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The Duplication Formula For The Cosine As A Tool For Analytical Description Of The Chaotic Behaviour Of Two Classes Of Non-Linear Maps.

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Abstract: *In this paper, the well known duplication formula for the cosine is used as a veritable tool to analyse the chaotic behaviour of two classical non- linear maps. The results are then extended to certain polynomial functions of higher dimensions which are related to the Chebyshev polynomials as well as to certain rational functions closely related to the duplication formulas of the circular and Jacobian elliptic functions.*

I. Introduction

In the study of iteration of complex functions, considerable attention is paid to the set of points z_0 in the complex plane such that the n-th iterate $f^n(z_0)$ of the function

$$f(z) = z^2 - \mu$$

evaluated at some point $z = z_0$ should remain bounded as $n \to \infty$. We have only to recall the book by Mandelbrot [6, 7] which has rendered the subject accessible to a much wider audience with its fine illustrations and graphical descriptions. It is also instructive to mention, among others, the papers by Barnsley et al [1], which has generated allot of recent wave of activities.

A similar problem to the one just highlighted is to analyse the behaviour of the n-th iterate $F^n(x_0)$ of the function

$$F(x) = Kx(1-x)$$

evaluated at the point $x = x_0$ in the unit interval. A comprehensive overview of the problem for this, as well as for other more general classes is provided by the survey articles by Blanchard [2] and Whitley[10] and also the book by Preston [8]. 'Chaos' is the general appellation received by the behaviour exhibited by the n-th iterate of $f \ and \ F$ as $n \to \infty$ in both of these situations.

The book by Devaney [4, p39-53] has derived explicit analytical formulas for the n-th iterates of $f \ and \ F$ in the special cases $\mu = 2 \ and \ K = 4$ and thus describing exactly the set of points for which $f^n(z_0)$ remains bounded as $n \to \infty$ on the interval $-2 \le z \le 2$. We shall in the sequel show that the set of points x_0 is dense in the interval $0 \le x \le 1$ as n runs over the positive integers in each of the following cases: the orbit of x_0 is periodic, eventually absorbed into a fixed point and the orbit of x_0 is dense in [0,1].

These results are extended to certain polynomial functions of higher degree which are related to the Chebyshev polynomials, and indeed to certain rational functions which are associated with the duplication formulas of the circular and Jacobian elliptic functions and in this connection, the work of Lattes [5] is highly valued.

II. Analytical Perspectives

In this section our ideal point of departure will be the function

$$X = F(x) = 4x(1-x)$$

which maps the interval $0 \le x \le 1$ onto itself. It is clear that by the simple variation given by

$z = \varphi(x) = 2 - 4x$ and $Z = \varphi(X) = 2 - 4X$, we see the close connection between F and f defined earlier. From this, the resulting function is simply

$$Z = f(z) = z^2 - 2$$

Evidently, $Z = \varphi(Z) = f(z)$ implies that $\varphi F = f\varphi$, and φ maps the interval [0, 1] onto the interval [-2,2]. Furthermore, by introducing the function, $z = h(u) = 2cosu$, we see that it maps the interval [0, π] onto [-2,2]. This suggests that by defining $g(u) = 2u$ then by an easy calculation we can show that

$$fh(u) = (2cosu)^2 - 2 = 2(2cos^2u - 1) = 2cos2u = hg(u)$$

That is to say $fh(z) = hg(z)$ throughout their common domain.

A simple iteration of this relation produces

$$f^2h = ffh = fhg = hgg = hg^2.$$

And inductively as well as the general result

$$f^nh = hg^n$$

for every positive integer n. But then $g^n(u) = 2^n u$ for each positive integer n, and this can be stated analytically in the form

$$f^n(2cosu) = 2cosg^n(u) = 2\cos(2^n u)$$

for $u\epsilon[0,\pi]$. This result is significant and so necessitates a theorem which we state without hesitation as follows;

Theorem 2.1 If $f(z) = z^2 - 2$ and $z\epsilon[-2,2]$, then,

$$f^n(2cosu) = 2\cos(2^n u),\ u\epsilon[0,\pi] \text{ for each positive integer n.}$$

A corollary is immediate from this theorem.

***Corollary* 2.2** *If* $F(z) = 4x(1-x),\ x\epsilon[0,1]$, then $F^n(sin^2 u) = sin^2(2^n u),\ u\epsilon[0,\frac{\pi}{2}]$

Proof. Since φ conjugates $F and\ f$ through the relation $\varphi F = f\varphi$ which implies $F^n = \varphi^{-1} f^n \varphi$ and we have only to observe that if we put $x = sin^2 u$, then we will have that $\varphi(x) = 2cos2u$ and

$$F^n(x) = \varphi^{-1}\left(f^n(2cos2u)\right) = \varphi^{-1}(2cos2^n 2u) = \frac{1-\cos(2^n 2u)}{2} = sin^2(2^n u).$$

We note however that since $f(z) = z^2 - 2$ is analytic throughout the complex plane, it follows that our analytical formula of **Theorem 2.1** which was shown to be valid in its domain must remain valid for all complex values of u by the principle of the permanence of analytical forms. Besides, if we make the substitution $u = v + iw$ and recall that,

$$|cosu|^2 = cos^2 v + sinh^2 w$$

We then deduce that

$|\cos(2^n u)|^2 = cos^2(2^n v) + sinh^2(2^n w)$, for each positive integer n, from which it follows that if the imaginary part $w\ of\ u$ is non-zero, then the orbits of f tend to infinity with n.

Furthermore, the relation

$$cosu = cosv\ coshw - isinv\ sinhw$$

indicates that the transformation $U = 2cosu$ maps the line $v = \pi$ to the negative real axis defined by $U \leq -2$. Consequently, $f^n(U) \to \infty$ as $n \to \infty$, with the exception of the point $U = -2$. Likewise, the cosine mapping sends the line $v = 0$ to the real axis defined by $U \geq 2$, and for such values, $f^n(U) \to \infty$ as $n \to \infty$ with the exception of $U = 2$. Thus the only points z_0 in the complex plane for which $f^n(z_0)$ remains bounded as n runs over the positive integers are those in [-2,2]

These results can now be carried over to certain higher degree polynomials$f(z)$. In this connection we retain the function $z = h(u) = 2cosu$ but now introduce $g(u) = ku$, where k is a positive integer.

Evidently,

$$hg(u) = 2cosku = 2T_k(cosu),$$

where T_k is the Chebyshev polynomial of degree k defined by

$$T_k(z) = \cos(karccosz).$$

Obviously, the leading coefficient of T_k, is given by 2^{k-1}. So if we require a polynomial f_k with leading coefficient 1, then we must put $f_k(z) = 2T_k(z/2)$ and it readily follows that

$$f_k h(u) = 2T_k(cosu) = 2cosku = hg(u).$$

The case $k = 2$ now follows similarly and leads us to the next theorem while a few examples of these polynomials are readily available in Whittaker [9].

Theorem 2.2 If T_k denotes the k-th Chebyshev polynomial and$f_k(z) = 2T_k(z/2)$, then

$$f_k^n(2cosu) = 2\cos(2^n u)$$

for all positive integers n and all complex values of u.

III. Evolving A Dynamical Model

A cursory look at the works of various authors such as Collet and Eckmann[3] as well as Whitley [10] reveal a different perspective of the difficulties outlined above in that they considered the natural iterates f^n $of\ f$ as only the numerous phases of a dynamical system. However, it is apparent from **Theorem 2.1** that we can indeed imbed f^n in a continuously evolving system-f^t for real values of the time t according to the formula

$$f^t(2cosz) = 2\cos(2^t z).$$

We then re-cast this equation into a more convenient form by means of the change of time scale defined by $t = s/log2$, and the system is then governed by the formula

$$f^s(2cosz) = 2\cos(e^s z).$$

If we introduce

$$x(s) = 2\cos(ae^s),\quad y(s) = 2ae^s\sin(ae^s),\quad z(s) = 2a^2e^{2s}\cos(ae^s) \tag{1}$$

where a is some constant, then by repeated differentiation we obtain the equations

$$x' = -y,\qquad y' = y + z,\qquad z' = 2z - a^2e^{2s}y \tag{2}$$

By the use of the relation $z = a^2e^{2s}x$ in z' of equation (2), we arrive at the following independent system of non-linear ordinary differential equations, hence forth, (ODEs) in what follows.

$$x' = -y, \qquad y' = y + z, \qquad z' = 2z - \frac{yz}{x} \tag{3}$$

The system (3) can further be reduced to the single third order equation

$$x''' = 3x'' + \frac{x'x''}{x} - \frac{x'^2}{x} - 2x'$$

We then examine an arbitrary trajectory coordinate $(x(s), y(s), z(s))$ of the system (3) with initial values

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0.$$

Since all the right hand sides of equation (3) are continuously differentiable everywhere except at $x = 0$, it follows from the classical existence and uniqueness theorems of ODEs that the trajectory passing through (x_0, y_0, z_0) must be unique provided that $x_0 \neq 0$.

From

$\frac{d}{ds}(xz + y^2) = x'z + xz' + 2yy' = -yz + x\left(2z \frac{yz}{x}\right) + 2y(y + z) = 2(xy + y^2)$, by (3), we infer that

$$x(s)z(s) + (y(s))^2 = (x_0 z_0 + {y_0}^2)e^{2s} \tag{4}$$

We shall assume that $x_0, z_0 > 0$ in what follows and observe that

$$\frac{d}{ds}\left(\frac{z}{x}\right) = \frac{xz' - zx'}{x^2} = \frac{x(2z - yz/x) - z(-y)}{x^2} = 2\frac{z}{x}$$

which implies that

$$\frac{z(s)}{x(s)} = \frac{z_0}{x_0} e^{2s}$$

and if we put $\mu^2 = \frac{x_0}{z_0}(x_0 z_0 + {y_0}^2)$ and eliminate e^{2s} in the foregoing, we will then obtain

$$\frac{z}{x} = \frac{1}{\mu^2}(xz + y^2)$$

Solving for z yields the result

$$z = \frac{xy^2}{\mu^2 - x^2} \tag{5}$$

Consequently, we infer that the trajectory must lie on the rational surface (5) in the (x, y, z) and so must be nowhere dense.

In seeking a general analytical solution of the system (3), we make use of the substitution

$z = \frac{z_0}{z_0} e^{2s} x, \; y = -x'$ into (4) yielding the single equation

$\frac{z_0}{x_0} e^{2s} x^2 + x'^2 = \frac{z_0}{x_0} \mu^2 e^{2s}$, which simplifies to the form

$$x^2 + \frac{x_0}{z_0} \mu^2 e^{-2s} x'^2 = \mu^2 \tag{6}$$

with an initial condition $x(0) = x_0$, and by simple integration or direct substitution, the solution to (6) becomes

$$x(s) = \mu\cos(\sqrt{\frac{z_0}{x_0}} \frac{e^s - 1}{\mu} + arccos\frac{x_0}{\mu})$$

The complete solution of (3) now follows from

$$y = -x' = \sqrt{\frac{z_0}{x_0}}\, e^s \sin(\sqrt{\frac{z_0}{x_0}} \frac{e^s - 1}{\mu} + arccos\frac{x_0}{\mu})$$

$$z = y' - y = \frac{z_0}{\mu x_0}\, e^{2s} \cos(\sqrt{\frac{z_0}{x_0}} \frac{e^s - 1}{\mu} + arccos\frac{x_0}{\mu})$$

and when s assumes such values that

$\sqrt{\frac{z_0}{x_0}} \frac{e^s - 1}{\mu} + arccos\frac{x_0}{\mu} = 2n\pi$ for some n, we will then have

$$x(s) = \mu, \qquad y(s) = 0, \qquad z(s) = \frac{z_0}{\mu x_0}\, e^{2s}.$$

It is pertinent to remark at this point that, if s denotes the surface defined by (5) whose domain lies in $x^2 < \mu^2$, then, the trajectories on s can cross over the x-axis when the they lie either directly above the point $(\mu, 0{,}0)$, or else directly below the point $(-\mu, 0{,}0)$. Since $y_0 = 0$ implies $z_0 = 0$ in (5), we must exclude all other initial values with $y_0 = 0$, so the surface s ought to have an open slit both above and below the segment joining the points $(\mu, 0{,}0)$ and $(-\mu, 0{,}0)$. Similarly the vertical half line must be adjoined above $(\mu, 0{,}0)$ as well as below $(-\mu, 0{,}0)$ so that s remains connected but no longer simply connected.

We shall choose $y_0\ and\ z_0$ for any prescribed values of $x_0\ and \mu$ so that

$$\sqrt{\frac{z_0}{x_0}} = \mu arccos\frac{x_0}{\mu}, \;\; y_0 = \sqrt{{\mu_0}^2 - {x_0}^2}\; arccos\frac{x_0}{\mu} \tag{7}$$

Consequently, our trajectories will now assume the form

$$x(s) = \mu\cos(e^s arccos\frac{x_0}{\mu}),$$

$$\left.\begin{aligned} y(s) &= \mu\left(arccos\frac{x_0}{\mu}\right)e^s\sin\left(e^s arccos\frac{x_0}{\mu}\right), \\ z(s) &= \mu\left(arccos\frac{x_0}{\mu}\right)^2 e^{2s}\cos\left(e^s arccos\frac{x_0}{\mu}\right). \end{aligned}\right\} \quad (8)$$

If one may then ask, "how do these trajectories exhibit chaotic behaviour"? We answer this poser by introducing a time delay of log2, and we clearly see that

$$x(s+log2) = \mu\cos\left(2e^s arccos\frac{x_0}{\mu}\right) = \mu\left(2cos^2\left(e^s arccos\frac{x_0}{\mu}\right)-1\right) = \frac{2}{\mu}(x(s))^2-\mu$$

$$\begin{aligned} y(s+log2) &= 2\mu\left(arccos\frac{x_0}{\mu}\right)e^s\sin\left(2e^s arccos\frac{x_0}{\mu}\right) \\ &= 4\mu\left(arccos\frac{x_0}{\mu}\right)\, e^s\sin\left(e^s arccos\frac{x_0}{\mu}\right)cos\left(e^s arccos\frac{x_0}{\mu}\right) = \frac{4}{\mu}x(s)y(s). \end{aligned}$$

$$\begin{aligned} z(s+log2) &= 4\mu\left(arccos\frac{x_0}{\mu}\right)^2 e^{2s}\cos\left(2e^s arccos\frac{x_0}{\mu}\right) \\ &= 4\mu\left(arccos\frac{x_0}{\mu}\right)^2 e^{2s}\left(cos^2\left(e^s arccos\frac{x_0}{\mu}\right)-sin^2\left(e^s arccooutings\frac{x_0}{\mu}\right)\right) \\ &= \frac{4}{\mu}\left(x(s)z(s)-(y(s))^2\right). \end{aligned}$$

This means, in particular, that if we put $f(x) = \left(\frac{2}{\mu}\right)x^2-\mu$ and $x(s) = x,$ then it becomes apparent that we will have $x(s+log2) = f(x)$, and further iteration of this relation leads to

$$x(s+nlog2) = f^n(x)$$

for every positive integer $n \, and \, s \geq 0$. Therefore, if s is measured in units of $log2$, then the $x-$ coordinates of the trajectory will exhibit the kind of chaotic behaviour earlier discussed.

IV. Chaotic Behaviour Of Orbits

Generally, the behaviour of the iterates $f^n(2cosu)$ for $0 \leq u \leq \pi$ of $f(z) = z^2-2$ as n runs over positive integers can be analyzed most conveniently by putting $r = u/\pi$ so that $0 \leq r \leq 1$. The fixed points of f clearly correspond to $r = 0, u = 0$ and $r = 2/3, u = 2\pi/3$. From the general formula

$$f^n(2cos\pi r) = 2\cos(2^n\pi r)$$

we conclude that the behaviour of $2^n r$ can best be described when we represent r by means of its dyadic expansion

$$r = \frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_k}{2^k}+\cdots = .a_1a_2\ldots a_k\ldots,$$

where a_k is either 0 or 1. We shall be interested in the necessary and sufficient conditions governing r for which the iterates $f^n(2cos\pi r)$ can be absorbed into one or the other of the fixed points.

Hence given the elementary formula

$$cos2\pi r = cos2\pi(1-r)$$

we infer that if $r < 1/2$, then we can compute the next iterate $cos2\pi r$ using the formula

$cos2\pi r = \cos(\pi T(r))$. where $T(r)$ is the shift defined to be the fractional part of $2r$, that is

$$T(r) = T(.a_1a_2\ldots) = .a_2a_3\ldots \quad (9)$$

However, if $r > 1/2$, then we must define $T(r)$ to be 1- fractional part of $2r$ according to the formula

$$T(r) = T(.a_1a_2\ldots) = .a'_2a'_3\ldots, \quad (10)$$

where $a'_k = 1-a_k$ for each index k and compute

$$cos2\pi r = \cos(\pi T(r)).$$

Equations (9) and (10) clearly can be compacted into a single formula depending upon whether $a_1 = 0$ or $a_1 = 0$. But certainly, the two agree in the ambiguous cases

$$T(.0111\ldots) = .111\ldots, \quad T(.111\ldots) = .000\ldots,$$

with the fixed points

$$r = 2/3 = .101010\ldots, \quad r = 0 = .000\ldots$$

clearly satisfying the relation $T(r) = r$. However, the relation

$$T(.010101\ldots) = .101010\ldots$$

reveals that the iterates of

$$r = 1/3 = .010101\ldots$$

are readily absorbed into the fixed point $r = 2/3$. From these observations we can easily deduce the following results.

A necessary and sufficient condition for $T^n(r)$ to be absorbed eventually into the fixed point $r = 2/3$ is that the tail of the dyadic expansion $r = .a_1a_2\ldots$ should be $\ldots 010101\ldots$. Similarly, a necessary and sufficient condition for $T^n(r)$ to be absorbed eventually into the fixed point $r = 0$ is that the tail of r should be $.0000\ldots$, that is that the dyadic expansion of r should be finite. But by taking $r = .100100\ldots$ we calculate

$T(r) = .110110...$ Then $T(T(r)) = T^2(r) = .010010..$, $T^3(r) = .100100 ...$ and we see clearly that for such r the iterates are eventually periodic and our aim will be to show that a necessary and sufficient condition for the iterates of r to be eventually periodic is that r should have a dyadic expansion with periodic tail of period p.

We have already dealt with the case $p \leq 2$ in the preceding analysis where the iterates are eventually absorbed into either $r = 0$ or $r = {}^2/_3$. If we denote the orbit $O(r)$ of r by $T^n(r)$ as r runs over the positive integers, then the example $r = .0111011 ...$ shows that such orbit need not include r in its periodic oscillations for $T(r) = .110110 ...$, and such orbit has already been charted. In the same vein, the period of $T^n(r)$ need not coincide with the period of the dyadic expansion as can be seen from the example $r = .11001100 ...$, $T(r) = .01100110 ...$, $T^n(r) = .11001100 ... = r$. Nevertheless, we need to show that the period $T^n(r)$ can not exceed the period of the dyadic expansion. It is worth noting that in order to understand the action of the function T we need observe that it consists of a truncation of the initial digit and, in other cases, a reversal of the remaining digits. This reversal will normally occur at the first appearance of a 1 in the dyadic expansion, and then at each digit a_k such that $a_{k-1} \neq a_k$. If p is the period of the dyadic expansion, the we have $a_{k+p} = a_k$ for sufficiently large indices k.If there is a reversal at a_k, then another is sure to occur at a_{k+p}, provided that a_k is not the first 1 in the expansion, if it is then another reversal need not occur at a_{k+p} as seen in the case of the orbit

$$O(.101101 ...) = \{.100100 ..., \ .110110 ..., .010010 ...\}$$

where clearly an even number of shifts and reversals is equivalent to an even number of shifts with no reversals, and if there were an even number of reversals within one complete period of the dyadic expansion, then the values of $T^n(r)$ would certainly repeat with the same period p.In consequence, a well known elementary arguement shows that the smallest period $T^n(r)$ as a function of n has to be a divisor of p. Then the converse of this result, that if $T^n(r)$ is eventually periodic, then the dyadic expansion or r must be periodic as well becomes obvious. These results are vital that we synthesize them into a theorem.

Theorem 4.1 Given that the dyadic expansion of r is eventually periodic with period p, then $T^n(r)$ will eventually be periodic and its period will be a divisor of p. Conversely if $T^n(r)$ is eventually periodic, the dyadic expansion of r will also be periodic.

Finally, we turn our attention to the orbits $O(r)$ which are infinite. By considering for example, the orbit $O(r)$ where $r = .1010010001 ...$ which is infinite and shows that its limit points are $1, {}^1/_2, ... {}^1/_{2^n}, ...,0$ so $O(r)$ is nowhere dense in [0,1].

Finally, we round up the paper with a vital result of Whittaker on the denseness of orbits for any two arbitrary close points in [0,1] in the following theorem.

Theorem 4.2 (Whittaker [9]). For any $r_0 \epsilon$ [0,1],we can always find a point r arbitrarily close to r_0 such that $O(r)$ is dense in [0,1].

For the proof of this result we shall refer the interested reader to Whittaker [9]. We wish to note however, that the proof is analogous to what obtains in Devaney [4 , p42] in the sense that the $O(r)$ is analogous to period $Per(\sigma)$ the period of the shift map σ in which there exists not only a dense orbit for σ in [0,1] but that $Per(\sigma)$ itself is dense in [0,1].

Remark. We remark here that to prove that $Per(\sigma)$ is dense, we need only to produce a sequence of periodic points which converge to an arbitrary point p in [0, 1]. These are clearly the cases under **Theorem 4.1**. However, any repeating sequence that is infinite can never be periodic and indeed they outnumber the periodic ones and besides, there are non-periodic orbits which wind densely about [0,1], a property that is a common characteristic of topologically transitive maps.

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