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# Fractional Order Finite Difference Scheme For Two Dimensional Space-Time Fractional Diffusion Equation

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***Abstract:*** *The aim of this paper is to develop a fractional order implicit finite difference scheme for two-dimensional space-time fractional diffusion equation. We also prove that the scheme is unconditionally stable and convergent. As an application of this scheme numerical solution for two-dimensional space-time fractional diffusion equation is obtained with the help of Mathematica software.*
***Keywords:*** *Finite difference scheme, Space-Time fractional diffusion equation, Mathematica.*

## I. Introduction

The fractional order integral and differentiation, which represent a rapidly growing field in both theory and in applications to real world problems. Fractional order partial differential equations, as generalizations of classical integer order partial differential equations are increasingly used to model problems in fluid flow, finance, physics and other areas of applications [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. We consider two dimensional space-time fractional diffusion equation (2D-STFDE) under Dirichlet boundary conditions. The two dimensional space-time fractional diffusion equation is obtained from the standard two dimensional diffusion equation by replacing the first order time derivative by fractional derivative of order $\alpha, 0 < \alpha < 1$ and the second order space derivatives by fractional derivatives with respective x and y of order $\beta$ and $\gamma$ , $1 < \beta < 2$ , $1 < \gamma < 2$ respectively.

Consider the two dimensional space-time fractional diffusion equation

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = d(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial x^{\beta}} + e(x,y)\frac{\partial^{\gamma} u(x,y,t)}{\partial y^{\gamma}}, 0 < t \le T, (x,y), \in \Omega \tag{1.1}$$

$$boundary condition\; u(x,y,t) = 0, (x,y) \in \partial\Omega \tag{1.2}$$

$$initial\ condition : u(x,y,0) = u_0(x,y) \tag{1.3}$$

Where $\Omega = \{(x,y)/ -a \le x \le a, -b \le x \le b\}, d(x,y) > 0, e(x,y) > 0.$

We assume that this fractional diffusion equation has a unique and sufficiently smooth solution under the initial and boundary conditions. The classical diffusion equation in two dimensions is given by $\alpha = 1$ and $\beta = \gamma = 2$. In equation (1.1), for the first order we use the Caputo fractional derivative of order $\alpha (0 < \alpha < 1)$. For every $\beta$ and $\gamma$ $(0 \le n-1 < \beta, \gamma < n)$ the Riemann-Liouville derivative exists and coincides with the $Günwald - Letnikov$ derivative. The relationship between the Riemann-Liouville and $Günwald - Letnikov$ definitions also has another consequences which is important for the numerical approximation of fractional order differential equations, formulation of applied problems, manipulation with fractional derivatives and formulation of physically meaningful initial and boundary value problems for fractional order differential equations. This allows the use of the Riemann-Liouville definition during problem formulation and then the $Günwald - Letnikov$ definitions for obtaining numerical solution. Therefore we use the shifted $Günwald$ formula at all time levels for approximating the second order space derivatives.

The plan of the paper is as follows: In section 2, the implicit space-time finite difference scheme is developed for two dimensional space-time fractional diffusion equation. The section 3, is devoted for stability of the scheme and the question of convergence is proved in section 4. Numerical solution of two dimensional space-time fractional diffusion equation is obtained using Mathematica software in the last section.

## II. Finite Difference Approximation For Two Dimensional Stfde

In this section we develop the fractional order implicit Euler finite difference scheme for two dimensional space-time fractional diffusion equation (STFDE) (1.1)-(1.3).

We apply Euler method to equation (1.1) on rectangular domain $\Omega$ with grid points $(x_i, y_j)$ where $x_i = i\Delta x$ and $y_j = j\Delta y$ for i = 0, 1, 2, ..., $N_x$ and j = 0, 1, 2, ..., $N_y$ respectively. Define $t_n = n\Delta t$ to integration time $0 \le t_n \le T$, $\Delta x = h > 0$ is the grid size in x-direction, $\Delta x = \dfrac{x_R - x_L}{N_x}$ with $x_i = x_L + i\Delta x$ for i = 0, 1, 2, ..., $N_x$, $\Delta y > 0$ is the grid size in y-direction, $\Delta y = \dfrac{y_R - y_L}{N_y}$ with $y_j = y_L + j\Delta y$ for j = 0, 1, 2, ..., $N_y$. Let $u_{i,j}^n$ be the numerical approximation to $u(x_i, y_j, t_n)$. Similarly we define $d_{i,j} = d(x_i, y_j)$ and $e_{i,j} = e(x_i, y_j)$.

In the differential equation (1.1) the time fractional derivative term is approximated by the following scheme:

$$\frac{\partial^\alpha u(x_i, y_j, t_{n+1})}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{u(x_i, y_j, t_{k+1}) - u(x_i, y_j, t_k)}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \frac{d\xi}{(t_{n+1} - \xi)^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{u(x_i, y_j, t_{k+1}) - u(x_i, y_j, t_k)}{\Delta t} \int_{(n-k)\Delta t}^{(n-k+1)\Delta t} \frac{d\eta}{\eta^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \frac{u(x_i, y_j, t_{n+1-k}) - u(x_i, y_j, t_{n-k})}{\Delta t} \int_{k\Delta t}^{(k+1)\Delta t} \frac{d\eta}{\eta^\alpha}$$

$$= \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \frac{u(x_i, y_j, t_{n+1-k}) - u(x_i, y_j, t_{n-k})}{\Delta t} [(k+1)^{1-\alpha} - k^{1-\alpha}]$$

$$= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} [u(x_i, y_j, t_{n+1}) - u(x_i, y_j, t_n)] +$$

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [u(x_i, y_j, t_{n+1-k}) - u(x_i, y_j, t_{n-k})][(k+1)^{1-\alpha} - k^{1-\alpha}]$$

$$= \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} [u_{i,j}^{n+1} - u_{i,j}^n] + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} b_k [u_{i,j}^{n-k+1} - u_{i,j}^{n-k}]$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}, k = 0,1,2,..., n.$

We discretize the spatial $\beta$-order fractional derivative using the *Grünwald* finite difference formula at all time levels. The standard *Grünwald* estimates generally yield unstable finite difference equation regardless of whatever the resulting finite difference method is an explicit or an implicit system for related discussion. Therefore, we start with a right shifted *Grünwald* approximation to the fractional derivative term and in this paper we prove that this leads to stable and convergent alternating directions implicit (ADI) implementation for the two-dimensional implicit Euler formulation. The right-shifted *Grünwald* formula for $1 < \beta \le 2$ is [7]

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(-\beta)} \lim_{N \to \infty} \frac{1}{h^\beta} \sum_{k=0}^{N} \frac{\Gamma(k-\beta)}{\Gamma(k+1)} u(x - (k-1)h, t)$$

where $N_x$ is the positive integer, $h = \dfrac{(x_R - x_L)}{N}$ and $\Gamma(.)$ is the gamma function.

We also define the normalized *Grünwald* weights by

$$g_{\beta,k} = \frac{\Gamma(k-\beta)}{\Gamma(-\beta)\Gamma(k+1)}, k = 0,1,...$$

For $D_x^\beta u(x_i, y_j, t_{n+1})$, $D_y^\gamma u(x_i, y_j, t_{n+1})$, we adopt the right shifted *Grünwald* formula at all time levels for approximating the second order space derivatives by implicit type numerical approximation to equation (1.1), we get

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}[u_{i,j}^{n+1}-u_{i,j}^{n}]+\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}b_k[u_{i,j}^{n-k+1}-u_{i,j}^{n-k}]=d_{i,j}\ \delta_{\beta,x}u_{i,j}^{n+1}+e_{i,j}\ \delta_{\gamma,y}u_{i,j}^{n+1}$$

where the above fractional partial differentiation operators are defined as

$$\delta_{\beta,x}u_{i,j}^{n+1}=\frac{1}{(\Delta x)^{\beta}}\sum_{k=0}^{i+1}g_{\beta,k}u_{i-k+1,j}^{n+1}\ \text{and}\ \delta_{\gamma,y}u_{i,j}^{n+1}=\frac{1}{(\Delta y)^{\gamma}}\sum_{k=0}^{j+1}g_{\beta,k}u_{i,j-k+1}^{n+1}$$

which are an $O(\Delta x)$ and $O(\Delta y)$ approximation to the $\beta$-order and $\gamma$-order fractional derivatives respectively.

Therefore the approximated equation is

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}[u_{i,j}^{n+1}-u_{i,j}^{n}]+\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}b_k[u_{i,j}^{n-k+1}-u_{i,j}^{n-k}]=\frac{d_{i,j}}{(\Delta x)^{\beta}}\sum_{k=0}^{i+1}g_{\beta,k}u_{i-k+1,j}^{n+1}$$
$$+\frac{e_{i,j}}{(\Delta y)^{\gamma}}\sum_{k=0}^{j+1}g_{\gamma,k}u_{i,j-k+1}^{n+1}$$

$$u_{i,j}^{n+1}-u_{i,j}^{n}+\sum_{k=1}^{n}b_k[u_{i,j}^{n-k+1}-u_{i,j}^{n-k}]=\frac{d_{i,j}(\Delta t)^{-\alpha}\Gamma(2-\alpha)}{(\Delta x)^{\beta}}\sum_{k=0}^{i+1}g_{\beta,k}u_{i-k+1,j}^{n+1}$$
$$+\frac{e_{i,j}(\Delta t)^{-\alpha}\Gamma(2-\alpha)}{(\Delta y)^{\gamma}}\sum_{k=0}^{j+1}g_{\gamma,k}u_{i,j-k+1}^{n+1}$$

$$u_{i,j}^{n+1}-u_{i,j}^{n}+\sum_{k=1}^{n}b_k[u_{i,j}^{n-k+1}-u_{i,j}^{n-k}]=r_1\sum_{k=0}^{i+1}g_{\beta,k}u_{i-k+1,j}^{n+1}+r_2\sum_{k=0}^{j+1}g_{\gamma,k}u_{i,j-k+1}^{n+1}$$

$$\text{where}\ r_1=\frac{d_{i,j}(\Delta t)^{\alpha}\Gamma(2-\alpha)}{(\Delta x)^{\beta}}\ \text{and}\ r_2=\frac{e_{i,j}(\Delta t)^{\alpha}\Gamma(2-\alpha)}{(\Delta y)^{\gamma}}$$

$$u_{i,j}^{n+1}-r_1\sum_{k=0}^{i+1}g_{\beta,k}u_{i-k+1,j}^{n+1}-r_2\sum_{k=0}^{j+1}g_{\gamma,k}u_{i,j-k+1}^{n+1}=u_{i,j}^{n}-\sum_{k=1}^{n}b_k[u_{i,j}^{n-k+1}-u_{i,j}^{n-k}]$$

For n = 0, we have

$$(1+\beta r_1+\gamma r_2)u_{i,j}^{1}-r_1\sum_{k=0,k\neq 1}^{i+1}g_{\beta,k}u_{i-k+1,j}^{1}-r_2\sum_{k=0,k\neq 1}^{j+1}g_{\gamma,k}u_{i,j-k+1}^{1}=u_{i,j}^{0} \tag{2.1}$$

For n = 1, 2, ..., we get

$$(1+\beta r_1+\gamma r_2)u_{i,j}^{n+1}-r_1\sum_{k=0,k\neq 1}^{i+1}g_{\beta,k}u_{i-k+1,j}^{n+1}-r_2\sum_{k=0,k\neq 1}^{j+1}g_{\gamma,k}u_{i,j-k+1}^{n+1}$$

$$=(1-b_1)u_{i,j}^{n}+\sum_{k=1}^{n-1}(b_k-b_{k+1})u_{i,j}^{n-k}+b_n u_{i,j}^{0} \tag{2.2}$$

The initial condition is approximated as

$u_{i,j}^{0}=\phi_{i,j}=\phi(x_i,y_j)$, i = 0, 1, 2, ... , $N_x$, j = 0, 1, 2, ... , $N_y$.

The Dirichlet boundary conditions on the rectangle in the form $u(x,0,t)=u(0,y,t)=u(L,y,t)=u(x,L,t)=0$ are approximated as $u_{i,0}^{n}=u_{0,j}^{n}=u_{L,j}^{n}=u_{i,L}^{n}=0$ respectively.

Therefore the fractional approximated IBVP is

$$u_{i,j}^{n+1} - r_1 \sum_{k=0}^{i+1} g_{\beta,k} u_{i-k+1,j}^{n+1} - r_2 \sum_{k=0}^{j+1} g_{\gamma,k} u_{i,j-k+1}^{n+1} = u_{i,j}^{n} - \sum_{k=1}^{n} b_k [u_{i,j}^{n-k+1} - u_{i,j}^{n-k}] \quad (2.3)$$

$$initial\ condition\ u_{i,j}^{0} = \phi_{i,j}, i = 0,1,2,..., N_x, j = 0,1,2,..., N_y \quad (2.4)$$

$$boundary\ conditions\ u_{i,0}^{n} = u_{0,j}^{n} = u_{L,j}^{n} = u_{i,L}^{n} = 0 \quad (2.5)$$

where $r_1 = \frac{d_{i,j}(\Delta t)^{\alpha}\Gamma(2-\alpha)}{(\Delta x)^{\beta}}$ *and* $r_2 = \frac{e_{i,j}(\Delta t)^{\alpha}\Gamma(2-\alpha)}{(\Delta y)^{\gamma}}$.

The above two-dimensional implicit Euler method has local truncation error of the form $0(\Delta t)^{\alpha} + 0(\Delta x) + 0(\Delta y)$.

The finite-difference equations (2.1) and (2.2) are expressed in the matrix form:

$$AU^{1} = U^{0} \quad (2.6)$$

$$AU^{n+1} = (1-b_1)U^{n} + \sum_{k=1}^{n}(b_k - b_{k+1})U^{n-k} + b_n U^{0} \quad (2.7)$$

where

$$U^{n} = [U_1^n, U_2^n, \cdots, U_{(N_x-1)(N_y-1)}]^T$$
$$= [u_{1,1}^n, u_{2,1}^n, ..., u_{N_x-1,1}^n,$$
$$u_{1,2}^n, u_{2,2}^n, ..., u_{N_x-1,2}^n,$$
$$\cdots\cdots$$
$$u_{1,N_y-1}^n, u_{2,N_y-1}^n, ..., u_{N_x-1,N_y-1}^n]^T$$

and $A = [a_{ij}]$ is a square matrix of coefficients

$$A = \begin{pmatrix} D & E_0 & & & & \\ E_2 & D & E_0 & & & \\ E_3 & E_2 & D & E_0 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & & & & & \\ E_k & E_{k-1} & \cdots & & E_2 & D \end{pmatrix},$$

and

$$D = \begin{pmatrix} \lambda_1 & -r_1 g_{\beta,0} & & & \\ -r_1 g_{\beta,2} & \lambda_1 & -r_1 g_{\beta,0} & & \\ -r_1 g_{\beta,3} & -r_1 g_{\beta,2} & \lambda_1 & -r_1 g_{\beta,0} & \\ \vdots & \vdots\ddots & & \ddots & \ddots \\ & & & & \\ -r_1 g_{\beta,N_x} & -r_1 g_{\beta,N_x-1} & \cdots & -r_1 g_{\beta,2} & \lambda_1 \end{pmatrix}$$

,

where $\lambda_1 = 1 + \beta r_1 + \gamma r_2$ and $E_k$ is a diagonal matrix with diagonal element of each $E_k$ is $-r_1 g_{\gamma,k}$ for k = 0, 2, 3, ..., $N_x$.

The system of algebraic equations (2.3)-(2.5) is solved by using Mathematica software in section 5.

## III. Stability

This section is devoted for the stability criteria of the space-time fractional implicit finite difference scheme (2.3)-(2.5) for the STFDE (1.1)-(1.3).

Lemma 3.1: In equation (2.1) and (2.2), the coefficients $b_k$ and $g_{\beta,k}$ $for\ k = 0,1,2,....$ satisfy

(i) $b_k > b_{k+1}$, k = 0, 1, 2,...

(ii) $b_0 = 1, b_k > 0$, k = 0, 1, 2,...

(iii) $g_{\beta,1} = -\beta, g_{\beta,k} \geq 0 (k \neq 1), \sum_{k=0}^{\infty} g_{\beta,k} = 0$

(iv) For any positive integer n, $\sum_{k=0}^{n} g_{\beta,k} < 0.$

(v) $g_{\gamma,1} = -\gamma, g_{\gamma,k} \geq 0 (k \neq 1), \sum_{k=0}^{\infty} g_{\gamma,k} = 0$

(vi) For any positive integer n, $\sum_{k=0}^{n} g_{\gamma,k} < 0.$

**Theorem 3.1** The fractional order implicit finite difference scheme for two dimensional fractional diffusion equation (1.1) - (1.3) defined by equation (2.3) - (2.5) is unconditionally stable.

Proof: We assume that $\bar{u}_{i,j}^n, i = 0,1,2,..., N_x, j = 0,1,2,..., N_y$ is the exact solution of equation (1.1), the error $\varepsilon_{i,j}^n = \bar{u}_{i,j}^n - u_{i,j}^n, i = 0,1,2,..., N_x, j = 0,1,2,..., N_y$ satisfies equations (2.1) and (2.2), therefore, we have

$$(1 + \beta r_1 + \gamma r_2)\varepsilon_{i,j}^1 - r_1 \sum_{k=0,k\neq 1}^{i+1} g_{\beta,k}\varepsilon_{i-k+1,j}^1 - r_2 \sum_{k=0,k\neq 1}^{j+1} g_{\gamma,k}\varepsilon_{i,j-k+1}^1 = \varepsilon_{i,j}^0 \tag{3.1}$$

$$(1 + \beta r_1 + \gamma r_2)\varepsilon_{i,j}^{n+1} - r_1 \sum_{k=0,k\neq 1}^{i+1} g_{\beta,k}\varepsilon_{i-k+1,j}^{n+1} - r_2 \sum_{k=0,k\neq 1}^{j+1} g_{\gamma,k}\varepsilon_{i,j-k+1}^{n+1}$$

$$= (1 - b_1)\varepsilon_{i,j}^n + \sum_{k=1}^{n-1}(b_k - b_{k+1})\varepsilon_{i,j}^{n-k} + b_n\varepsilon_{i,j}^0 \tag{3.2}$$

where $i = 0,1,2,..., N_x - 1, j = 0,1,2,..., N_y - 1.$

Now, equations (3.1) and (3.2) can be written in the matrix form:

$$\left.\begin{aligned} &AE^1 = E^0 \\ &AE^{n+1} = (1 - b_1)E^n + \sum_{k=1}^{n-1}(b_k - b_{k+1})\, E^{n-k} + b_n\, E^0 \\ &E^0 = 0 \end{aligned}\right\} \tag{3.3}$$

where

$$E^n = [E_{1,1}^n, E_{2,1}^n, ..., E_{N_x-1,1}^n,$$
$$E_{1,2}^n, E_{2,2}^n, ..., E_{N_x-1,2}^n,$$

$\dots$
$E^n_{1,N_y-1}, E^n_{2,N_y-1}, ..., E^n_{N_x-1,N_y-1}]^T$

We have to analyze the stability by mathematical induction: Let

$$\| E^1 \|_\infty = \left|\varepsilon^1_{l,m}\right| = \max_{1\le i\le N_x-1, 1\le j\le N_y-1} \left|\varepsilon^1_{i,j}\right|$$

When n = 1 and by lemma 3.1, we have

$$\begin{aligned}
\| E^1 \|_\infty = \left|\varepsilon^1_{l,m}\right| &\le (1+\beta r_1+\gamma r_2)\left|\varepsilon^1_{l,m}\right| - r_1 \sum_{k=0,k\ne1}^{l+1} g_{\beta,k}\left|\varepsilon^1_{l,m}\right| - r_2 \sum_{k=0,k\ne1}^{m+1} g_{\gamma,k}\left|\varepsilon^1_{l,m}\right| \\
&\le (1+\beta r_1+\gamma r_2)\left|\varepsilon^1_{l,m}\right| - r_1 \sum_{k=0,k\ne1}^{l+1} g_{\beta,k}\left|\varepsilon^1_{l-k+1,m}\right| - r_2 \sum_{k=0,k\ne1}^{m+1} g_{\gamma,k}\left|\varepsilon^1_{l,m-k+1}\right| \\
&\le \left|(1+\beta r_1+\gamma r_2)\varepsilon^1_{l,m} - r_1 \sum_{k=0,k\ne1}^{l+1} g_{\beta,k}\varepsilon^1_{l-k+1,m} - r_2 \sum_{k=0,k\ne1}^{m+1} g_{\gamma,k}\varepsilon^1_{l,m-k+1}\right| \\
&= \left|\varepsilon^0_{l,m}\right| \\
&\le \| E^0 \|_\infty
\end{aligned}$$

Suppose $\| E^{n+1} \|_\infty = \left|\varepsilon^{n+1}_{l,m}\right| = \max_{1\le i\le N_x-1, 1\le j\le N_y-1} \left|\varepsilon^{n+1}_{i,j}\right|$ ,

we assume that $\| E^n \|_\infty \le \| E^0 \|_\infty$, n = 1, 2, ... , k and using lemma 3.1, we have

$$\begin{aligned}
\| E^{n+1} \|_\infty = \left|\varepsilon^{n+1}_{l,m}\right| &\le (1+\beta r_1+\gamma r_2)\left|\varepsilon^{n+1}_{l,m}\right| - r_1 \sum_{k=0,k\ne1}^{l+1} g_{\beta,k}\left|\varepsilon^{n+1}_{l,m}\right| - r_2 \sum_{k=0,k\ne1}^{m+1} g_{\gamma,k}\left|\varepsilon^{n+1}_{l,m}\right| \\
&\le (1+\beta r_1+\gamma r_2)\left|\varepsilon^{n+1}_{l,m}\right| - r_1 \sum_{k=0,k\ne1}^{l+1} g_{\beta,k}\left|\varepsilon^{n+1}_{l-k+1,m}\right| - r_2 \sum_{k=0,k\ne1}^{m+1} g_{\gamma,k}\left|\varepsilon^{n+1}_{l,m-k+1}\right| \\
&\le \left|(1+\beta r_1+\gamma r_2)\varepsilon^{n+1}_{l,m} - r_1 \sum_{k=0,k\ne1}^{l+1} g_{\beta,k}\varepsilon^{n+1}_{l-k+1,m} - r_2 \sum_{k=0,k\ne1}^{m+1} g_{\gamma,k}\varepsilon^{n+1}_{l,m-k+1}\right| \\
&\le \left|(1-b_1)\varepsilon^n_{l,m} + b_n\varepsilon^0_{l,m} + \sum_{k=1}^{n-1}(b_k-b_{k+1})\varepsilon^{n-k}_{l,m}\right| \\
&\le (1-b_1)\| E^n \|_\infty + b_n \| E^0 \|_\infty + \sum_{k=1}^{n-1}(b_k-b_{k+1})\| E^{n-k} \|_\infty \\
&\le (1-b_1)\| E^0 \|_\infty + b_n \| E^0 \|_\infty + \sum_{k=1}^{n-1}(b_k-b_{k+1})\| E^0 \|_\infty \\
&\le \| E^0 \|_\infty
\end{aligned}$$

Hence by mathematical induction, we have prove that

$\| E^n \|_\infty \le \| E^0 \|_\infty$ *for all n.*

Hence the proof is completed.

## IV. Convergence

In this section we discuss the convergence of the finite difference scheme.

**Theorem 4.1** The solution of the fractional order implicit finite difference scheme for two dimensional space-time fractional diffusion equation (1.1) - (1.3) defined by (2.3) - (2.5) is convergent.

Proof: Let $u^n_{i,j}, i = 0,1,2,..., N_x - 1, j = 0,1,2,..., N_y - 1$ be the numerical solution of the two dimensional space-time fractional diffusion equation (1.1)-(1.3) at mesh point $(x_i, y_j, t_n)$. We define $e_{i,j}^n = \bar{u}_{i,j}^n - u_{i,j}^n$, where $\bar{u}_{i,j}^n$ is the exact solution of the two dimensional space-time fractional diffusion equation (1.1)-(1.3).

Let

$$\begin{aligned} e_{i,j}^n &= [e_{1,1}^n, e_{2,1}^n, ..., e_{N_x-1,1}^n, \\ & e_{1,2}^n, e_{2,2}^n, ..., e_{N_x-1,2}^n, \\ & \ldots \\ & e_{1,N_y-1}^n, e_{2,N_y-1}^n, ..., e_{N_x-1,N_y-1}^n]^T \end{aligned}$$

Using $e^0 = 0$ and $e_{i,j}^n = \bar{u}_{i,j}^n - u_{i,j}^n$, satisfying equations (2.1) and (2.2), we get

$$(1+\beta r_1 + \gamma r_2)e_{i,j}^1 - r_1 \sum_{k=0,k\neq 1}^{i+1} g_{\beta,k} e_{i-k+1,j}^1 - r_2 \sum_{k=0,k\neq 1}^{j+1} g_{\gamma,k} e_{i,j-k+1}^1 = e_{i,j}^0$$

$$(1+\beta r_1 + \gamma r_2)e_{i,j}^{n+1} - r_1 \sum_{k=0,k\neq 1}^{i+1} g_{\beta,k} e_{i-k+1,j}^{n+1} - r_2 \sum_{k=0,k\neq 1}^{j+1} g_{\gamma,k} e_{i,j-k+1}^{n+1}$$

$$= (1-b_1)e_{i,j}^n + \sum_{k=1}^{n-1}(b_k - b_{k+1})e_{i,j}^{n-k} + b_n e_{i,j}^0$$

where $i = 0,1,2,..., N_x - 1,\ j = 0,1,2,..., N_y - 1$.

We have mathematical induction to prove this theorem. Considering $\tau^n$ is the truncation error at time level $t_n$ and using lemma 3.1, we get the convergence analysis as follows:

For n = 1 , let $\left|e_{l,m}^1\right| = \max_{1\le i\le N_x-1, 1\le j\le N_y-1} \left|e_{i,j}^1\right|$, we have

$$\begin{aligned} \left|e_{l,m}^1\right| &\le (1+\beta r_1 + \gamma r_2)\left|e_{l,m}^1\right| - r_1 \sum_{k=0,k\neq 1}^{l+1} g_{\beta,k}\left|e_{l,m}^1\right| - r_2 \sum_{k=0,k\neq 1}^{m+1} g_{\gamma,k}\left|e_{l,m}^1\right| + \left|\tau^1\right| \\ &\le (1+\beta r_1 + \gamma r_2)\left|e_{l,m}^1\right| - r_1 \sum_{k=0,k\neq 1}^{l+1} g_{\beta,k}\left|e_{l-k+1,m}^1\right| - r_2 \sum_{k=0,k\neq 1}^{m+1} g_{\gamma,k}\left|e_{l,m-k+1}^1\right| + \left|\tau^1\right| \\ &\le \left|(1+\beta r_1 + \gamma r_2)e_{l,m}^1 - r_1 \sum_{k=0,k\neq 1}^{l+1} g_{\beta,k} e_{l-k+1,m}^1 - r_2 \sum_{k=0,k\neq 1}^{m+1} g_{\gamma,k} e_{l,m-k+1}^1 + \tau^1\right| \\ &= \left|e_{l,m}^0 + \tau^1\right| \\ &\le \left|e_{l,m}^0\right| + \left|\tau^1\right| \end{aligned}$$

Suppose $\left|e^n\right| \le \left|e^0\right| and \left|e_{l,m}^{n+1}\right| = \max_{1\le i\le N_x-1, 1\le j\le N_y-1} \left|e_{i,j}^{n+1}\right|$, we have

$$\begin{aligned} \left|e^{n+1}\right| &\le (1+\beta r_1 + \gamma r_2)\left|e_{l,m}^{n+1}\right| - r_1 \sum_{k=0,k\neq 1}^{l+1} g_{\beta,k}\left|e_{l,m}^{n+1}\right| - r_2 \sum_{k=0,k\neq 1}^{m+1} g_{\gamma,k}\left|e_{l,m}^{n+1}\right| + \left|\tau^{n+1}\right| \\ &\le (1+\beta r_1 + \gamma r_2)\left|e_{l,m}^{n+1}\right| - r_1 \sum_{k=0,k\neq 1}^{l+1} g_{\beta,k}\left|e_{l-k+1,m}^{n+1}\right| - r_2 \sum_{k=0,k\neq 1}^{m+1} g_{\gamma,k}\left|e_{l,m-k+1}^{n+1}\right| + \left|\tau^{n+1}\right| \end{aligned}$$

$$\le \left|(1+\beta r_1+\gamma r_2)e_{l,m}^{n+1} - r_1 \sum_{k=0,k\neq 1}^{l+1} g_{\beta,k} e_{l-k+1,m}^{n+1} - r_2 \sum_{k=0,k\neq 1}^{m+1} g_{\gamma,k} e_{l,m-k+1}^{n+1} + \tau^{n+1}\right|$$

$$\le \left|(1-b_1)e_{l,m}^{n} + b_n e_{l,m}^{0} + \sum_{k=1}^{n-1}(b_k - b_{k+1})e_{l,m}^{n-k} + \tau^{n+1}\right|$$

$$\le (1-b_1)\left|e_{l,m}^{n}\right| + b_n\left|e_{l,m}^{0}\right| + \sum_{k=1}^{n-1}(b_k - b_{k+1})\left|e_{l,m}^{n-k}\right| + \left|\tau^{n+1}\right|$$

$$\le (1-b_1)\left|e_{l,m}^{0}\right| + b_n\left|e_{l,m}^{0}\right| + \sum_{k=1}^{n-1}(b_k - b_{k+1})\left|e_{l,m}^{0}\right| + \left|\tau^{n+1}\right|$$

$$\le \left|e_{l,m}^{0}\right| + \left|\tau^{n+1}\right|$$

Hence by mathematical induction, we have prove

$$\left|e^n\right| \le \left|e_{l,m}^0\right| + \left|\tau^n\right|, \text{ for all } n.$$

Since, $\lim_{(\Delta x,\Delta y,\Delta t)\to(0,0,0)} \left|\tau^n\right| = 0,$ and $\left|e_{l,m}^0\right| = \left|e^0\right|$

Hence $\left|e^n\right| \to 0, as\,(\Delta x,\Delta y,\Delta t) \to (0,0,0)$ .

The proof is completed.

## V. Numerical Solutions

In this section we obtain the numerical solution of two dimensional space-time fractional diffusion equation by a discretization scheme developed in equations (2.3)-(2.5). In our test problem we consider the value of $\alpha = 0.8, \beta = 1.8\, and\, \gamma = 1.8,$

$u(x,y,0) = sin\pi x sin\pi y,\ 0 \le x,y \le 1\, and\, u(x,y,t) = 0$ . Also here we consider the square grids with $\Delta x = \Delta y = 0.20,\, and\, \Delta t = 0.01.$ The approximate solution of the test problem is

$$u1,1^1 = 0.31965,\ u2,1^1 = 0.514383,\ u3,1^1 = 0.514383,\ u4,1^1 = 0.31965,$$

$$u1,2^1 = 0.513734,\ u2,2^1 = 0.827546,\ u3,2^1 = 0.827546,\ u4,2^1 = 0.513734,$$

$$u1,3^1 = 0.513734,\ u2,3^1 = 0.827546,\ u3,3^1 = 0.827546,\ u4,3^1 = 0.513734,$$

$$u1,4^1 = 0.31965,\ u2,4^1 = 0.514383,\ u3,4^1 = 0.514383,\ u4,4^1 = 0.31965,$$

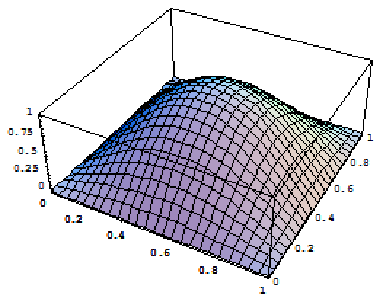


*Fig.*5.1: Initial solution of the test problem.

## VI. Conclusions

(i) We develop a fractional order finite difference scheme for two dimensional space-time fractional diffusion equation.

(ii) The numerical example is presented to show that the numerical results are in good agreement with our theoretical analysis. Therefore these solution techniques can be applied to other fractional partial differential equations.

(iii) This is unconditionally stable finite difference scheme.

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