p = (2a + 1)(2b + 1)$; $x = (2b + 1)(2c + 1)$; GCD$(2a + 1, 2b + 1) = 1$; and GCD$(2a + 1, 2c + 1) = 1$. Under the above conditions, we can find the desired identities of which any two consecutive terms are mutually prime to each other.

Examples: Then from (11),

(1) For $d = 3$, $p = 3·5 = 15$, and $x = 5·7 = 35$, we get: $59 + 62 + … + 161 = 164 + 167 + … + 221$.
(2) For $d = 5$, $p = 3·7 = 21$, and $x = 3·5 = 15$, we get: $17 + 22 + … + 117 = 122 + 127 + … + 147$.
(3) For $d = 7$, $p = 3·7 = 21$, and $x = 3·9 = 27$, we get: $– 68 – 61 – … + 114 = 121 + 128 + … + 156$.

(ii) GCD(d, x) = 1; $d \neq p$

For $d = (2a + 1)$, $p = 2^{2b+1}(2a + 1)$, $x = 2^b\{2^{b+1}(2a + 1) + 2n – 1\}$, we show below three examples.

Examples: (1) For $d = 3$, $p = 6$ and $x = 7$, we get: $– 7 – 4 – 1 + 2 + … + 11 = 14$.
(2) For $d = 5$, $p = 10$ and $x = 13$, we get: $– 18 – 13 – … + 42 = 47 + 52 + 57$.
(3) For $d = 3$, $p = 24$ and $x = 34$, we get: $– 44 – 41 – … + 55 = 58 + 61 + … + 85$.

(iii) Even Values of d; $d \neq p$

Example: (1) $d = 2(2b + 1)$; $p = 2^{2a}(2b + 1)(2c + 1)$; $x = 2^{a+n}(2c + 1)(2d + 1)$; and then for $d = 6$, $p = 12$ and $x = 14$, we get: $– 31 – 25 – … + 41 + 47 = 53 + 59$.

Example: (2) $d = 2(2b + 1)$; $p = 2^{2a+1}(2b + 1)(2c + 1)$; $x = 2^{a+n}(2c + 1)(2d + 1)$; and then for $d = 6$, $p = 24$ and $x = 28$, we get: $55 + 61 + … + 227 = 233 + 239 + 245 + 251$.

Example: (3) $d = 2^{a+1}(2b + 1)$; $p = 2^{2a}(2b + 1)(2c + 1)$; $x = (2c + 1)(2d + 1)$; and then for $d = 4$, $p = 12$ and $x = 15$, we get: $–19 – 15 – … + 33 + 37 = 41 + 45 + 49$.

In all three examples, GCD$(2b + 1, 2c + 1) = 1$; GCD$(2b + 1, 2d + 1) = 1$; and GCD$(2c + 1, 2d + 1) = 1$.

VI. Some Examples Starting with 1

$1 + 2 = 3$.
$1 + 4 + 7 + 10 + 13 = 16 + 19$.
$1 + 7 + 13 + … + 55 = 61 + 67 + 73 + 79$.
$1 + 2 + 3 + … + 14 = 15 + 16 + 17 + 18 +1 9 + 20$.
$1 + 17 + 33 + 49 + … + 417 = 433 + 449 + 465 + … + 593$.

VII. Identity of Degree 2 with the Bases of the Terms in AP

By the process of derivation of (5), we get the identity of degree 2 with the bases of the terms in AP: For

$$T_4 = \frac{1}{p}\left[x^2 - 2xp + \frac{1}{2}p(p+1) + \sqrt{\left\{x^2 - 2xp + \frac{1}{2}p(p-1)\right\}^2 + p\left\{x^2(2x+1) - 2xp(2x-p+1) - \frac{1}{6}p\,(p-1)(2p-1)\right\}}\,\right].$$

$$T_4^2 + (T_4+1)^2 + (T_4+2)^2 + \cdots + (T_4+x-1)^2 = (T_4+\mathrm{x})^2 + (T_4+x+1)^2 + \cdots + (T_4+2x-p-1)^2 \quad (21)$$

Putting $p = 1$, we get: $T_4 = (2x – 1)(x – 1)$. Since $x \geq 2$, we can write: $x = n + 1$. Then $T_4 = n(2n + 1)$; and (21) is reduced as:

$$\{n(2n + 1)\}^2 + \{n(2n + 1) + 1\}^2 + \{n(2n + 1) + 2\}^2 + … + \{n(2n + 1) + n\}^2$$
$$= \{2n(n + 1) + 1\}^2 + \{2n(n + 1) + 2\}^2 + … + \{n(2n + 3)\}^2 . \quad (22)$$

For n = 1, 2, 3, 4, …, in succession, we get:

$3^2 + 4^2 = 5^2$.
$10^2 + 11^2 + 12^2 = 13^2 + 14^2$.
$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.
$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$.
… …

It is remarkable that the above successive relations start with the square of the successive alternate triangular numbers: 3 = 1 + 2; 10 = 1 + 2 + 3 + 4; 21 = 1 + 2 + 3 + 4 + 5 + 6; 36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8; and so on.

VIII. Equalities Involving Integers in AP with Addition of Same Integers in All Terms

$1^2 + 2^2 \neq 3^2$. But $(1^2 + 4) + (2^2 + 4) = (3^2 + 4)$.

Similarly,

$(3^2 + 63) + (5^2 + 63) + (7^2 + 63) + (9^2 + 63) = (11^2 + 63) + (13^2 + 63)$.
$(4^2 + 36) + (5^2 + 36) + (6^2 + 36) = (7^2 +36) + (8^2 + 36)$.
$(4^2 + 66) + (5^2 + 66) + (6^2 + 66) + (7^2 + 66) + (8^2 + 66) = (9^2 + 66) + (10^2 +66) + (11^2 + 66)$.
$\{(– 2)^2 +10\} + (1^2 + 10) + (4^2 + 10) + (7^2 +10) = (10^2 +10)$.
$(9^2 +144) + (10^2 +144) + (11^2 +144) + (12^2 +144) = (13^2 +144) + (14^2 +144) + (15^2 +144)$.
$(3^2 + 98) + (4^2 +98) + … + (12^2 + 98) = (13^2 +98) + (14^2 +98) + (15^2 +98) + (16^2 +98) + (17^2 +98)$.

References

[1]. Arithmetic and geometric progressions, http://www.mathcentre.ac.uk.
[2]. Arithmetic Sequences and Series, http://www.regentsprep.org/regents/math/algtrig/atp2/arithseq.htm