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An Enumeration of the Transitivity, Primitivity and Faithfulness of the Wreath Products of Permutation Groups.

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Abstract: *The conditions under which the Wreath Products of Permutation groups would be transitive, primitive and faithful were highlighted by Suleiman Ahmad (2006) see [2]. These properties, under the highlighted conditions are established in [1].In this communication, we enumerate the establishment of such properties under the given conditions; our goal is achievable by relying on existing theorems and enumerating an elaborate example.*

Keywords: *Group actions, Wreath products, Automorphism, Transitive Permutation groups, $K-$transitive, Primitivism, Faithfulness.*

I. Introduction

Over the years, the Wreath products of permutation groups have become an interesting area of study. These were first reported by Audu M.S in [3]. Ezenwanne I.U in [4] also discussed extensively on transitivity, primitivity and Faithfulness of the wreath products of permutation groups. Apine.E [5] considered permutation groups of prime-power. Ahmad, Suleiman (2006) [2], highlighted the conditions under which the wreath products of permutation group could be transitive, primitive and faithful. In this communication, we enumerate the establishment of such properties under the given conditions.

Notations

C^{Δ} : The set of all maps of Δ into the permutation group C.
$\Gamma x \Delta$: Direct products of two sets Γ and Δ
CwrD : The wreath product of C by D.

Preliminary

We shall thus, state and prove several theorems and make obvious remarks which shall lead to the attainment of our goal.

Theorem1

If G doubly Transitive, the G is Primitive.

Proof: Let G be doubly transitive on Ω. Suppose that Δ is a non empty set of imprimitivity of G and $|\Delta| \succ 1$, we show that $\Delta = \Omega$. Now as $|\Delta| \succ 1$, there exist $\alpha, \beta \in \Delta$ *with* $\alpha \neq \beta$. Choose $\delta \in \Omega,\ \delta \neq \alpha.$ Then there exist $\delta \in \Omega,\ \delta \neq \alpha. g \in G \big| \alpha^g = \alpha$ *and* $\beta^g = \delta$ (this is 2-tansitivity). Thus $\alpha, \delta \in \Delta^g$. As $\alpha \in \Delta,$ we have that $\Delta^g = \delta$; *i.e* $\delta \in \Delta$. This holds for all $\delta \neq \alpha$ *in* Ω.Thus $\Delta = \Omega$ and G is primitive on Ω.

Theorem2

Let G be a primitive permutation group on Ω and $N \underline{\Delta} G$. Then $N = 1$ *or* N is transitive on Ω. Or equivalently if the transitive group G contains an intransitive normal subgroup $N \neq 1$, then G is imprimitive.

Proof: Let $\alpha \in \Omega,$ now $G\alpha$ is maximal proper subgroup of G and $G\alpha \leq NG\alpha \leq G$. Hence $G = NG\alpha$ *i.e* $(N \leq G\alpha)$ *or* $NG\alpha = G.$ If $N \leq G\alpha$ then $N^g \leq (G\alpha)$ for all $g \in G.$ Since $N \underline{\Delta} G$, $NG = N$ for all $g \in G.$ This means that $N \leq G_\alpha$ for all $g \in G.$ Since G is transitive on Ω, we have that

$N \leq G_\beta$ $for\, all\, \beta \in G$ and hence N fixes every point on Ω. But only the identity fixes every point on Ω. According to $N = 1$. If $NG\alpha = G$, then $|G| = |Ng\alpha| = \dfrac{|N||G\alpha|}{|G\alpha \cap N|} = \dfrac{|N||G\alpha|}{|N\alpha|}$.

Hence $\left|\alpha^g\right| = |G : G\alpha|$

$$= |N : N\alpha| = \left|\alpha^N\right|.$$

But $\alpha^N \leq \Omega$ and $\alpha^g = \Omega.$ Thus $\alpha^N = \Omega$ so N is transitive on Ω.

Theorem 3

If (G,Ω) is $K-$transitive, $K \geq 2$,then (G,Ω) is primitive.

Proof:

If a block B contains two two distinct elements x, y, we can always find an element $g \in G$ fixing Ω and taking y to any other element of Ω. Thus $B = \Omega$.

Theorem 4

If (G,Ω) is primitive and $1 \prec N\underline{\Delta G,}$ then (N,Ω) is transitive.

Proof:

The orbits x^N of N on Ω are blocks of (G,Ω), since $\underline{(\ (,}x^N)^g = x^{Ng}$. Since (G,Ω) is primitive, we must therefore have $x^N = \{x\}$ or $x^N = \Omega.$ But the former is impossible for all $x \in \Omega$, since it would imply that $N = 1$, thus $x^N = \Omega$ for some $x \in \Omega$, so (N,Ω) is transitive.

Example

Consider the permutation groups:

$C = \{(1),(123),(132)\}$, $D = \{(1),(45)\}$ acting on the sets $\Gamma = \{1,2,3\}$ and $\Delta = \{4,5\}$ respectively.

Let $P = C^\Delta = \{f \,|\Delta \to C\}$ then

$|P| = |C|^{|\Delta|} = 3^2 = 9\, and\, |W| = |PD| = 9 \times 2 = 18$.

Let $d_1 = (1)$ and $d_2 = (45),$

We can easily verify that P is a group with respect to the operations $(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$ where $\delta \in \Delta$. We define the action of D on P as $f^d(\delta) = f(\delta d^{-1})$, where $d \in D, \delta \in \Delta,$ then D act on P as groups. We now define $W = C_{wr}D,$ the semi-direct product of $P\, by\, D$ in that order; $i.e\, W = \{fd\,|\, f \in P, d \in D\}.$ Now W is a group with respect to the operation $(f_1 d_1)(f_2 d_2) = (f_1 d_2 d_1^{-1})(d_1 d_2)$.

The Mappings are as follows:

$f_1 : 4 \to (1) : 5 \to (1)$

$f_2 : 4 \to (123) : 5 \to (123)$

$f_1 : 4 \to (132) : 5 \to (132)$

$f_1 : 4 \to (1) : 5 \to (123)$

$f_1 : 4 \to (1) : 5 \to (132)$

$f_1 : 4 \to (123) : 5 \to (1)$

$f_1 : 4 \to (132) : 5 \to (1)$

$f_1 : 4 \to (123) : 5 \to (132)$

$f_1 : 4 \to (132) : 5 \to (123$

The elements of W are:

$(f_1d_1); (f_2d_1), (f_3d_1), (f_4d_1), (f_5d_1), (f_6d_1), (f_7d_1), (f_8d_1), (f_9d_1)$

$(f_1d_2), (f_2d_2), (f_3d_2), (f_4d_2), (f_5d_2), (f_6d_2), (f_7d_2), (f_8d_2), (f_9d_2)$

Since $|W| = |PD| = 9\times 2 = 18.$

Now, define the action of W on $\Gamma\times\Delta$ as $(\alpha,\delta) fd = (\alpha f(\delta), \delta d)$.

Further, $\Gamma\times\Delta = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$.

We obtain the following Permutations by the action of W on $\Gamma\times\Delta$.

$$(\Gamma\times\Delta) f_1d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_2d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (2,4),(2,5),(3,4),(3,5),(1,4),(1,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_3d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (2,4),(2,5),(3,4),(3,5),(1,4),(1,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_4d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (1,4),(2,5),(2,4),(3,5),(3,4),(1,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_5d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (1,4),(3,5),(2,4),(1,5),(3,4),(2,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_6d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (2,4),(1,5),(3,4),(2,5),(1,4),(3,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_7d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (3,4),(1,5),(1,4),(2,5),(2,4),(3,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_8d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (2,4),(3,5),(3,4),(1,5),(1,4),(2,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_9d_1 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (3,4),(2,5),(1,4),(3,5),(2,4),(1`,5) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_1d_2 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (1,5),(1,4),(2,5),(2,4),(3,5),(3`,4) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_2d_2 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (2,5),(2,4),(3,5),(3,4),(1,5),(1`,4) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_3d_2 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (3,5),(3,4),(1,5),(1,4),(2,5),(2`,4) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_4d_2 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (3,5),(3,4),(1,5),(1,4),(2,5),(2`,4) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_5d_2 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (1,5),(3,4),(2,5),(1,4),(3,5),(2`,4) \end{Bmatrix}$$

$$(\Gamma\times\Delta) f_6d_2 = \begin{Bmatrix} (1,4),(1,5),(2,4),(2,5),(3,4),(3,5) \\ (2,5),(1,4),(3,5),(2,4),(1,5),(3`,4) \end{Bmatrix}$$

$$(\Gamma\times\Delta)\,f_7d_2=\left\{\begin{matrix}(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\\(3,5),(1,4),(1,5),(2,4),(2,5),(3`,4)\end{matrix}\right\}$$

$$(\Gamma\times\Delta)\,f_8d_2=\left\{\begin{matrix}(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\\(2,5),(3,4),(3,5),(1,4),(1,5),(2`,4)\end{matrix}\right\}$$

$$(\Gamma\times\Delta)\,f_9d_2=\left\{\begin{matrix}(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\\(3,5),(2,4),(1,5),(3,4),(2,5),(1`,4)\end{matrix}\right\}$$

We rename the symbol of $(\Gamma\times\Delta)$ as:

$(1,4)\rightarrow 1;\quad (1,5)\rightarrow 2$

$(2,4)\rightarrow 3;\quad (2,5)\rightarrow 4$

$(2,4)\rightarrow 5;\quad (3,5)\rightarrow 6$

We give the permutation in cycle form as follows:

(1), (135)(246), (153)(264), (246), (264), (135), (153), (1356)(264), (153)(246), (12)(34)(56), (145236), (163254), (123456), (165432), (125634), (143652), (14)(25)(36), (16)(23)(45).

II. Conclusion

From the foregoing, we conclude that by the actions of W on $\Gamma\times\Delta$;

i. That the permutation group obtained is transitive as C and D are transitive on Γ and Δ respectively.

ii. W is faithful on $\Gamma\times\Delta$, since C and D are faithful on Γ and Δ respectively.

iii. W is imprimitive on $\Gamma\times\Delta$.

iv. On taking the Wreath product of C and D, we ended up with a biger new group which has order 18.

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