IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. II (Mar. - Apr. 2016), PP 27-34
www.iosrjournals.org

The First Triangular Representation of The Symmetric Groups with *p* divides (n-1)

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Abstract : *In this paper we will introduce new type of representations of the symmetric groups we call them the triangular representations of the symmetric groups and we will study the first of them which we call it the first triangular representation of the symmetric groups when p divides (n-1).*
Keywords: *symmetric group,group algebra KS_n ,KS_n −module,Specht module ,exact sequence .*

I. Introduction

When Prof. W.Specht was a student under the supervision of Prof. I.Schur, he began investigating representation theory of the symmetric group .During that time it was well known that standard Young tableaux of a given partition λ of a positive integer n form a basis of an ordinary irreducible representationspace of $\mathrm{S_n}$. The problem that W.Specht was facing in his investigating in that time is that the symmetric group acts in a natural way on tableaux, but the result of the application of a permutation to a standard tableau can be a nonstandard tableau, and it is by no means clear how a nonstandard tableau can be written as a linear combination of standard ones. For this reason W.Specht introduced in 1935 polynomials corresponding to the tableaux (known nowadays as Specht polynomials),and it is obvious how a given polynomial can be written as a linear combination of other polynomials (see [1]).

In 1962 H.K. Farahatstudied the representation which deals with the partition$\lambda = (n-1,1)$ of the positive integer n and called it the natural representation of the symmetric groups [2].

In 1969 M. H. Peel renamed the natural representation of the symmetric groups by the first natural representation of thesymmetric groups and studied the second representation of the symmetric group which deal with the partition $\lambda = (n-2,2)$of the positive integer n [3].

In 1971 Peel introduced ther^{th} Hook representationswhich deal with the partitions $\lambda = (n-r, 1^r)$; r≥ 1.[4]

Now in this paper we will introduce new representations of the symmetric groups we call them the triangular representations of the symmetric groups and we will study the first of them which we call it the first triangular representationof the symmetric groups.

Throughout this paper let **K** be a field of characteristic p (which may be zero or a prime number not equal 2), and $x_1, x_2, \dots, x_n$ be linearly independent commuting variables over **K**.

II Preminaries

Definition 1:Let $\mathbf{S_n}$ be the set of all permutationsτ onthe set $\{x_1, x_2, \dots, x_n\}$and$\mathbf{K}[x_1, x_2, \dots, x_n]$ be the ring of polynomials in $x_1, x_2, \dots, x_n$ with coefficients in**K**. Then each permutation $\tau \in \mathbf{S_n}$ canbe regarded as a bijective function from$\mathbf{K}[x_1, x_2, \dots, x_n]$ onto $\mathbf{K}[x_1, x_2, \dots, x_n]$ defined by $\big(f(x_1, x_2, \dots, x_n)\big) = f\big(\tau(x_1), \tau(x_2), \dots, \tau(x_n)\big)$ $\forall f(x_1, x_2, \dots, x_n) \in \mathbf{K}[x_1, x_2, \dots, x_n]$. Then$\mathbf{KS_n}$forms a group algebra with respect to addition of functions, product of functions by scalars and composition of functions which is called the group algebra of the symmetric group$\mathbf{S_n}$[3].

Definition 2: Let n be a positive integer then the sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$is called a partition of n if$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and$\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$.Then the set $D_\lambda = \{(i,j) | i = 1,2, \dots, l; 1 \leq j \leq \lambda_i\}$ is called λ −diagram .And any bijective function $t : D_\lambda \to \{x_1, x_2, \dots, x_n\}$ is called aλ–tableau. Aλ–tableau may be thought as an array consisting of$l$rows and $\lambda_1$columns of distinct variables$t((i,j))$ where the variables occur in the first λ_i positions of the i^{th} row and each variable $t((i,j))$ occurs in the i^{th}row and the j^{th} column ((i,j)-position) of the array.

$t((i,j))$will be denoted by $t(i,j)$ for each $(i,j) \in D_\lambda$.

The set of all λ-tableaux will be denoted by T_λ. i.e $T_\lambda = \{t | t \; is \; a \; \lambda - tableau\}$.

Then the function $g: T_\lambda \to K[x_1, x_2, \dots, x_n]$ which is defined by $g(t) = \prod_{i=1}^{l} \prod_{j=1}^{\lambda_i} (t(i,j))^{i-1}$, $\forall\, t \in T_\lambda$. is called the row position monomial function of T_λ, and for each λ–tableau t, $g(t)$ is called the row position monomial of t .So $M(\lambda)$ is the cyclic KS_n −module generated by $g(t)$ over KS_n.[5]

III The First Triangular Representation of S_n

In the beginning we define some denotations which we need them in this paper.

1) Let $\sigma_1(n) = \sum_{i=1}^{n} x_i$.
2) Let $\sigma_2(n) = \sum_{1 \le i < j \le n} x_i x_j$.
3) Let $C_l(n) = x_l^2\, (\sigma_2(n) - \sum_{\substack{j=1 \\ j \ne l}}^{n} x_l x_j)$; $l = 1,2,\dots,n$.

   We denote $\overline{N}$ to be the KS_n module generated by $C_1(n)$ over KS_n. The set $B = \{C_i(n) \mid i = 1,2,\dots,n\}$ is a K-basis for $\overline{N} = KS_n C_1(n)$ and $dim_K \overline{N} = n$.
4) Let $u_{ij}(n) = C_i(n) - C_j(n)$; $i,j = 1,2,\dots,n$.
   we denote $\overline{N}_0$ the KS_n submodule of $\overline{N}$ generated by $u_{12}(n)$.
5) Let $\sigma_3(n) = \sum_{1 \le i < j \le n} \sum_{\substack{k=1 \\ k \ne i,j}}^{n} x_i x_j x_k^2$.

Then $\sum_{l=1}^{n} C_l(n) = \sigma_3(n)$ and $dim_K(K\sigma_1(n)) = dim_K(K\sigma_2(n)) = dim_K(K\sigma_3(n)) = 1$. $K\sigma_1(n), K\sigma_2(n)$ and $K\sigma_3(n)$ are all KS_n-modules, since $\tau\sigma_k(n) = \sigma_k(n)\ \forall\, k = 1,2,3$.

Definition 3: The KS_n- module $M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right)$ defined by

$$M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right) = KS_n x_1 x_2 \dots x_{r+1} x_{r+2}^2 \dots x_{2r+1}^2 x_{2r+2}^3 \dots x_n^{r+1}$$

is called the r^{th} triangular representation module of S_n over K ,where $n \ge \frac{(r+3)(r+2)}{2}$.

Remark: The first triangular representation module of S_n over K is the KS_n −module $M(n-3,2,1)$, the second triangular representation module of S_n over K is the KS_n −module $M(n-6,3,2,1)$, the third triangular representation module of S_n over K is the KS_n −module $M(n-10,4,3,2,1)$,and so on.

Lemma 1: The set $B(n-3,2,1) = \{ x_i x_j x_l^2 \mid 1 \le i < j \le n, 1 \le l \le n, l \ne i,j\}$ is a K-basis of $M(n-3,2,1)$, and $dim_K M(n-3,2,1) = \binom{n}{2}(n-2)$; $n \ge 6$.

Theorem1: The set $B_0(n-3,2,1) = \{ x_i x_j x_l^2 - x_1 x_2 x_3^2 \mid 1 \le i < j \le n, 1 \le l \le n, l \ne i,j,$ $(i,j,l) \ne (1,2,3)\}$ is a K-basis of $M_0(n-3,2,1)$, and $dim_K M_0(n-3,2,1) = \binom{n}{2}(n-2) - 1$; $n \ge 6$.

Proof: Since the KS_n-module $M_0(n-3,2,1)$ consist of all polynomials of the form

$\sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_i x_j x_l^2$ with $\sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} = 0$ and $k_{ijl} \in K.\, i.e.$

$$M_0(n-3,2,1) = \{ \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_i x_j x_l^2 \mid \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} = 0,\ k_{ijl} \in K\}$$

it is clear that $B_0(n-3,2,1) \subseteq M_0(n-3,2,1)$ .To prove $B_0(n-3,2,1)$ generates $M_0(n-3,2,1)$ over K.
Let $x \in M_0(n-3,2,1)$.

$$\Rightarrow x = \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_i x_j x_l^2;\ \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} = 0$$

$$\Rightarrow x = \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_i x_j x_l^2 - 0(x_1 x_2 x_3^2);\ \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} = 0$$

$$\Rightarrow x = \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_i x_j x_l^2 - (\sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl}) x_1 x_2 x_3^2$$

$$\Rightarrow x = \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_i x_j x_l^2 - \sum_{1 \le i < j \le n} \sum_{\substack{l=1 \\ l \ne i,j}}^{n} k_{ijl} x_1 x_2 x_3^2$$

$\Rightarrow x = \sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl}\,(x_i x_j x_l^2 - x_1 x_2 x_3^2)$.

$\Rightarrow x = (\sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl}\,(x_i x_j x_l^2 - x_1 x_2 x_3^2)$)with the term 123 excluded from the double summation since $k_{123}\,(x_1x_2x_3^2 - x_1x_2x_3^2) = 0$.Thus $B_0(n-3,2,1)$ generates$M_0(n-3,2,1)$ over K. $B_0(n-3,2,1)$is linearly independent since if

$$\sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j\\ (i,j,k)\ne(1,2,3)}}^{n} k_{ijl}(x_i x_j x_l^2 - x_1 x_2 x_3^2)=0$$

$$\Rightarrow \sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j\\ (i,j,k)\ne(1,2,3)}}^{n} k_{ijl} x_i x_j x_l^2 - (\sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j\\ (i,j,k)\ne(1,2,3)}}^{n} k_{ijl}) x_1 x_2 x_3^2 = 0$$

$$\Rightarrow \sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl} x_i x_j x_l^2 = 0, \text{ where } k_{123} = -\sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl} \text{ with } \sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\ne i,j\\ (i,j,k)\ne(1,2,3)}}^{n} k_{ijl} = 0$$

Hence $k_{ijl} = 0 \quad \forall\ i,j,l;\ 1 \le i < j \le n, 1 \le l \le n, l \ne i,j$,since $B(n-3,2,1)$ is linearly independent by lemma 1.
Thus$B_0(n-3,2,1)$ is a K-basis of $M_0(n-3,2,1)$, and

$$dim_K M_0(n-3,2,1) = \binom{n}{2}(n-2) - 1 = \frac{n(n-1)(n-2)}{2} - 1 = \frac{n^3 - 3n^2 + 2n - 2}{2}$$

Theorem 2:$\bar{N} = KS_nC_1(n)$and $M(n-1,1)$ are isomorphic over KS_n.
Proof : Let $\varphi : M(n-1,1) \to \bar{N}$ be defined as follows:
$\varphi(x_i) = C_i(n);\ i = 1,2,\dots,n$. Then for each$\tau = (x_i x_j) \in S_n$such that $\tau(x_i) = x_j$we get that$\varphi(\tau x_i) = \varphi(x_j) = C_j(n) = \tau C_i(n) = \tau\varphi(x_i)$.

Hence φ is aKS_n-homomorphism. Also $y = \sum_{i=1}^{n} k_i C_i(n)$ for any $y \in \bar{N}$.

Thus $\forall y \in \bar{N},\ \exists\ w = \sum_{i=1}^{n} k_i x_i \in M(n-1,1)\ \ s.t.\ \varphi(w) = \varphi(\sum_{i=1}^{n} k_i x_i) = \sum_{i=1}^{n} \varphi(k_i x_i) = \sum_{i=1}^{n} k_i \varphi(x_i) = \sum_{i=1}^{n} k_i C_i(n) = y$. Hence φ is an epimorphism.Thus $dim_K\ ker\varphi = dim_K M(n-1,1) - dim_K \bar{N} = n - n = 0$.
$\Rightarrow\ ker\varphi = 0$ .Then φ is monomorphism.Thus φ is a KS_n – isomorphism. Hence $M(n-1,1)$ and $\bar{N}$ are isomorphic overKS_n
Theorem 3:$\bar{N}_0 = KS_n u_{12}(n)$ and $M_0(n-1,1)$are isomorphic overKS_n.
Proof: From theorem (2) we have a KS_n–homomorphism$\varphi : M(n-1,1) \to \bar{N}\quad s.t.$
$\varphi(x_i) = C_i(n)\ ;\ i = 1,2,\dots,n$ .And since $M_0(n-1,1) = KS_n(x_2 - x_1) \subset M(n-1,1)$, then
$\varphi(x_i - x_1) = \varphi(x_i) - \varphi(x_1) = C_i(n) - C_1(n) = u_{i1}(n) \in \bar{N}_0$.Let$\psi = \varphi|_{M_0(n-3,2,1)}$.
Then $\psi : \mathrm{M}_0(\mathrm{n}-1,1) \to \bar{N}_0$
s.t. $\psi(x_i - x_1) = u_{i1}(n)\ ;\ i = 1,2,\dots,n$. Which is KS_n –homomorphism.
Also we have $\forall\ u_{ij} \in \bar{N}_0\ ,\ \exists\ x_i - x_j \in \mathrm{M}_0(\mathrm{n}-1,1)\quad s.t.$

$$\psi(x_i - x_j) = \psi(x_i - x_1 + x_1 - x_j) = \psi(x_i - x_1) - \psi(x_j - x_1) = u_{i1}(n) - u_{j1}(n)$$
$$= C_i(n) - C_1(n) - C_j(n) + C_1(n) = C_i(n) - C_j(n) = u_{ij}(n).$$

Thusψ is an epimorphism .Since$dim_K M_0(n-1,1) = n-1$and $dim_K \bar{N}_0 = n-1$.
Then $dim_K\ ker\psi = dim_K M_0(n-1,1) - dim_K \bar{N}_0 = 0$. Hence $ker\psi = 0$. Thusψ is a monomorphism. Therefore $\psi$is a KS_n – isomorphism .Thus$M_0(n-1,1)$ and$\bar{N}_0$ are isomorphic overS_n .

Corollary 1: TheKS_n-module$\bar{N}_0 = KS_n u_{12}(n)$ is irreducible overKS_n if p does not dividen.
Proof: From theorem (3) we have$\bar{N}_0 \simeq M_0(n-1,1)$ .Since $M_0(n-1,1)$ is irreducible over KS_n if$p$ dose not divide n by [4]. Hence$\bar{N}_0$ is irreducible over KS_n if$p$ dose not divide .

Proposition 1: Ifp does not divide n, then$\bar{N} = \bar{N}_0 \oplus K\sigma_3$.
Proof: By theorem (3) we have$\bar{N}_0 \simeq M_0(n-1,1)$, and by corollary (1) we have $\bar{N}_0$is irreducible submodule overKS_n when p does not divides n and $\sigma_3(n) \notin \bar{N}_0$when p does not divide n since the sum of the coefficients

of the $C_i(n)$ in $\sigma_3(n)$ is n. Since $dim_K K\sigma_3(n)$ =1. Then $K\sigma_3$(n) is irreducible submodule over KS_n. Hence $\overline{N}_0 \cap K\sigma_3(n) = \mathbf{0}$.$K\sigma_3 \subset \overline{N}$and $\overline{N}_0 \subset \overline{N}$.
But $dim_K\overline{N}_0 + dim_K K\sigma_3 = n-1+1 = n = dim_K\overline{N}$.
Hence$\overline{N}_0 \oplus K\sigma_3 = \overline{N}$ when$p$ does not divides$n$.

Proposition2: If p does not divide n, then $\overline{N}$ has the following two composition series
$0 \subset \overline{N}_0 \subset \overline{N}and0 \subset K\sigma_3(n) \subset \overline{N}$.

Proof: Since$p$ does not divide n, then by proposition (1) we have $\overline{N}$ =$\overline{N}_0 \oplus K\sigma_3$, and by theorem (3) we have$\overline{N}_0 \simeq M_0(n-1,1)$. Hence by corollary (1)we get that$\overline{N}_0$ is irreducible submodule when p does not divide n.
Hence $\frac{\overline{N}}{K\sigma_3(n)} = \frac{\overline{N}_0 \oplus K\sigma_3(n)}{K\sigma_3(n)} \simeq \overline{N}_0$.Thus$\frac{\overline{N}}{K\sigma_3(n)}$ is irreducible module when p does not divide n.
Since$dim_K K\sigma_3(n)$ =1 .Then $\sigma_3(n)$is irreducible submodule overKS_n .
But $\frac{\overline{N}}{\overline{N}_0} = \frac{\overline{N}_0 \oplus K\sigma_3(n)}{\overline{N}_0} \simeq K\sigma_3(n)$ .Therefore $\frac{\overline{N}}{\overline{N}_0}$ is irreducible module over KS_n.Thus we get the following two composite series
$0 \subset \overline{N}_0 \subset \overline{N}$ and $0 \subset K\sigma_3(n) \subset \overline{N}$.

Definitions 4: Let K be a field of characteristic $p \neq 2$. Then
1. the KS_n–homomorphism $d : M(n-3,2,1) \rightarrow M(n-2,2)$ is defined in terms of the partial operators by

$$d\left(x_i x_j x_l^2\right) = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\left(x_i x_j x_l^2\right),$$

2. the KS_n–homomorphism $\bar{d}$ which is the restriction of d to$M_0(n-3,2,1)$. i.e.

$$\bar{d}: M_0(n-3,2,1). \rightarrow M_0(n-3,2).$$

3. theKS_n–homomorphism : $M(n-3,2,1) \rightarrow K$ which is defined by

$$f\Big(\sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{i,j,l} x_i x_j x_l^2\Big) = \sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{i,j,l}.$$

Theorem 4 : The following sequence of $KS_n -$ modules is exact

$$0 \rightarrow Ker\ d \xrightarrow{i} M(n-3,2,1) \xrightarrow{d} M(n-2,2) \rightarrow 0 \qquad \ldots(1)$$

Proof: It is clear that the map d is onto since $\forall \sum_{1\le i\langle j\le n} k_{i,j} x_i x_j \in M(n-2,2)$,

$$\exists\ \frac{1}{2}\sum_{1\le i\langle j\le n} k_{i,j}\, x_i x_j x_l^2 \in M(n-3,2,1) \text{for some } l\ (l \neq i,j)\ s.t.\ d\Big(\frac{1}{2}\sum_{1\le i\langle j\le n} k_{i,j}\, x_i x_j x_l^2\Big) = \sum_{1\le i\langle j\le n} k_{i,j}\, x_i x_j \quad .$$

Since the inclusion map$i$ is 1-1 and $Im\ i = \ker\ d$. Hence the sequence (1) is exact.

Theorem5: The sequence (1) is split iff p does not divide (n–2).

Proof: Assume p does not divide (n–2). We can define a function

$$\varphi : M(n-2,2) \rightarrow M(n-3,2,1) \text{by} \varphi\left(x_i x_j\right) = \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2 \text{which is a } KS_n\text{-homomorphism.}$$

Since for any$\tau \in S_n$ then $\varphi\left(\tau(x_i x_j)\right) = \varphi\left(\tau(x_i)\tau(x_j)\right) = \frac{1}{2(n-2)}\sum_{\substack{l_1=1\\ l_1\neq i_1,j_1}}^{n} \tau(x_i)\tau(x_j)x_{l_1}^2$

Where $\tau(x_i) = x_{i_1}$, $\tau(x_j) = x_{j_1}$
$\Longrightarrow \varphi(\tau(x_i x_j)) = \frac{1}{2(n-2)}\tau(x_i x_j x_l^2) = \tau\Big(\frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2\Big) = \tau\ \varphi\left(x_i x_j\right)$

And $\varphi\left(x_i x_j\right) = d\Big(\frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2\Big) = \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\neq i,j}}^{n} d(x_i x_j x_l^2) = \frac{1}{2(n-2)}(2(n-2)x_i x_j) = x_i x_j$. Then $d\varphi = I$ on $M(n-2,2)$. Hence the sequence (1) is split .
Thus $M(n-3,2,1) = L \oplus \ker d$, where $L = \varphi(M(n-2,2))$
Now assume that the sequence (1) is split .Then there exist a
KS_n-homomorphism $\psi: M(n-2,2) \longrightarrow M(n-3,2,1)\ s.t.\ d\,\psi = I$ on $(n-2,2)$.
i.e. $d\ \psi(x_i x_j) = x_i x_j$.

Then ψ has the form $\psi(x_{i_1} x_{j_1}) = \sum_{1\le i\langle j\le n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} k_{ijl} x_i x_j x_l^2$, $1\le i_1 < j_1 \le n$.

Thus we get $d\,\psi(x_{i_1}x_{j_1}) = d(\sum_{1\le i\langle j\le n}\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl}x_ix_jx_l^2) = \sum_{1\le i\langle j\le n}(2\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl})x_ix_j = x_{i_1}x_{j_1}$

Which implies that $2(\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl}) = 0$ if $(i,j) \ne (i_1,j_1)$ and $2\ (\sum_{\substack{l_1=1\\ l_1\ne i_1,j_1}}^{n} k_{i_1j_1l}) = 1$ if $(i,j) = (i_1,j_1)$.

Moreover if $\tau = (x_rx_s) \in S_n\,; 1 \le r < s \le n\ s.t.\ \tau\left(x_{i_1}x_{j_1}\right) = x_{i_1}x_{j_1}$.

Then $\psi(\tau\,(x_{i_1}x_{j_1})) = \psi\,(x_{i_1}x_{j_1}) = \tau\,\psi\,(x_{i_1}x_{j_1}) \Longrightarrow \psi\,(x_{i_1}x_{j_1}) - \tau\,\psi\,(x_{i_1}x_{j_1}) = 0$

$$\Longrightarrow \sum_{1\le i\langle j\le n}\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl}x_ix_jx_l^2 - \tau\,(\sum_{1\le i\langle j\le n}\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl}x_ix_jx_l^2) = 0 \Longrightarrow \sum_{1\le i\langle j\le n}\sum_{\substack{l=1\\ l\ne i,j}}^{n} (k_{ijl}x_ix_jx_l^2 - k_{ijl}\tau(\,x_ix_jx_l^2)) = 0 \Longrightarrow$$

$$\sum_{\substack{j=r+1\\ j\ne s}}^{n}\sum_{\substack{l=1\\ l\ne r,s,j}}^{n}(k_{rjl}-k_{sjl})x_rx_jx_l^2 + \sum_{j=s+1}^{n}\sum_{\substack{l=1\\ l\ne r,s,j}}^{n}(k_{sjl}-k_{rjl})x_sx_jx_l^2 + \sum_{i=1}^{r-1}\sum_{\substack{l=1\\ l\ne i,r,s}}^{n}(k_{irl}-k_{isl})x_ix_rx_l^2 + \sum_{\substack{i=1\\ i\ne r}}^{s-1}\sum_{\substack{l=1\\ l\ne i,r,s}}^{n}$$

$$(k_{isl}-k_{irl})x_ix_sx_l^2 + \sum_{\substack{i=1\\ i\ne r,s}}^{n-1}\sum_{\substack{j=i+1\\ j\ne r,s}}^{n}(k_{ijr}-k_{ijs})x_ix_jx_r^2 + \sum_{\substack{i=1\\ i\ne r,s}}^{n-1}\sum_{\substack{j=i+1\\ j\ne r,s}}^{n}(k_{ijs}-k_{ijr})x_ix_jx_s^2 + \sum_{i=1}^{r-1}(k_{irs} -$$

$$k_{isr})x_ix_rx_s^2 + \sum_{\substack{i=1\\ i\ne r}}^{s-1}(k_{isr}-k_{irs})x_ix_sx_r^2 + \sum_{\substack{j=r+1\\ j\ne s}}^{n}(k_{rjs}-k_{sjr})x_rx_jx_s^2 + \sum_{j=s+1}^{n}(k_{sjr}-k_{rjs})x_sx_jx_r^2 = 0$$

$$\Longrightarrow \sum_{\substack{j=r+1\\ j\ne s}}^{n}\sum_{\substack{l=1\\ l\ne r,s,j}}^{n}(k_{rjl}-k_{sjl})(x_rx_jx_l^2 - x_sx_jx_l^2) + \sum_{i=1}^{r-1}\sum_{\substack{l=1\\ l\ne r,s,i}}^{n}(k_{irl}-k_{isl})(x_ix_rx_l^2 - x_ix_sx_l^2) +$$

$$\sum_{\substack{1\le i\langle j\le n\\ i,j\ne r,s}}(k_{ijr}-k_{ijs})(x_ix_jx_r^2 - x_ix_jx_s^2) + \sum_{i=1}^{r-1}(k_{irs}-k_{isr})(x_ix_rx_s^2 - x_ix_sx_r^2) +$$

$$\sum_{\substack{j=r+1\\ j\ne s}}^{n}(k_{rjs}-k_{sjr})\left(x_rx_j\,x_s^2 - x_sx_j\,x_r^2\right) = 0$$

Then by equating the coefficient of the above equation we get

$k_{rjl} = k_{sjl}\ \forall\ r < j < n\ \ni j \ne s$ and $\forall\ 1 \le l \le n \ni l \ne r,s,j$

$k_{irl} = k_{isl}\ \forall\ 1 \le i < r$ and $\forall\ 1 \le l \le n \ni l \ne r,s,i$

$k_{ijr} = k_{ijs}\ \forall\ 1 \le i < j \le n\ \ni\ i,j \ne r,s$

$k_{irs} = k_{isr}\ \forall\ 1 \le i < r$

$k_{rjs} = k_{sjr}\ \forall\ r < j \le n\ ,j \ne s$.

So for any $r,s\,; 1 \le r < s \le n$ we get $k_{rjl} = k_{sjl} = k_{irl} = k_{isl} = k_{irs} = k_{isr} = k_{rjs} = k_{sjr} = k$ for any i,j,l

But we have $\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl} = 0$ when $(i,j) \ne (i_1,j_1)$ which implies that $\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl} = \sum_{\substack{l=1\\ l\ne i,j}}^{n} k = 0.$

i.e. $(n-2)k = 0 \Longrightarrow p|(n-2)\ or\ \ k = 0$

From other side we get for any $r,s\,; 1 \le r < s \le n$ that $k_{ijr} = k_{ijs} = k_1$ .But we have $\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl} = 1$ when $(i,j) = (\,i_1,j_1)$.Which implies that $\sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{ijl} = \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_1 = 1$ i.e. $(n-2)k_1 = 1 \Longrightarrow p \nmid (n-2)$ and $k_1 \ne 0.$

Hence we get that $p \nmid (n-2)$,$k_1 \ne 0$ and $k = 0$.

i.e. if the sequence (1) is split ,then $p \nmid (n-2)$.

Corollary 2: The dimension of ker$d$ over K of the KS_n −homomorphism $d: M(n-3.2.1) \to M(n-2,2)$ is $\frac{n(n-1)(n-3)}{2}$

Proof: Since $d: M(n-3,2,1) \to M(n-2,2)$ is onto map .Then we have

$$\frac{M(n-3,2,1)}{kerd} \simeq M(n-2,2)\ .\text{Thus } dim_K\,kerd = dim_K M(n-3,2,1) - dim_K\,M(n-2,2)$$
$$= \frac{n(n-1)(n-2)}{2} - \frac{n(n-1)}{2} = \frac{n(n-1)(n-3)}{2}.$$

Lemma 2:$dim_K\ \ S(n-3,2,1) = \frac{n(n-2)(n-4)}{3}$.

Proposition 3: $S(n-3,2,1)$ is a proper submodule of $kerd$.

Proof: Recall that $S(n-3,2,1) = KS_n\,\Delta(x_1,x_2,x_3)\Delta(x_4,x_5)$.
Let $y = \Delta(x_1,x_2,x_3)\,\Delta(x_4,x_5). = (x_2-x_1)(x_3-x_1)(x_3-x_2)(x_5-x_4)$
$= (x_2x_3 - x_1x_2 - x_1x_3 + x_1^2)(x_3x_5 - x_3x_4 - x_2x_5 + x_2x_4)$
$\Longrightarrow y = x_2x_5x_3^2 - x_2x_4x_3^2 - x_3x_5x_2^2 + x_3x_4x_2^2 - x_1x_2x_3x_5 + x_1x_2x_3x_4 + x_1x_5x_2^2 - x_1x_4x_2^2 - x_1x_5x_3^2 + x_1x_4x_3^2 + x_1x_2x_3x_5 - x_1x_2x_3x_4 + x_3x_5x_1^2 - x_3x_4x_1^2 - x_2x_5x_1^2 + x_2x_4x_1^2$.
Then $d(y) = 2x_2x_5 - 2x_2x_4 - 2x_3x_5 + 2x_3x_4 + 2x_1x_5 - 2x_1x_4 - 2x_1x_5 + 2x_1x_4 + 2x_3x_5 - 2x_3x_4 - 2x_2x_5 + 2x_2x_4 = 0$.
Then$y \in ker\,d$. Hence$S(n-3,2,1) \subset ker\,d$.
But $dim_K\ S(n-3,2,1) = \frac{n(n-2)(n-4)}{3} \leq \frac{n(n-1)(n-3)}{2} = dim_K\,ker\,d$
Therefore$S(n-3,2,1)$ is a proper submodule of$ker\,d$.
Corollary 3: The following sequence ofKS_n- modules is exact

$$0 \to Kerd \xrightarrow{i} M_0(n-3,2,1) \xrightarrow{\bar d} M_O(n-2,2) \to 0 \qquad \ldots(2)$$

Proof: Since $M_0(n-3,2,1) \subset M(n-3,2,1)$ and the K- basis of $M_0(n-3,2,1)$ is $\{x_ix_jx_l^2 - x_1x_2x_3^2 \mid 1 \leq i < j \leq n, 1 \leq l \leq n; l \neq i,j, (i,j,l) \neq (1,2,3)\}$
Then $(x_ix_jx_l^2 - x_1x_2x_3^2) = 2x_ix_j - 2x_1x_2 \in M_0(n-2,2)$.
Hence $d|_{M_0(n-3,2,1)} : M_0(n-3,2,1) \longrightarrow M_0(n-2,2)$. Let $\bar d = d|_{M_0(n-3,2,1)}$.
Then $\bar d : M_0(n-3,2,1) \longrightarrow M_0(n-2,2)$ such that $\bar d(x_ix_jx_l^2 - x_1x_2x_3^2) = 2x_ix_j - 2x_1x_2$.
$\forall\ \alpha\,(x_ix_j - x_1x_2) \in M_0(n-2,2)$, $\exists\ \frac{\alpha}{2}(x_ix_jx_l^2 - x_1x_2x_3^2) \in M_0(n-3,2,1)$ s.t.

$$\bar d\left(\frac{\alpha}{2}(x_ix_jx_l^2 - x_1x_2x_3^2)\right) = \alpha\,(x_ix_j - x_1x_2)$$

and by linearity of$\bar d$we get $\bar d$ is on to. Thus the following sequence

$$0 \to Ker\,\bar d \xrightarrow{i} M_0(n-3,2,1) \xrightarrow{\bar d} M_0(n-2,2) \to 0$$

is exact since the inclusion map$i$ is one-to-one and Im i = Ker $\bar d$.Moreover $Ker\bar d \subset Kerd$. So by counting the dimension of $Ker\bar d$we get
$dim_K\,Ker\bar d$=$dim_K M_0(n-3,2,1) - dim_K M_0\,(n-2,2)$=$\frac{n(n-1)(n-2)}{2} - 1 - \frac{n(n-1)}{2} + 1$=$\frac{n(n-1)(n-3)}{2}$= $dim_K\,Kerd$
Hence $er\bar d = Kerd$. Thus we get the following sequence

$$0 \to Kerd \xrightarrow{i} M_0(n-3,2,1) \xrightarrow{\bar d} M_0(n-2,2) \to 0$$

which is exact sequence
Corollary 4: The sequence (2) is split iff$p$ dose not divide (n-2).
Proof: Assume p dose not divide (n-2). By theorem (5) we have aKS_n -homomorphism

$$\varphi: M(n-2,2) \to M(n-3,2,1) \quad s.t.$$

$\varphi(x_ix_j) = \frac{1}{2(n-2)} \sum_{\substack{l=1 \\ l\neq i,j}}^{n} x_ix_jx_l^2$Then $\varphi(x_ix_j - x_1x_2) = \varphi(x_ix_j) - \varphi(x_1x_2)$

$$= \frac{1}{2(n-2)} \sum_{\substack{l=1 \\ l\neq i,j}}^{n} x_ix_jx_l^2 - \frac{1}{2(n-2)} \sum_{\substack{l=1 \\ l\neq 1,2}}^{n} x_1x_2x_l^2 = \frac{1}{2(n-2)} \Big(\sum_{\substack{l=1 \\ l\neq i,j}}^{n} x_ix_jx_l^2 - \sum_{\substack{l=1 \\ l\neq 1,2}}^{n} x_1x_2x_l^2\Big) \in M_0(n-3,2,1)$$

i.e. $\varphi|_{M_0(n-2,2)} : M_0(n-2,2) \to M_0(n-3,2,1)$.
Let $\bar\varphi = \varphi|_{M_0(n-2,2)}$ .Hence$\bar\varphi$ is aKS_n -homomorphism s.t.
$d\,\bar\varphi(x_ix_j - x_1x_2) = \bar d\big(\frac{1}{2(n-2)}\big(\sum_{\substack{l=1 \\ l\neq i,j}}^{n} x_ix_jx_l^2 - \sum_{\substack{l=1 \\ l\neq 1,2}}^{n} x_1x_2x_l^2\,\big)\big) = \frac{1}{2(n-2)}\big(\overline{d}\big(\sum_{\substack{l=1 \\ l\neq i,j}}^{n} x_ix_jx_l^2 - \sum_{\substack{l=1 \\ l\neq 1,2}}^{n} x_1x_2x_l^2\big)\big)$=
$\frac{1}{2(n-2)}(2(n-2)x_ix_j - 2(n-2)x_1x_2 = \frac{1}{2(n-2)}(2(n-2)(x_ix_j - x_1x_2)) = x_ix_j - x_1x_2$.
Then $\bar d\bar\varphi = I$ on $M_0(n-2,2)$ .Thus the sequence (2) is split i.e.

$$M_0(n-3,2,1) = Ker\,d\ \oplus \bar\varphi\big(M_0(n-2,2)\big).$$

Now assume the sequence (2) is split .Then there exist a KS_n –homomorphism $\bar\psi = \psi|_{M_0(n-2,2)}$ where ψ as its defined in Theorem (5) s .t.$\bar d\bar\psi = I$. *i.e.*

$$x_ix_j - x_1x_2 = \bar d\bar\psi(x_ix_j - x_1x_2) = d\psi\,(x_ix_j - x_1x_2) = d\psi\,(x_ix_j) - d\psi\,(x_1x_2)$$

$$= d\Big(\sum_{\substack{l=1 \\ l\neq i,j}}^{n} k\,x_ix_jx_l^2\Big) - d\Big(\sum_{\substack{l=1 \\ l\neq 1,2}}^{n} k_1x_1x_2x_l^2\Big) = 2(n-2)kx_ix_j - 2(n-2)k_1x_1x_2$$

Then by equating the coefficient we get $2(n-2)k = 1$ and$2(n-2)k_1 = 1$. In each case we get that p dose not divide $(n-2)$.

Corollary 5: If $p \neq 2$ and $p|(n-1)$ then we get the following composition series
1) $0 \subset \bar{N}_0 \subset \bar{N} \subset L_0$
2) $0 \subset K\sigma_3(n) \subset \bar{N} \subset L_0$
where $L_0 \simeq M_0(n-2,2)$
Proof: Since $p|(n-1)$. Then $p \nmid n$ and $p \nmid (n-2)$. Since $p \nmid (n-2)$.Then by corollary (4) we have $M_0(n-3,2,1) = ker\ d \oplus L_0$ where $L_0 \simeq M_0(n-2,2)$.

$\bar{N}_0 = KS_nC_1(n)$; $C_1(n) = \sum_{1\langle i \langle j \leq n} x_i x_j x_1^2$ .Then the sum of coefficients of $C_1(n)$ is $\frac{(n-1)(n-2)}{2}$ and since $p|(n-1)$ then $\bar{N}_0 \subset M_0(n-3,2,1)$. But $d(C_1(n)) = 2\sum_{1\langle i \langle j \leq n} x_i x_j \neq 0$.

Hence $\bar{N}_0 \cap \ker d = 0$, which implies that $\bar{N}_0 \subset L_0$. Since $p \nmid n$. Then by proposition (1) we have $\bar{N} = \bar{N}_0 \oplus K\sigma_3(n)$;$\bar{N}_0 = KS_n u_{12}(n)$. and by theorem (3) we get $\bar{N}_0 \simeq N_0 \simeq M_0(n-1,1)$, where $N_0 = KS_n b_{12}(n) \subset M_0(n-2,2)$.
Let $g_1: K \to K\sigma_2(n)$ defined by $g_1(k) = k\sigma_2(n)$ theng_1 is aKS_n −isomorphism. Thus $K \simeq K\sigma_2(n)$.
Let $g_2: K \to K\sigma_3(n)$ defined by $g_2(k) = k\sigma_3(n)$ theng_2 is a KS_n −isomorphism. Thus $K \simeq K\sigma_3(n)$.
Hence $K\sigma_3(n) \simeq K\sigma_2(n)$. Since $N_0 \simeq M_0(n-1,1)$(see [4]) and$\bar{N}_0 \simeq M_0(n-1,1)$ by theorem (3), then $\bar{N} = \bar{N}_0 \oplus K\sigma_3(n) \simeq N_0 \oplus K\sigma_2(n)$, which implies that
$\frac{L_0}{\bar{N}} = \frac{L_0}{\bar{N}_0 \oplus K\sigma_3(n)} \simeq \frac{M_0(n-2,2)}{N_0 \oplus K\sigma_2(n)} = \frac{N_0 \oplus S(n-2,2)}{N_0 \oplus K\sigma_2(n)} \simeq \frac{S(n-2,2)}{K\sigma_2(n)}$ .But $\frac{S(n-2,2)}{K\sigma_2(n)}$ is irreducible overKS_n when $p\,|(n-1)$ (see[3]).
By proposition (2) if$p \nmid n$ we have the following composition series
$0 \subset \bar{N}_0 \subset \bar{N}$ and $0 \subset K\sigma_3(n) \subset \bar{N}$. Hence we get the following two composition series
$0 \subset \bar{N}_0 \subset \bar{N} \subset L_0$ and $0 \subset K\sigma_3(n) \subset \bar{N} \subset L_0$
Theorem 6: The following sequence over a field K is exact .

$$0 \to M_0(n-3,2,1) \xrightarrow{i} M(n-3,2,1) \xrightarrow{f} K \to 0 \qquad \ldots(3)$$

Proof: Since the inclusion map i is one –to one and $f : M(n-3,2,1) \to K$ s.t.

$f\ (\sum_{1\leq i\langle j\leq n} \sum_{\substack{k=1\\k\neq i,j}}^{n} k_{ijl}\ x_i\, x_j x_k^2) = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\l\neq i,j}}^{n} k_{ijl}$ is onto since $\forall\ k \in K, \exists\ kx_i x_j x_l^2 \in M(n-3,2,1)$ s .t.

$f\left(k\ \ x_i\, x_j x_k^2\right) = k$ .Moreover we have $ker\, f = \{y \in M(n-3,2,1) | f(y) = 0\}$

$=\{ \sum_{1\leq i\langle j\leq n} \sum_{\substack{k=1\\k\neq i,j}}^{n} k_{ijl}\ x_i\, x_j x_k^2\ \ | f(\sum_{1\leq i\langle j\leq n} \sum_{\substack{l=1\\l\neq 1,2}}^{n} k_{ijl} x_i x_j x_l^2) = 0\ \}\ = \{ \sum_{1\leq i\langle j\leq n} \sum_{\substack{k=1\\k\neq i,j}}^{n} k_{ijl}\ x_i\, x_j x_k^2\ \ | \sum_{1\langle j\leq n} \sum_{\substack{l=1\\l\neq 1,2}}^{n}$

$k_{ijl}\ = 0\ \}$
$= M_0(n-3,2,1) = Im\ i$.Hence the sequence (3) is exact .
Theorem 7: If $p \neq 2$ and p divides $(n-1)$ then we have the following series:
1) $0 \subset \bar{N}_0 \subset \bar{N} \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.
2) $0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus ker\, d \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$
3) $0 \subset K\sigma_3 \subset K\sigma_3 \oplus ker\, d \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1)\ \subset\ M(n-3,2,1)$.
4) $0 \subset K\sigma_3 \subset \bar{N} \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.
5) $0 \subset ker\, d \subset \bar{N}_0 \oplus \ker d \subset\ \bar{N} \oplus\ ker\, d \subset M_0(n-3,2,1) \subset\ M(n-\ 3,2,1)$.
6) $0 \subset ker\, d \subset\ K\sigma_3 \oplus ker\, d \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \subset\ M(n-3,2,1)$.
7) $0 \subset \bar{N}_0 \subset \bar{N} \subset L_0 \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

Proof : Since $\bar{N} = KS_nC_1(n)$ where $C_1(n) = x_1^2(\sigma_2(n) - \sum_{j=2}^{n} x_i x_j) = \sum_{1\langle i\langle j\leq n} x_i x_j x_1^2$

Then the sum of coefficient of $C_1(n)$ is $\frac{(n-1)(n-2)}{2}$ which implies that $C_1(n) \in M_0(n-3,2,1)$ when p divides $(n-1)$. Thus we get that $\bar{N} \subset M_0(n-3,2,1)$. Since $u_{12}(n) = C_1(n) - C_2(n)$ .Then $\bar{N}_0 = Ku_{12}(n) = KS_n(C_1(n) - C_2(n))$ and hence $\bar{N}_0 \subset \bar{N}$ .Since $p \neq 2$ and p divides $(n-1)$.
Then p does not divide n which implies by corollary (1) that $\bar{N}_0$ is irreducible submodule over KS_n ,and p does not divide $(n-2)$ which implies that $K\sigma_3 \not\subset \ker d$.
Since $\bar{N} = \bar{N}_0 \oplus K\sigma_3$ when p does not divide n by proposition(1) ,and both $\bar{N}_0$ and $K\sigma_3$ are irreducible modules.

Then $\overline{N} \cap \ker d = 0$ .Therefore we get the following series if $p \neq 2$ and $p \mid (n-1)$

1) $0 \subset \overline{N}_0 \subset \overline{N} \subset \overline{N} \oplus \ker d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.
2) $0 \subset \overline{N}_0 \subset \overline{N}_0 \oplus \ker d \subset \overline{N} \oplus \ker d \subset M_0(n-3,2,1) \subset M(n-3,2,1)$
3) $0 \subset K\sigma_3 \subset K\sigma_3 \oplus \ker d \subset \overline{N} \oplus \ker d \subset M_\circ(n-3,2,1) \subset M(n-3,2,1)$.
4) $0 \subset K\sigma_3 \subset \overline{N} \subset \overline{N} \oplus ker\, d \subset M_\circ(n-3,2,1) \subset M(n-3,2,1)$.
5) $0 \subset ker\, d \subset \overline{N}_\circ \oplus ker\, d \subset \overline{N} \oplus ker\, d \subset M_\circ(n-3,2,1) \subset M(n-3,2,1)$.
6) $0 \subset ker\, d \subset K\sigma_3 \oplus ker\, d \subset \overline{N} \oplus ker\, d \subset M_\circ(n-3,2,1) \subset M(n-3,2,1)$.
7) $0 \subset \overline{N}_0 \subset \overline{N} \subset L_0 \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

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