IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 2 Ver. I (Mar. - Apr. 2016), PP 121-136
www.iosrjournals.org

Applications Geometry Riemannian Manifolds

Mohamed M. Osman
[1]*(Department Of Mathematics Faculty Of Science - University Of Al-Baha – K.S.A)*

Abstract: *In This Paper Some Fundamental Theorems , Definitions In Riemannian Geometry Manifolds In The Space R^n To Pervious Of Differentiable Manifolds Which Are Used In An Essential Way In Basic Concepts Of Applications Riemannian Geometry Examples Of The Problem Of Differentially Projection Mapping Parameterization System By Strutting Rank k .*

Keywords : *Basic Notions On Differential Geometry – Tangents Spaces And Vector Fields – Differential Geometry – Cotangent Space And Vector Bundles – Tensor Fields – Differentiable Manifolds Charts - Surface N-Dimensional.*

I. Introduction

A Riemannian Manifolds Is A Generalization Of Curves And Surfaces To Higher Dimension , It Is Euclidean In E^n In That Every Point Has A Neighbored, Called A Chart Homeomorphism To An Open Subset Of R^n , The Coordinates On A Chart Allow One To Carry Out Computations As Though In A Euclidean Space , So That Many Concepts From R^n , Such As Differentiability, Point Derivations , Tangents , Cotangents Spaces , And Differential Forms Carry Over To A Manifold. In This We Given The Basic Definitions And Properties Of A Smooth Manifold And Smooth Maps Between Manifolds , Initially The Only Way We Have To Verify That A Space , We Describe A Set Of Sufficient Conditions Under Which A Quotient Topological Space Becomes A Manifold Is Exhibit A Collection Of C^∞ Compatible Charts Covering The Space Becomes A Manifold , Giving Us A Second Way To Construct Manifolds , A Topological Manifolds C^∞ Analytic Manifolds , Stating With Topological Manifolds , Which Are Hausdorff Second Countable Is Locally Euclidean Space , We Introduce The Concept Of Maximal C^∞ Atlas , Which Makes A Topological Manifold Into A Smooth Manifold , A Topological Manifold Is A Hausdorff , Second Countable Is Local Euclidean Of Dimension n , If Every Point p In M Has A Neighborhood U Such That There Is A Homeomorphism φ From U Onto A Open Subset Of R^n , We Call The Pair A Coordinate Map Or Coordinate System On U , We Said Chart (U, φ) Is Centered At $p \in U$, $\varphi(p) = 0$, And We Define The Smooth Maps $f : M \to N$ Where (M, N) Are Differential Manifolds We Will Say That f Is Smooth If There Are Atlases (U_α, h_α) On M And (V_β, g_β) On N . In This Paper, The Notion Of A Differential Manifold Is Necessary For The Methods Of Differential Calculus To Spaces More General Than De R^n , A Differential Structure On A Manifolds M Induces A Differential Structure On Every Open Subset Of M , In Particular Writing The Entries Of An $(n \times k)$ Matrix In Succession Identifies The Set Of All Matrices With $R^{n,k}$, An $n \times k$ Matrix Of Rank k Can Be Viewed As A K-Frame That Is Set Of k Linearly Independent Vectors In R^n , $(V_{n,k} K \le n)$ Is Called The Steels Manifold ,The General Linear Group $GL(n)$ By The Foregoing $V_{n,k}$ Is Differential Structure On The Group n Of Orthogonal Matrices, We Define The Smooth Maps Function $f : M \to N$ Where M, N Are Differential Manifolds We Will Say That f Is Smooth If There Are Atlases (U_α, h_α) On M , (V_B, g_B) On N , Such That The Maps $\left(g_B f h_\alpha^{-1} \right)$ Are Smooth Wherever They Are Defined f Is A Homeomorphism If Is Smooth And A Smooth Invers . A Differentiable Structures Is Topological Is A Manifold It An Open Covering U_α Where Each Set U_α Is Homeomorphism, Via Some Homeomorphism h_α To An Open Subset Of Euclidean Space R^n , Let M Be A Topological Space , A Chart In M Consists Of An Open Subset $U \subset M$ And A Homeomorphism h Of U Onto An Open Subset Of R^m , A C^r Atlas On M Is A Collection (U_α, h_α) Of Charts Such That The U_α Cover M And h_B, h_α^{-1} The Differentiable Vector Fields On A Differentiable Manifold M , Let X And Y Be A Differentiable Vector Field On A Differentiable Manifolds M Then There Exists A Unique Vector Field Z Such That Such That , For All $f \in D, Zf = (XY - YX) f$ If That $p \in M$ And Let $x : U \to M$ Be A Parameterization At Specs .

II. A Basic Notions On Differential Geometry

In This Section Is Review Of Basic Notions On Differential Geometry:

2.1 First Principles

Hausdrff Topological 2.1.1

A Topological Space M Is Called (Hausdorff) If For All $x, y \in M$ There Exist Open Sets Such That $x \in U$ And $y \in V$ And $U \cap V = \phi$

Definition 2.1.2

A Topological Space M Is Second Countable If There Exists A Countable Basis For The Topology On M .

Definition 2.1.3: Locally Euclidean Of Dimension (M)

A Topological Space M Is Locally Euclidean Of Dimension N If For Every Point $x \in M$ There Exists On Open Set $U \in M$ And Open Set $w \subset R^n$ So That U And W Are (Homeomorphism).

Definition 2.1.3

A Topological Manifold Of Dimension N Is A Topological Space That Is Hausdorff, Second Countable And Locally Euclidean Of Dimension N.

Definition 2.1.4

A Smooth Atlas A Of A Topological Space M Is Given By: (I) An Open Covering $\{U\}_{i \in I}$ Where $U_i \subset M$

Open And $M = \cup_{i \in I} U_i$.(Ii) A Family $\{\phi_i : U_i \to W_i\}_{i \in I}$ Of Homeomorphism ϕ_i Onto Open Subsets $W_i \subset R^n$ So That If $U_i \cap U_j \neq \phi$ Then The Map $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ Is (A Diffoemorphism)

Definition 2.1.5

If $(U_i \cap U_j) \neq \phi$ Then The Diffoemorphism $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ Is Known As The (Transition Map).

Definition 2.1.6

A Smooth Structure On A Hausdorff Topological Space Is An Equivalence Class Of Atlases, With Two Atlases A And B Being Equivalent If For $(U_i, \varphi_i) \in A$ And $(V_j, \Psi_j) \in B$ With $U_i \cap V_j \neq \varphi$ Then The Transition $\phi_i(U_i \cap V_j) \to \Psi_j(U_i \cap V_j)$ Map Is A Diffoemorphism (As A Map Between Open Sets Of R^n).

Definition 2.1.7

A Smooth Manifold M Of Dimension N Is A Topological Manifold Of Dimension N Together With A Smooth Structure

Definition 2.1.8

Let M And N Be Two Manifolds Of Dimension m, n Respectively A Map $F : M \to N$ Is Called Smooth At $p \in M$ If There Exist Charts $(U, \phi), (V, \Psi)$ With $p \in U \subset M$ And $F(p) \in V \subset N$ With $F(U) \subset V$ And The Composition $\Psi \circ F \circ \phi^{-1} : \phi(U) \to \Psi(V)$ Is A Smooth (As Map Between Open Sets Of R^n Is Called Smooth If It Smooth At Every $p \in M$.

Definition 2.1.9

A Map $F : M \to N$ Is Called A Diffeomorphism If It Is Smooth Objective And Inverse $F^{-1} : N \to M$ Is Also Smooth.

Definition 2.1.10

A Map F Is Called An Embedding If F Is An Immersion And (Homeomorphism) Onto Its Image.

Definition 2.1.11

If $F : M \to N$ Is An Embedding Then $F(M)$ Is An Immersed (Sub Manifolds) Of N .

2.2 Tangent Space And Vector Fields

Let $C^\infty(M, N)$ Be Smooth Maps From M And N , Let $C^\infty(M)$ Smooth Functions On M Is Given A Point $p \in M$ Denote, $C^\infty(p)$ Is Functions Defined On Some Open Neighbourhood Of p And Smooth At p .

Definition 2.2.1

(I) The Tangent Vector X To The Curve $c:(-\varepsilon,\varepsilon)\to M$ At $t=0$ Is The Map $c(0):C^{\infty}(c(0))\to R$ Given By The Formula

(1) $$X(f)=c(0)(f)=\left(\frac{d(f\circ c)}{dt}\right)_{t=0}\ :\ \forall f\in C^{\infty}c(0)$$

(Ii) A Tangent Vector X At $p\in M$ Is The Tangent Vector At $t=0$ Of Some Curve $\alpha:(-\varepsilon,\varepsilon)\to M$ With $\alpha(0)=p$ This Is $X=\alpha'(0):C^{\infty}(p)\to R$.

Remark 2.2.2

A Tangent Vector At p Is Known As A Liner Function Defined On $C^{\infty}(p)$ Which Satisfies The (Leibniz Property)

(2) $$X(fg)=X(f)g+fX(g)\ \ ,\forall\ f,g\in C^{\infty}(p)\qquad .$$

2.3 Differential Geometrics

Given $F\in C^{\infty}(M,N)$ And $p\in M$, $X\in(T_pM)$ Choose A Curve $\alpha:(-\varepsilon,\varepsilon)\to M$ With $\alpha(0)=p$ And $\alpha'(0)=X$ This Is Possible Due To The Theorem About Existence Of Solutions Of Liner First Order Odes , Then Consider The Map $F_{*p}:T_pM\to T_{F(p)}N$ Mapping $X\to F_{*p}(X)=(F\circ\alpha)'(0)$, This Is Liner Map Between Two Vector Spaces And It Is Independent Of The Choice Of α .

Definition 2.3.1

The Liner Map F_{*p} Defined Above Is Called The Derivative Or Differential Of F At p While The Image $F_{*p}(X)$ Is Called The Push Forward X At $p\in M$.

Definition 2.3.2: Cotangent Space And Vector Bundles And Tensor Fields

Let M Be A Smooth N-Manifolds And $p\in M$ .We Define Cotangent Space At p Denoted By T_p^*M To Be The Dual Space Of The Tangent Space At $p:T_p^*(M)=\{f:T_pM\to R\}$, f Smooth Element Of T_p^*M Are Called Cotangent Vectors Or Tangent Convectors At p .(I) For $f:M\to R$ Smooth The Composition $T_p^*M\to T_{f(p)}R\cong R$ Is Called df_p And Referred To The Differential Of f .Not That $df_p\in T_p^*M$ So It Is A Cotangent Vector At p (Ii) For A Chart (U,ϕ,x^i) Of M And $p\in U$ Then $\{dx^i\}_{i=1}^{n}$ Is A Basis Of T_p^*M In Fact $\{dx^i\}$ Is The Dual Basis Of $\left\{\frac{d}{dx^i}\right\}_{i=1}^{n}$.

Definition 2.3.3

A Smooth Real Vector Bundle Of Rank k Denoted (E,M,π) Is A Smooth Manifold E Of Dimension $n+1$ The

Total Space A Smooth Manifold M Of Dimension n The Manifold Dimension $n+k$ And A Smooth Subjective Map $\pi:E\to M$ (Projection Map) With The Following Properties: (I) There Exists An Open Cover $\{V_\alpha\}_{\alpha\in I}$ Of M And Diffoemorphism $\Psi_\alpha:\pi^{-1}(V_\alpha)\to V_\alpha\times R^k$.(Ii) For Any Point $p\in M$,$\Psi_\alpha(\pi^{-1}(p))=\{p\}\times R^k\cong R^k$ And We Get A Commutative Diagram (In This Case $\pi_1:V_\alpha\times R^k\to V_\alpha$ Is Projection Onto The First Component .(Iii) Whenever $V_\alpha\cap V_\beta\neq\phi$ The Diffoemorphism.

(3) $$\psi_\alpha\circ\Psi_\beta^{-1}:(V_\alpha\cap V_\beta)\times R^k\to(V_\alpha\cap V_\beta)\times R^k$$

Takes The Form $\Psi_\alpha\circ\Psi_\beta^{-1}(p,a)=(p,A_{\alpha\beta}(p)(a)),a\in R^k$ Where $A_{\alpha\beta}:V_\alpha\cap V_\beta\to GL(k,R)$ Is Called Transition Maps.

Definition 2.3.4 : Bundle Maps And Isomorphism's

Suppose (E,M,π) And $(\tilde{E},\tilde{M},\tilde{\pi})$ Are Two Vector Bundles A Smooth Map $F:E\to\tilde{E}$ Is Called A Smooth Bundle Map From (E,M,π) To $(\tilde{E},\tilde{M},\tilde{\pi})$. (I) There Exists A Smooth Map $f:M\to\tilde{M}$ Such That The Following Diagram Commutes That $\pi(F(q))=f(\pi(q))$ For All $p\in M$ (Ii) F Induces A Linear Map From E_p To $\tilde{E}_{f(p)}$ For Any $p\in M$.

Definition 2.3.5 : Projective Spaces

The $n-$ Dimensional Real (Complex) Projective Space, Denoted By $P_n(R)$ or $P_n(C)$), Is Defined As The Set Of 1-Dimensional Linear Subspace Of R^{n+1} or C^{n+1}), $P_n(R)$ or $P_n(C)$ Is A Topological Manifold.

Definition 2.3.6

For Any Positive Integer n, The $n-$ Torus Is The Product Space $T^n = (S^1 \times ... \times S^1)$ .It Is A $n-$ Dimensional Topological Manifold. (The 2-Torus Is Usually Called Simply The Torus).

Definition2.3.7

The Boundary Of A Line Segment Is The Two End Points; The Boundary Of A Disc Is A Circle. In General The Boundary Of A $n-$ Manifold Is A Manifold Of Dimension $(n-1)$, We Denote The Boundary Of A Manifold M As ∂M. The Boundary Of Boundary Is Always Empty, $\partial\partial M = \phi$

Lemma 2.3.8

Every Topological Manifold Has A Countable Basis Of Compact Coordinate Balls. Every Topological Manifold Is Locally Compact.

Definitions 2.3.9

Let M Be A Topological Space n -Manifold. If $[(U,\varphi),(V,\psi)]$ Are Two Charts Such That $(U \cap V) \neq \phi$, The Composite Map $\psi \circ \varphi^{-1} : [\varphi\ (U \cap V)] \to [\psi\ (U \cap V)]$ Is Called The Transition Map From φ To ψ .

Definition 2.3.10

An Atlas A Is Called A Smooth Atlas If Any Two Charts In A Are Smoothly Compatible With Each Other. A Smooth Atlas A On A Topological Manifold M Is Maximal If It Is Not Contained In Any Strictly Larger Smooth Atlas. (This Just Means That Any Chart That Is Smoothly Compatible With Every Chart In A Is Already In A.

Definition 2.3.11

A Smooth Structure On A Topological Manifold M Is Maximal Smooth Atlas. (Smooth Structure Are Also Called Differentiable Structure Or C^{∞} Structure By Some Authors).

Definition 2.3.12

A Smooth Manifold Is A Pair $(M$, A), Where M Is A Topological Manifold And A Is Smooth Structure On M . When The Smooth Structure Is Understood, We Omit Mention Of It And Just Say M Is A Smooth Manifold.

Definition 2.3.13

Let M Be A Topological Manifold.

(I) Every Smooth Atlases For M Is Contained In A Unique Maximal Smooth Atlas. (Ii) Two Smooth Atlases For M Determine The Same Maximal Smooth Atlas If And Only If Their Union Is Smooth Atlas.

Definition 2.3.14

Every Smooth Manifold Has A Countable Basis Of Pre-Compact Smooth Coordinate Balls. For Example The General Linear Group The General Linear Group $GL\ (n, R)$ Is The Set Of Invertible $n \times n$ -Matrices With Real Entries. It Is A Smooth n^2 -Dimensional Manifold Because It Is An Open Subset Of The n^2 - Dimensional Vector Space $M\ (n, R)$, Namely The Set Where The (Continuous) Determinant Function Is Nonzero.

Definition 2.3.15

Let M Be A Smooth Manifold And Let p Be A Point Of M . A Linear Map $X : C^{\infty}(M) \to R$ Is Called A Derivation At p If It Satisfies:

(4) $$X\ (fg) = f\ (p) Xg + g\ (p) Xf$$

For All $f, g \in C^{\infty}(M)$. The Set Of All Derivation Of $C^{\infty}(M)$ At p Is Vector Space Called The Tangent Space To M At p , And Is Denoted By [T_pM]. An Element Of T_pM Is Called A Tangent Vector At p .

Lemma 2.3.16

Let M Be A Smooth Manifold, And Suppose $p \in M$ And $X \in T_pM$. If f Is A Const And Function, Then $Xf = 0$. If $f(p) = g(p) = 0$, Then $X\ (fp) = 0$.

Definition2.3.17

If γ Is A Smooth Curve (A Continuous Map $\gamma : J \to M$, Where $J \subset R$ Is An Interval) In A Smooth Manifold M , We Define The Tangent Vector To γ At $t_{\circ} \in J$ To Be The Vector $\gamma'(t_{\circ}) = \gamma_*\left(\frac{d}{dt}|_{t_{\circ}}\right) \in T_{\gamma(t_{\circ})}M$, Where $d/_{dt}\,|_{t_{\circ}}$ Is The Standard Coordinate Basis For $T_{t_{\circ}}R$. Other Common Notations For The Tangent Vector To γ Are $\left[\gamma^{*}(t_{\circ}), \frac{d\gamma}{dt}(t_{\circ})\right]$ And $\left[\frac{d\gamma}{dt}|_{t=t_{\circ}}\right]$. This Tangent Vector Acts On Functions By:

(5) $$\gamma'(t_\circ)\ f = \left(\gamma_* \frac{d}{dt}|_{t_\circ}\right) f = \frac{d}{dt}|_{t_\circ}\ (f\circ\gamma) = \left(\frac{d(f\circ\gamma)}{dt}\right)(t_\circ).$$

Lemma 2.3.18

Let M Be A Smooth Manifold And $p \in M$. Every $X \in (T_p M)$ Is The Tangent Vector To Some Smooth Curve In M .

Definition 2.3.19

A Lie Group Is A Smooth Manifold G That Is Also A Group In The Algebraic Sense, With The Property That The Multiplication Map $m:(G\times G)\to G$ And Inversion Map $m:(G\to G)$, Given By $m\ (g,h) = g\ h,\ \ i\ (g) = g^{-1}$, Are Both Smooth. If G Is A Smooth Manifold With Group Structure Such That The Map $G\times G \to G$ Given By $(g,h)\to gh^{-1}$ Is Smooth, Then G Is A Lie Group. Each Of The Following Manifolds Is A Lie Group With Indicated Group Operation. The General Linear Group $GL\ (n,R)$ Is The Set Of Invertible $(n\times n)$ Matrices With Real Entries. It Is A Group Under Matrix Multiplication, And It Is An Open Sub-Manifold Of The Vector Space $M\ (n,R)$, Multiplication Is Smooth Because The Matrix Entries Of A Aid B . Inversion Is Smooth Because Cramer's Rule Expresses The Entries Of A^{-1} As Rational Functions Of The Entries Of A . The $n-$ Torus $T^n = (S^1\times ...\times S^1)$ Is A $n-$ Dimensional A Belgian Group.

Definition 2.3.20 Lie Brackets

Let V And W Be Smooth Vector Fields On A Smooth Manifold M . Given A Smooth Function $f: M\to R$, We Can Apply V To f And Obtain Another Smooth Function Vf , And We Can Apply W To This Function, And Obtain Yet Another Smooth Function $(W\ V)f = W(Vf)$. The Operation $f\to W\ V\ f$, However, Does Not In General Satisfy The Product Rule And Thus Cannot Be A Vector Field, As The Following For Example Shows Let $V = \left(\frac{\partial}{\partial x}\right)$ And $W = \left(\frac{\partial}{\partial y}\right)$ On R^n , And Let $f(x,y) = x,\ g(x,y) = y$. Then Direct Computation Shows That $V\ W\ (f\ \ g) = 1$, While $(f\ V\ \ W\ \ g + g\ \ V\ \ W\ \ f) = 0$, So $V\ W$ Is Not A Derivation Of $C^\infty(R^2)$. We Can Also Apply The Same Two Vector Fields In The Opposite Order, Obtaining A (Usually Different) Function $W\ V\ f$. Applying Both Of This Operators To f And Subtraction, We Obtain An Operator $[V,W]: C^\infty(M)\to C^\infty(M)$, Called The Lie Bracket Of V And W , Defined By $[V,W]\ f\ \ =\ \ (V\ W)\ f - (WV)\ f$. This Operation Is A Vector Field. The Smooth Of Vector Field Is Lie Bracket Of Any Pair Of Smooth Vector Fields Is A Smooth Vector Field.

Lemma 2.3.21: Properties Of The Lie Bracket

The Lie Bracket Satisfies The Following Identities For All $V,W,X \in\ (M)$. Linearity: $\forall a,b\in R$,

(6) $$\left\{\begin{array}{l}[\ aV + bW\ ,X\ ] = a\ [V,X] + b\ [W,X] \\ [\ X,aV + bW\ ] = a\ [X,V] + b\ [X,W].\end{array}\right\}$$

(I) Ant Symmetry $[V,W] = -\ [W,V]$.(Ii)Jacobi Identity $[V,[\ W,X\ ]] + [\ W,[\ X,V\ ]\ ] + [\ X,[V,W\ ]\ ] = 0$. For $f,g\in C^\infty(M)$:

(7) $$[\ f\ V, g\ \ W\ ] = f\ g\ [\ V,W\ ] + [(\ f\ V\ g)\ W - (\ g\ W\ \ f)\ \ V]$$

2.4 Convector Fields

Let V Be A Finite – Dimensional Vector Space Over R And Let V^* Denote Its Dual Space. Then V^* Is The Space Whose Elements Are Linear Functions From V To R, We Shall Call Them Convectors. If $\sigma\in V^*$ Then $\sigma: V\to R$ For The Any $v\in V$, We Denote The Value Of σ On v By $\sigma(v)$ Or By $\langle v,\sigma\rangle$. Addition And Multiplication By Scalar In V^* Are Defined By The Equations $(\sigma_1+\sigma_2)\ (v) = \ \sigma_1(v) + \sigma_2(v)$, $(\alpha\sigma)\ (v) = \alpha\ \ (\sigma\ \ (v)\ )$. Where $v\in V\ \ ,\sigma,\alpha\sigma\ \in V^*$ And $\alpha\in R$.

Proposition 2.4.1 : Convectors

Let V Be A Finite- Dimensional Vector Space. If $(E_1,..., E_n)$ Is Any Basis For V ,Then The Convectors $(\omega^1,...,\omega^n)$ Defined By:

(8) $$\omega^i(E_j) = \delta^i_j = \begin{cases}1 & if \quad i = j \\ 0 & if \quad i\neq j\end{cases}$$

Form A Basis For V^* ,Called The Dual Basis To (E_j) .Therefore, $\dim V^* = \dim V$.

Definition 2.4.2 Convectors On Manifolds

$AC^{r}-$Convector Field σ On M , $r\geq 0$, Is A Function Which Assigns To Each $\beta\in M$ A Convector $\sigma_{p}\in T_{P}^{*}(M)$ In Such A Manner That For Any Coordinate Neighborhood (U,ϕ) With Coordinate Frames $E_{1},..,E_{n}$, The Functions $\sigma(E_{i})$, $i=1,.....,n$, Are Of Class C^{r} On U . For Convenience, "Convector Field" Will Mean $C^{\infty}-$Convector Field.

Remark 2.4.3

It Is Important To Note That A $C^{r}-$ Convector Field σ Defines A Map $\sigma:\mathcal{H}(M)\rightarrow C^{r}(M)$, Which Is Not Only $R-$ Linear But Even $C^{r}(M)-$ Linear, More Precisely, If $f,g\in C^{r}(M)$ Any X , Y Are Vector Fields On M , Then $\sigma\ (fX+gY)=f\sigma\ (X)+g\sigma\ (Y)$. For These Functions Are Equal At Each $p\in M$.

Definition 2.4.4: Tensors Vector Spaces

We Now Proceed To Define Tensors. Let $k\in N$ Given A Vector Space $(V_{1},.....,V_{k})$ One Can Define A Vector Space $(V_{1}\otimes.....\otimes V_{k})$ Called Their Tensor Product. The Element Of This Vector Space Are Called Tensors With The Situation Where The Vector Space $V_{1},.....,V_{k}$ Are All Equal To The Same Space. In Fact The Tensor Space $T^{k}V$ We Define Below Corresponds To $(V^{*}{}_{1}\otimes.....\otimes V^{*}{}_{k})$ In The General Notation. And We Define $V^{k}=(V\times....\times V)$ Be The Cartesian Product Of k Copies Of V .A Map φ From V^{k} To A Vector Space U Is Called Multiline If In Each Variable Separately I.E. (With The Other Variables Held Fixed) .

Definition 2.4.5

Let $V^{K}=(V\times.....\times V)$ Be The Cartesian Product Of k Copies Of V . A Map φ From V^{k} To A Vector Space U Is Called Multiline If It Is Linear In Each Variable Separately (I.E. With The Other Variables Held Fixed)

Definition 2.4.6

A (Covariant) K-Tensor On V Is A Multiline Map $T:V^{k}\rightarrow R$. The Set Of K-Tensors On V Is Denoted $T^{k}(V)$. In Particular, A 1-Tensor Is A Linear Form, $T^{1}(V)=V^{*}$. It Is Convenient To Add The Convention That $T^{0}(V)=R$. The Set $T^{k}(V)$ Is Called Tensor Space, It Is A Vector Space Because Sums And Scalar Products Of Multiline Maps Are Again Multiline.

2.5Alternating Tensors

Let V Be A Real Vector Space. In The Preceding Section The Tensor Spaces $T^{k}V$ Were Defined , Together With The Tensor Product $(S,T)\rightarrow S\otimes T$, $T^{k}(V)\times T^{l}(V)\rightarrow T^{k+l}(V)$ There Is An Important Construction Of Vector Spaces Which Resemble Tensor Powers Of V , But For Which There Is A More Refined Structure, These Are The So-Called Exterior Powers V , Which Play An Important Role In Differential Geometry Because The Theory Of Differential Forms Is Built On Them. They Are Also Of Importance In Algebraic Topology And Many Other Fields. A Multiline Map $\varphi:V^{k}=V\times....\times V\rightarrow U$ Where $k\geq 1$ Is Said To Be Alternating If For All $v_{1},......,v_{k}$ Are Inter-Changed That Is $\varphi(v_{1},......,v_{i},....,v_{k})=-\varphi(v_{i},.....,v_{j},.....,v_{i},....,v_{k})$ Since Every Permutation Of Numbers $1,......,k$ Can Be Decomposed Into Transpositions, It Follows That $\varphi(v_{\sigma 1},....,v_{\sigma k})=\text{sgn}\ \varphi(v_{1},.....,v_{k})$ For All Permutations $\sigma\in S_{k}$ Of The Numbers $(1,.....,k)$.For Example Let $V=R^{3}$ The Vector Product $(v_{1},v_{1})\rightarrow v_{1}\times v_{2}\in V$ Is Alternating For $(V\times V)\rightarrow V$ .And Let $V=R$ The $(n\times n)$ Determinant Is Multiyear And Alternating In Its Columns, Hence It Can Be Viewed As An Alternating Map $(R^{n})^{n}\rightarrow R$.

Definition 2.5.1

An Alternating K-Form Is An Alternating K-Tensor $V^{k}\rightarrow R$ The Space Of These Is Denoted $A^{k}(V)$, It Is A Linear Subspace Of $T^{k}(V)$

Theorem 2.5.2

Assume Dim $[V]=n$ With $(e_{1},....,e_{n})$ A Basis. Let $(\zeta_{1},....,\zeta_{n})\in V^{*}$ Denote The Dual Basis . The Elements $(\zeta_{1}\otimes....\otimes\zeta_{i,k})$ Where $I=(i_{1},....,i_{k})$ Is An Arbitrary Sequence Of K Numbers In $\{1,....,n\}$,Form A Basis For $T^{k}(V)$.

Proof:

Let $T_{1}=(\zeta_{i1}\otimes......\otimes\zeta_{ik})$. Notice That If $J=(j_{1},.....,j_{k})$ Is Another Sequence Of K Integers, And We Denote By e_{j} The Element $(e_{j1},....,e_{jk})\in V^{k}$ Then $T_{I}(e_{j})=\delta_{Ij}$ That Is $T_{I}(e_{j})=1$ If $J=I$ And 0 Otherwise. If Follows That They T_{I} Are Linearly Independent, For If A Liner Combination $T=\left(\sum_{I}a_{I}T_{I}\right)$ Is Zero, Then $a_{j}=T(e_{j})=0$. It Follows From The Multilinearity That A K-Tensor Is Uniquely Determined By Its Values

On All Elements In V^k Of The Form e_j. For Any Given K-Tensor T We Have That The K-Tensor $\sum_I T(e_I)T_I$ Agrees With T On All e_j Hence $\sum_I T(e_I)T_I$ And We Conclude That The T_I Span $T^K(V)$.

2.6 The Wedge Product

In Analogy With The Tensor Product $(S,T) \to S \otimes T$ Form $T^k(V) \times T^l(V) \to T^{k+1}(V)$, There Is A Construction Of A Product $A^k(V) \times A^l(V) \to A^{k+1}$ Since Tensor Products Of Alternating Tensors Are Not Alternating, It Does Not Suffice Just To Take $S \otimes T$.

Definition 2.6.1

Let $S \in A^k(V)$ And $T \in A^l(V)$. The Wedge Product $(S \wedge T) \in A^{k+1}(V)$ Is Defined By $(S \wedge T) = ALt\ (S \otimes T)$.Notice That In The Case $k = 0$,Where $A^k(V) = R$, The Wedge Product Is Just Scalar Multiplication.

Example 2.6.2

Let $(\eta_1, \eta_2) \in A^1(V) = V^*$ Then By Definition $(\eta_1 \wedge \eta_2) = 1/2(\eta_1 \otimes \eta_2 - \eta_2 \otimes \eta_1)$ Since The Operator. Alt Is Linear The Wedge Product Depends Linearly On The Factors S And T. It Is More Cumbersome To Verify The Associative Rule For $\wedge$. In Order To Do This We Need The Following.

Lemma 2.6.3

Let $R \in A^k(V)$, $S \in A^l(V)$ And $T \in A^m(V)$ Then $(R \wedge S) \wedge T = R \wedge (S \wedge T) = Alt\ (R \otimes S \otimes T)$

(9) $$R \wedge (S \wedge T) = Alt\ (R \otimes Alt\ (S \otimes T) = Alt\ (R \otimes S \otimes T)$$

The Wedge Product Is Associative, We Can Write Any Product $(T_1 \wedge ..... \wedge T_r)$ Of Tensor $T_i \in A^{k_i}(V)$ Without Specifying Brackets. In Fact It Follows By Induction From That $(T_1 \wedge ..... \wedge T_r) = Alt\ (T_1 \otimes ...... \otimes T_r)$ Regardless Of How Brackets Are Inserted In The Wedge Product In Particular, It Follows From $(\eta_1 \wedge ..... \wedge \eta_k)(v_1, ..., v_k) = \frac{1}{k!} \det\ [\eta_i(v_j)_{i,j}]$ For All $(v_1, ..., v_k) \in V$ And $(\eta_1, ..., \eta_2) \in V^*$ Are Viewed As 1-Forms, The Basic Elements ζ_I Are Written In This Fashion As $\zeta_I = (\zeta_{i1} \wedge ..... \wedge \zeta_{ik})$ Where $I = (i_1, ..., i_k)$ Is An Increasing Sequence Form $(1, ..., n)$ This Will Be Our Notation For ζ_I From Now On. The Wedge Product Is Not Commutative. Instead, It Satisfies The Following Relation For Interchange Of Factors. In This Defined A Tensor ϕ On V Is By Definition A Multiline V^* Denoting The Dual Space To V, r Its Covariant Order And s Its Contra Variant Order , Assume $(r \geq 0) or\ (s \geq 0)$ Thus ϕ Assigns To Each R-Tape Of Elements Of V And s Tupelo Of Elements Of V^* A Real Number And If For Each k, $(1 \leq k) \leq (r+s)$ We Hold Every Variable Except The ϕ Fixed The $(k-th)$ Satisfies The Linearity Condition

(10) $$\phi\ (v_1, ..., \alpha\ v_k + \alpha' v_k', ..) = (v_1, ..., v_k) + \alpha' \phi(v_1', ..., v_k)$$

For All $(\alpha, \alpha') \in R$ And $v_k, v_k' \in V$ Or V Respectively For A Fixed r, s We Let $f_s^r(V)$ Be The Collection Of All Tensors On V Of Covariant Order s And Contra Variant Order r, We Know That As A Function From ($V \times .... \times V \times V^* \times ... \times V$) To Order R They May Be Added And Multiplied By Scalars Elements R With This Addition And Scalar Multiplication $f_s^r(V)$ Is A Vector Space So That If $(\phi_1, \phi_2) \in f_s^r(V)$ And $(\alpha_1, \alpha_2) \in R$ Then $(\alpha_1\phi_1 + \alpha_2\phi_2)$ Defined In The Way Alluded To Above That Is By.

(11) $$(\alpha_1\phi_1 + \alpha_2\phi_2)\quad (v_1, v_2, ...) = \alpha_1\phi_1\ (v_1, v_2, ...) + \alpha_2\ \phi_2\ (v_1, v_2, ...)$$

Is Multiline And Therefore In $f_s^r(V)$ This $f_s^r(V)$ Has A Natural Vector Space Structure. In Properties Come Naturally Interims Of The Metric Defined Those Spaces Are Known Interims Differential Geometry As Riemannian Manifolds A Convector Tensor On A Vector V Is Simply A Real Valued $\varphi(v_1, v_2, ..., v_r)$ Of Several Vector Variables $(v_1, ..., v_r)$ Of V The Multiline Number Of Variables Is Called The Order Of The Tensor , A Tensor Field φ Of Order r On Linear In Each On A Manifold M Is An Assignment To Each Point $p \in M$ Of Tensor φ_p On The Vector Space T_pM Which Satisfies A Suitable Regularity Condition C^0, C^∞ Or C^r As P On M.

Definition2.6.4

With The Natural Definitions Of Addition And Multiplication By Elements Of R The Set $f_s^r(V)$ Of All Tensors Of Order r, s On V Forms A Vector Space Of Dimension n^{r+s}.

Definition2.6.5

We Shall Say That $\phi \in f_s^r(V)$, V A Vector Space Is Symmetric If For Each $1 \leq i, j \leq r$,We $\phi(v_1, v_2, ..., v_j, ..., v_i, ..., v_r)$ Similarly If Interchanging The $i-th$ And $j-th$ Variables $1 \leq i, j \leq r$ Changes The

Sign, $-\phi(v_1, v_2, ..., v_j, ..., v_i, ..., v_r)$ Then We Say ϕ Is Skew Or Anti Symmetric Or Alternating Covariant Tensors Are Often Called Exterior Forms, A Tensor Field Is Symmetric Respective Alternating If It Has This Property At Each Point.

Theorem 2.6.8

The Product $(f^r(V) \times f^s(V)) \to (f^{r+s}(V))$ Just Defined Is Bilinear Associative If $(w^1, ..., w^n)$ Is Abasis1 $V^* = f^1(V)$ Then $(w^{i(1)} \otimes, ..., \otimes w^{i(r)})$ And $1 \le (i_1, ..., i_r) \le n$ Is A Basis Of $f^r(V)$ Finally $F^* : W \to V$ Is Linear, Then

Proof:

Each Statement Is Proved By Straightforward Computation To Say That Bilinear Means That α, β Are Numbers $(\phi_1, \phi_2) \in f^r(V)$ And $\psi \in f^r(V)$ Then $(\alpha\phi_1 + \beta\phi_2) \otimes \psi = \alpha(\phi_1 \otimes \psi) + \beta(\phi_2 \otimes \psi)$ Similarly For The Second Variable This Is Checked By Evaluating Side On $r+s$ Vectors Of V In Fact Basis Vectors Suffice Because Of Linearity Associatively Is Similarly $(\phi \otimes \psi) \otimes \varphi = \varphi(\psi \otimes \phi)$, The Defined In Natural Way This Allows Us To Drop The Parentheses To Both $(w^{i(1)} \otimes, ..., \otimes w^{i(r)})$ From A Basis It Is Sufficient To Note That If $(e_1, ..., e_n)$ Is The Basis Of V Dual To $(w^1 \otimes .... \otimes w^n)$ Then The Tensor Previously $\Omega^{(i1,...,ir)}$ Defined Is Exactly $(w^{i(1)} \otimes, ..., \otimes w^{i(r)})$ This Follows From The Two Definitions.

(12) $$\Omega^{(i1,...,ir)}(e_{j(1)}, ..., e_{j(r)}) = \begin{cases} 0 & if\ (i_1, ..., i_r) \neq (j_1, ..., j_r) \\ 1 & if\ (i_1, ..., i_r) = (j_1, ..., j_r) \end{cases}$$

(13) $$(w^{i(1)} \otimes, ..., \otimes w^{i(r)})[(e_{j(1)}, ..., e_{j(r)})] = (w^{i(1)}(e_{j(1)})\, w^{i(2)}(e_{j(2)}), .., w^{i(r)}) = [\delta^{i(1)}_{j(1)}, .., \delta^{i(r)}_{j(r)}]$$

Which Show That Both Tensors Have The Same Values On Any Order Set Of r Basis Vectors And Are Thus Equal Finally Given $F^* : W \to V$ If $(w_1, ..., w_{r+s})$ Then

(14) $$\begin{Bmatrix} F^*(\varphi \otimes \psi)(w_1, ..., w_{r+s}) = \varphi \otimes \psi\,(F^*(w_1), ..., F^*(w_{r+s})) \\ = \varphi(F^*(w_1), ..., F^*(w_r))\, \psi\,(F^*(w_1), ..., F^*(w_{r+s})) \\ = (F^*\varphi) \otimes (F^*\psi)(w_1, ..., w_{r+s}) \end{Bmatrix}$$

Which Proves $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$ And Completes Tensor Field.

Remark 2.6.9

The Rule For Differentiating The Wedge Product Of A P-Form α_p And Q-Form β_q Is

(2.8) $$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$$

Definition 2.6.10

Let $f : M \to N$ Be A C^∞ Map Of C^∞ Manifolds, Then Each C^∞ Covariant Tensor Field φ On N Determines A C^∞ Covariant Tensor Field $F^*\varphi$ On M By The Formula $(F^*\varphi)_p(X_{1p}, ..., X_{rP}) = \varphi_{F(p)}(F^*X_{1p}, ..., F^*X_{rp})$ The Map $F^* : f^r(N) \to f^r(M)$ So Defined Is Linear And Takes Symmetry Alternating Tensor To Symmetric Alternating Tensors.

Lemma 2.6.11

Let $\Omega \neq 0$ Be An Alternating Covariant Tensor V Of Order N=Dim. V And Let $(e_1, ..., e_n)$ Be A Basis Of V Then For Any Set Of Vectors $(v_1, ..., v_n)$ With $v_i = \sum \alpha_i^j e_j$ We Have, $\Omega(v_1, ..., v_n) = \det |\alpha_i^j|$.

Example 2.6.12

(I) Possible P-Forms α_p In Two Dimensional Space Are.

(15) $$\begin{Bmatrix} \alpha_0 = f(x, y) \\ \alpha_1 = u(x, y)\,dx + v(x, y)\,dy \\ \alpha_2 = \phi(x, y)\,dx \wedge dy \end{Bmatrix}$$

The Exterior Derivative Of Line Element Givens The Two Dimensional Curl Times The Area $d\,[u\,(x, y)\,dx + v(x, y)\,dy\,] = (\partial_x v - \partial_y u)\,dx \wedge dy$.

(Ii) The Three Space P-Forms α_p Are.

(16) $$\begin{Bmatrix} \alpha_0 = f(x) \\ \alpha_1 = v_1\,dx^1 + v_2\,dx^2 + v_3\,dx^3 \\ \alpha_2 = w_1\,dx^2 \wedge dx^3 + w_2\,dx^3 \wedge dx^1 + w_3\,dx^1 \wedge dx^2 \\ \alpha_3 = \varphi(x)\,dx^1 \wedge dx^2 \wedge dx^3 \end{Bmatrix}$$

We See That

(17) $$\begin{Bmatrix} d\alpha_1 = (\varepsilon_{ijk}\partial_j v_k)\frac{1}{2}\varepsilon_{ijm} dx^1 \wedge dx^m \\ d\alpha_2 = (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) dx^1 \wedge dx^1 \wedge dx^3 \end{Bmatrix}$$

Where ε_{ijk} Is The Totally Anti-Symmetric Tensor In 3-Dimensions.The Isomorphism Vectors Tensor Field We Saw In The Equation $\tilde{V} \equiv g(\vec{V},\cdot) \equiv g(\cdot,\vec{V})$ And $\vec{V} \equiv g^{-1}(\tilde{V},\cdot) \equiv g^{-1}(\cdot,\tilde{V})$ The Link Between The Vector And Dual Vector Spaces Is Provided By g And g^{-1} If $\vec{A} = \vec{B}$ Components $A^{\mu} = B^{\mu}$ Then $\tilde{A} = \tilde{B}$ Components $B_{\mu} = g_{\mu\nu} B^{\nu}$ So Where $A_{\mu} \equiv g_{\mu\nu} A^{\nu}$ And $B_{\mu} \equiv g_{\mu\nu} B^{\nu}$ So Why Do We Bother One-Forms When Vector Are Sufficient The Answer Is That Tensors May By Function Of Both One-Form And Vectors , There Is Also An Isomorphism A Mongo Tensors Of Different Rank , We Have Just Argued That The Tensor Space Of Rank (1.0) Vectors And (0.1) Are Isomorphic , In Fact All $2^{(m+n)}$ Tensor Space Of Rank $(m+n)$ With Fixed $(m+n)$ Are Isomorphic, The Metric Tensor Like Together These Spaces As Exempla Field By Equation $T_{\mu\nu}{}^{\lambda} \equiv g(\vec{e}_{\mu}, T^{k}{}_{\nu\lambda}\vec{e}_k)$ We Could Now Use The Inverse Metric

(18) $$T_{\mu\nu}^{\lambda} \equiv g^{-1}(\vec{e}^{\lambda}, T_{\mu\nu k}\vec{e}^{k}) \quad g^{\lambda k} T_{\mu\nu k} \equiv g^{\lambda k} g_{\mu p} T_{\nu k}^{p}$$

The Isomorphism Of Different Tensor Space Allows Us To Introduce A Notation That Unifies Them, We Could Affect Such A Unification By Discarding Basis Vectors And One-Forms Only With Components, In General Isomorphism Tensor Vector $\overline{A}$ Defined By.

(19) $$\overline{A} = \overline{A}_{\mu}\vec{e}^{-\mu} = \overline{A}_{\mu} g^{\mu\nu}\vec{e}_{\nu} \equiv \overline{A}^{\mu}\vec{e}_{\mu}$$

And $\overline{A} = \overline{A}_{\mu}\overline{e}^{\mu}$ Is Invariant Under A Change Of Basis Because $\vec{e}^{\mu}$ Transforms Like A Basis One-Form.

2.7: Tensor Fields

The Introduced Definitions Allows One To Introduce The Tensor Algebra $A_R(T_pM)$ Of Tensor Spaces Obtained By Tensor Products Of Space R And (T_pM) , $(T^*{}_pM)$. Using Tensor Defined On Each Point $p \in M$ One May Define Tensor Fields.

Definition 2.7.1

Let M Be A N-Dimensional Manifold. A Differentiable Tensor Field T Is An Assignment $p \to t_p$ Where Tensors $t_p \in A_R(T_pM)$ Are Of The Same Kind And Have Differentiable Components With Respect To All The Canonical Bases Of $A_R(T_pM)$ Given By Product Of Bases $\left\{\frac{\partial}{\partial x^K}\,|_p\right\} k = 1,\ldots,n \subset T_pM$ And $dx^k_p\, k = (1,\ldots,n) \subset T_p^*M$ Induced By All Of Local Coordinate System M .In Particular A Differentiable Vector Field And A Differentiable 1-Form (Equivalently Called Coveter Field) Are Assignments Of Tangent Vectors And 1-Forms Respectively As Stated Above.

For Tensor Fields The Same Terminology Referred To Tensor Is Used .For Instance, A Tensor Field t Which Is Represented In Local Coordinates By $t^i_j(p)\frac{\partial}{\partial x^i}\Big|_p \otimes (dx^j)\,|_p$ Is Said To Of Order (1,1) .It Is Clear That To Assign On A Differentiable Manifold M A Differentiable Tensor Field T (Of Any Kind And Order) It Necessary And Sufficient To Assign A Set Of Differentiable Functions . $(x^1,\ldots, x^n) \to T^{i1,\ldots,im}{}_{j1,\ldots,jk}(x^1,\ldots,x^n)$. In Every Local Coordinate Patch (Of The Whole Differentiable Structure M Or, More Simply, Of An Atlas Of M) Such That They Satisfy The Usual Rule Of Transformation Of Comports Of Tensors Of Tensors If $(x^1,\ldots,x^n)$ And $(y^1,\ldots,y^n)$ Are The Coordinates Of The Same Point $p \in M$ In Two Different Local Charts.

(20) $$T^{i1,\ldots,im}{}_{j1,\ldots,jk}\left(\frac{\partial}{\partial x^{i1}}\bigg|\right)_p \otimes \ldots \otimes \left(\frac{\partial}{\partial x^{im}}\right)\bigg|_p \otimes (dx^{j1})\,|_p \otimes \ldots \otimes (dx^{jk})\,|_p$$

(I) It Is Obvious That The Differentiability Requirement Of The Comports Of A Tensor Field Can Be Choked Using The Bases Induced By A Single Atlas Of Local Charts. It Is Not Necessary To Consider All The Charts Of The Differentiable Structure Of The Manifold.

(Ii) If X Is A Differentiable Vector Field On A Differentiable Manifold, M Defines A Derivation At Each Point $p \in M : if\ f \in D(M)$, $X_p(f) = X^i(p)\frac{\partial}{\partial x^i}\Big|_p$ Where $(x^1,\ldots,x^n)$ Are Coordinates Defined About p . More Generally Every Differentiable Vector Field X Defines A Linear Mapping From $D(M)$ To $D(M)$ Given By $f \to X(f)$ For Everywhere $X(f) \in D(M)$ Is Defined As $X(f)(P) = X_p(f)$ For Every $p \in M$.(Iii) For (Contra Variant) Vector Field X On A Differentiable Manifold M , A Requirement Equivalent To The Differentiability

Is The Following The Function $X(f):P \to X_p(f)$, (Where We Use X_p As A Derivation) Is Differentiable For All Of $f \in D(M)$. Indeed It X Is A Differentiable Contra Variant Vector Field And If $f \in D(M)$, One Has That $X(f):P \to X_p(f)$ Is A Differentiable Function Too As Having A Coordinate Representation.

(21) $$[X(f) \circ \phi^{-1}]: \phi(U) \in (x^1,\dots,x^n) \to X^i(x^1,\dots,x^n)\frac{\partial f}{\partial x^i}\Big|_{(x^1,\dots,x^n)}$$

In Every Local Coordinate Chart (U,ϕ) And All The Involved Function Being Differentiable. Conversely $p \to X_p(f)$ Defines A Function In $D(M)$, $X(f)$ For Every $f \in D(M)$ The Components Of $p \to X_p(f)$ In Every Local Chart (U,ϕ) Must Be Differentiable. This Is Because In A Neighborhood Of $q \in U$, $X^i(q) = X(f^{(1)})$.Where The Function $f^{(1)} \in D(M)$ Vanishes Outside U And Is Defined As $r \to x^i(r)$, $h(r)$ In U Where x^i Is The Its Component Of ϕ (The Coordinate x^i) And h A Hat Function Centered On q With Support In U. Similarly The Differentiability Of A Covariant Vector Field w Is Equivalent To The Differentiability Of Each Function $p \to \langle X_p . w_p \rangle$ For All Differentiable Vector Fields X .(Iv) If $f \in D(M)$ The Differential Of f In p, df_p Is The 1-Form Defined By $df_p = \frac{\partial f}{\partial x^i}\Big|_p (dx^i)\,|_p$ In Local Coordinates About p. The Definition Does Not Depend On The Chosen Coordinates .As A Consequence, The Point $p \in M$, $p \to df_p$ Defines A Covariant Differentiable Vector Field Denoted By df And Called The Differential Of f. (V) The Set Of Contra Variant Differentiable Vector Fields On Any Differentiable Manifold M Defines A Vector Space With Field Given By R Is Replaced By $D(M)$, The Obtained Algebraic Structure Is Not A Vector Space Because $D(M)$ Is A Commutative Ring With Multiplicative And Addictive Unit Elements But Fails To Be A Field. However The Incoming Algebraic Structure Given By A Vector Space With The Field Replaced By A Commutative Ring With Multiplicative And Addictive Unit Elements Is Well Known And It Is Called Module.

A Sub Manifolds Of Others Of R^n For Instance S^2 Is Sub Manifolds Of R^3 It Can Be Obtained As The Image Of Map Into R^3 Or As The Level Set Of Function With Domain R^3 We Shall Examine Both Methods Below First To Develop The Basic Concepts Of The Theory Of Riemannian Sub Manifolds And Then To Use These Concepts To Derive A Equantitive Interpretation Of Curvature Tensor , Some Basic Definitions And Terminology Concerning Sub Manifolds, We Define A Tensor Field Called The Second Fundamental Form Which Measures The Way A Sub Manifold Curves With The Ambient Manifold , For Example X Be A Sub Manifold Of Y Of $\pi : E \to X$ And $g : E_1 \to Y$ Be Two Vector Brindled And Assume That E Is Compressible , Let $f : E \to Y$ And $g : E_1 \to Y$ Be Two Tubular Neighborhoods Of X In Y Then There Exists A C^{p-1}.

2.8 : Differentiable Manifolds And Tangent Space

In This Section Is Defined Tangent Space To Level Surface γ Be A Curve Is In R^n, $\gamma : t \to (\gamma^1(t), \gamma^2(t),\dots, \gamma^n(t))$ A Curve Can Be Described As Vector Valued Function Converse A Vector Valued Function Given Curve , The Tangent Line At The Point $\frac{d\gamma}{dt}(t) = \left(\frac{d\gamma^1}{dt}t_0,\dots,\frac{d\gamma^n}{dt}t_0\right)$ We Many k Bout Smooth Curves That Is Curves With All Continuous Higher Derivatives Cons The Level Surface $f\ (x^1, x^2,\dots, x^n) = c$ Of A Differentiable Function f Where x^i To $i-th$ Coordinate The Gradient Vector Of f At Point $P = (x^1(P), x^2(P),\dots, x^n(P))$ Is $\nabla f = \left(\frac{\partial f}{\partial x^1},\dots,\frac{\partial f}{\partial x^n}\right)$ Is Given A Vector $u = (u^1,\dots,u^n)$ The Direction Derivative $D_u f = \nabla f \cdot \bar{u} = \left(\frac{\partial f}{\partial x^1}u^1 + \dots + \frac{\partial f}{\partial x^n}u^n\right)$, The Point P On Level Surface $f\ (x^1, x^2,\dots, x^n)$ The Tangent Is Given By Equation $\frac{\partial f}{\partial x^1}(P)\,[(x^1 - x^1)\,](P) + \dots + \frac{\partial f}{\partial x^n}(P)\,[(x^n - x^n)\,](P) = 0$. For The Geometric Views The Tangent Space Shout Consist Of All Tangent To Smooth Curves The Point P, Assume That Is Curve Through $t = t_0$ Is The Level Surface $f(x^1, x^2,\dots, x^n) = c$ That Is $f(\gamma^1(t), \gamma^2(t),\dots, \gamma^n(t)) = c$ By Taking Derivatives On Both $\left(\frac{\partial f}{\partial x^1}(P)(\gamma'(t_0)\right) + \dots + \left(\frac{\partial f}{\partial x^n}(P)\gamma^n(t))\right) = 0$ And So The Tangent Line Of γ Is Really Normal Orthogonal To

Where γ Runs Over All Possible Curves On The Level Surface Through The Point P .The Surface M Be A C^∞ Manifold Of Dimension n With $k \geq 1$ The Most Intuitive To Define Tangent Vectors Is To Use Curves , $p \in M$ Be Any Point On M And Let $\gamma :]-\varepsilon,\varepsilon[\to M$ Be A C^1 Curve Passing Through p That Is With $\gamma(M) = p$ Unfortunately It M Is Not Embedded In Any R^N The Derivative $\gamma'(M)$ Does Not Make Sense

,However For Any Chart (U,φ) At p The Map $(\varphi\circ\gamma)$ At A C^1 Curve In R^n And Tangent Vector $v=(\varphi\circ v)'(M)$ Is Will Defined The Trouble Is That Different Curves The Same v Given A Smooth Mapping $f:N\to M$ We Can Define How Tangent Vectors In T_pN Are Mapped To Tangent Vectors In T_qM With (U,φ) Choose Charts $q=f(p)$ For $p\in N$ And (V,ψ) For $q\in M$ We Define The Tangent Map Or Flash-Forward Of f As A Given Tangent Vector $X_p=[\gamma]\in T_pN$ And $d\,f_*:T_pM$, $f_*([\gamma]=[f\circ\gamma])$. A Tangent Vector At A Point p In A Manifold M Is A Derivation At p , Just As For R^n The Tangent At Point p Form A Vector Space $T_p(M)$ Called The Tangent Space Of M At p, We Also Write $T_p(M)$ A Differential Of Map $f:N\to M$ Be A C^∞ Map Between Two Manifolds At Each Point $p\in N$ The Map F Induce A Linear Map Of Tangent Space Called Its Differential At p, $F_*:T_pN\to T_{F(p)}N$ As Follows It $X_p\in T_pN$ Then Is The Tangent Vector In $T_{F(p)}M$ Defined $(F_*(X_p))f=X_p(f\circ F)\in R$, $f\in C^\infty(M)$. The Tangent Vectors Given Any C^∞ -Manifold M Of Dimension n With For Any $p\in M$,Tangent Vector To M At p Is Any Equivalence Class Of C^1 -Curves Through p On M Modulo The Equivalence Relation Defined In The Set Of All Tangent Vectors At p Is Denoted By T_pM We Will Show That Is A Vector Space Of Dimension n Of M .The Tangent Space T_pM Is Defined As The Vector Space Spanned By The Tangents At p To All Curves Passing Through Point p In The Manifold M , And The Cotangent T_p^*M Of A Manifold At $p\in M$ Is Defined As The Dual Vector Space To The Tangent Space T_pM , We Take The Basis Vectors $E_i=\left(\frac{\partial}{\partial x^i}\right)$ For T_pM And We Write The Basis Vectors T_p^*M As The Differential Line Elements $e^i=dx^i$ Thus The Inner Product Is Given By $\langle\partial/\partial x,dx^i\rangle=\delta_i^j$.

2.8. : Definition

Let M_1 And M_2 Be Differentiable Manifolds A Mapping $\varphi:(M_1\to M_2)$ Is A Differentiable If It Is Differentiable , Objective And Its Inverse φ^{-1} Is Diffoemorphism If It Is Differentiable φ Is Said To Be A Local Diffoemorphism At $p\in M$ If There Exist Neighborhoods U Of p And V Of $\varphi(p)$ Such That $\varphi:(U\to V)$ Is A Diffoemorphism , The Notion Of Diffeomorphism Is The Natural Idea Of Equivalence Between Differentiable Manifolds , Its An Immediate Consequence Of The Chain Rule That If $\varphi:(M_1\to M_2)$ Is A Diffoemorphism Then $d\varphi:(T_pM)_1\to(T_{\varphi(p)}M_2)$. Is An Isomorphism For All $\varphi:(M_1\to M_2)$ In Particular , The Dimensions Of M_1 And M_2 Are Equal A Local Converse To This Fact Is The Following $d\varphi:T_pM_1\to T_{\varphi(p)}M_2$ Is An Isomorphism Then φ Is A Local Diffoemorphism At p From An Immediate Application Of Inverse Function In R^n, For Example Be Given A Manifold Structure Again A Mapping $f^{-1}:M\to N$ In This Case The Manifolds N And M Are Said To Be Homeomorphism , Using Charts (U,φ) And (V,ψ) For N And M Respectively We Can Give A Coordinate Expression $\tilde{f}:M\to N$

Definition 2.8.2

Let (M_1^{-1}) And (M_2^{-1}) Be Differentiable Manifolds And Let $\varphi:(M_1\to M_2)$ Be Differentiable Mapping For Every $p\in M_1$ And For Each $v\in(T_pM_1)$ Choose A Differentiable Curve $\alpha:(-\varepsilon,\varepsilon)\to M_1$ With $\alpha(M)=p$ And $\alpha'(0)=v$ Take $(\alpha\circ\beta)=\beta$ The Mapping $d\varphi_p:T_\varphi(p)M_2$ By Given By $d\varphi(v)=\beta'(M)$ Is Line Of α And $\varphi:(M_1^{-1})\to(M_2^{-1})$ Be A Differentiable Mapping And At $p\in M_1$ Be Such $d\varphi:(T_pM_1)\to(T_\varphi M_2)$ Is An Isomorphism Then φ Is A Local Homeomorphism

Theorem 2.8.3

Let G Be Lie Group Of Matrices And Suppose That Log Defines A Coordinate Chart The Near The Identity Element Of G , Identify The Tangent Space $g=(T_1G)$ At The Identity Element With A Linear Subspace Of Matrices , Via The Log And Then A Lie Algebra With $[B_1,B_2]=B_1B_2-B_2B_1$ The Space g Is Called The Lie Algebra Of G .

Proof:

It Suffices To Show That For Every Two Matrices $B_1,B_2\in g$ The $[B_1,B_2]$ Is Also An Element Of g As $[B_1,B_2]$ Is Clearly Anti Commutative And The (Jacobs Identity) Holds For $A(t)=(B_1t)_{\exp}(B_2t)_{\exp}(-B_1t)_{\exp}(-B_2t)_{\exp}$. Define For $[t]\le\varepsilon$ With Sufficiently Small ε A Path $A(T)$ In G Such That $A(O)=I$ Using For Each Factor The Local Formula

(22) $$(Bt)_{\exp}=I+Bt+1/2B^2t^2+O(t^2)\quad A(t)=I+[B_1,B_2]\;t^2+O(t),t\to 0$$

Hence $B(t) = \log A(t) = [B_1, B_2]t^2 + O(t^2)$ Expr $B(t) = A(t)$ Hold For Any Sufficiently That Lie Bracket $[B_1, B_2] \in g$ On Algebra Is An Infinitesimal Version Of The Commutation $(g_1, g_1)(g_1^{-1}, g_2^{-1})$ In The Corresponding (Lie Group).

Theorem 2.8.4

The Tangent Bundle (TM) Has A Canonical Differentiable Structure Making It Into A Smooth 2N-Dimensional Manifold, Where N=Dim. The Charts Identify Any $U_p \in U(T_pM) \subseteq (TM)$ For An Coordinate Neighborhood $U \subseteq M$, With $(U \times R^n)$ That Is Hausdorff And Second Countable Is Called (The Manifold Of Tangent Vectors)

Definition 2.8.5

A Smooth Vectors Fields On Manifolds M Is Map $X : M \to TM$ Such That :(I) $X(P) \in T_pM$ For Every G (Ii) In Every Chart X Is Expressed As $a_i(\partial / \partial x_i)$ With Coefficients $a_i(x)$ Smooth Functions Of The Local Coordinates x_i .

III. Differentiable Manifolds Chart

In This Section, The Basically An M-Dimensional Topological Manifold Is A Topological Space M Which Is Locally Homeomorphism To R^m , Definition Is A Topological Space M Is Called An M-Dimensional (Topological Manifold) If The Following Conditions Hold: (I) M Is A Hausdorff Space.(Ii) For Any $p \in M$ There Exists A Neighborhood U Of P Which Is Homeomorphism To An Open Subset $V \subset R^m$. (Iii) M Has A Countable Basis Of Open Sets Coordinate Charts (U, φ) Axiom (Ii) Is Equivalent To Saying That $p \in M$ Has A Open Neighborhood $U \in P$ Homeomorphism To Open Disc D^m In R^m , Axiom (Iii) Says That M Can Covered By Countable Many Of Such Neighborhoods , The Coordinate Chart (U, φ) Where U Are Coordinate Neighborhoods Or Charts And φ Are Coordinate . A Homeomorphisms , Transitions Between Different Choices Of Coordinates Are Called Transitions Maps $\varphi_{ij} = \varphi_j \circ \varphi_i$, Which Are Again Homeomorphisms By Definition , We Usually Write $p = \varphi^{-1}(x), \varphi : (U \to V) \subset R^n$ As Coordinates For U , And $p = \varphi^{-1}(x), \varphi^{-1} : (V \to U) \subset M$ As Coordinates For U , The Coordinate Charts (U, φ) Are Coordinate Neighborhoods, Or Charts , And φ Are Coordinate Homeomorphisms , Transitions Between Different Choices Of Coordinates Are Called Transitions Maps $\varphi_{ij} = \varphi_j \circ \varphi_i$ Which Are Again Homeomorphisms By Definition , We Usually $x = \varphi(p), \varphi : U \to V \subset R^n$ As A Parameterization U A Collection $A = \{(\varphi_i, U_i)\}_{i \in I}$ Of Coordinate Chart With $M = \cup_i U_i$ Is Called Atlas For M .The Transition Maps φ_{ij} A Topological Space M Is Called (Hausdorff) If For Any Pair $p, q \in M$, There Exist Open Neighborhoods $p \in U$ And $q \in U'$ Such That $U \cap U' \neq \phi$ For A Topological Space M With Topology $\tau \in U$ Can Be Written As Union Of Sets In β , A Basis Is Called A Countable Basis β Is A Countable Set .

Definition 3.1.1

A Topological Space M Is Called An M-Dimensional Topological Manifold With Boundary $\partial M \subset M$ If The Following Conditions.

(I) M Is Hausdorff Space.(Ii) For Any Point $p \in M$ There Exists A Neighborhood U Of p Which Is Homeomorphism To An Open Subset $V \subset H^m$.(Iii) M Has A Countable Basis Of Open Sets, Can Be Rephrased As Follows Any Point $p \in U$ Is Contained In Neighborhood U To $D^m \cap H^m$ The Set M Is A Locally Homeomorphism To R^m Or H^m The Boundary $\partial M \subset M$ Is Subset Of M Which Consists Of Points p .

Definition 3.1.2

A Function $f : X \to Y$ Between Two Topological Spaces Is Said To Be Continuous If For Every Open Set U Of Y The Pre-Image $f^{-1}(U)$ Is Open In X .

Definition 3.1.3

Let X And Y Be Topological Spaces We Say That X And Y Are Homeomorphism If There Exist Continuous Function Such That $f \circ g = id_y$ And $g \circ f = id_x$ We Write $X \cong Y$ And Say That f And g Are Homeomorphisms Between X And Y , By The Definition A Function $f : X \to Y$ Is A Homeomorphisms If And Only If .(I) f Is A Objective .(Ii) f Is Continuous (Iii) f^{-1} Is Also Continuous.

3.2 Differentiable Manifolds

A Differentiable Manifolds Is Necessary For Extending The Methods Of Differential Calculus To Spaces More General R^n A Subset $S \subset R^3$ Is Regular Surface If For Every Point $p \in S$ The A Neighborhood

V Of P Is R^3 And Mapping $x : u \subset R^2 \to V \cap S$ Open Set $U \subset R^2$ Such That. (I) x Is Differentiable Homomorphism. (Ii) The Differentiable $(dx)_q : (R^2 \to R^3)$, The Mapping x Is Called A Aparametnzation Of S At P The Important Consequence Of Differentiable Of Regular Surface Is The Fact That The Transition Also Example Below If $x_\alpha : U_\alpha \to S^1$ And $x_\beta : U_\beta \to S^1$ Are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$, The Maps $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \to R^2$ And $x_\alpha^{-1} \circ x_\beta = x_\beta^{-1}(w) \to R$.

Are Differentiable Structure On A Set M Induces A Natural Topology On M It Suffices To $A \subset M$ To Be An Open Set In M If And Only If $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ Is An Open Set In R^n For All α It Is Easy To Verify That M And The Empty Set Are Open Sets That A Union Of Open Sets Is Again Set And That The Finite Intersection Of Open Sets Remains An Open Set. Manifold Is Necessary For The Methods Of Differential Calculus To Spaces More General Than De R^n, A Differential Structure On A Manifolds M Induces A Differential Structure On Every Open Subset Of M , In Particular Writing The Entries Of An $(n \times k)$ Matrix In Succession Identifies The Set Of All Matrices With $R^{n,k}$, An $n \times k$ Matrix Of Rank k Can Be Viewed As A K-Frame That Is Set Of k Linearly Independent Vectors In R^n, $V_{n,k} K \leq n$ Is Called The Steels Manifold ,The General Linear Group $GL(n)$ By The Foregoing $V_{n,k}$ Is Differential Structure On The Group ^{n} Of Orthogonal Matrices, We Define The Smooth Maps Function $f : M \to N$ Where M, N Are Differential Manifolds We Will Say That f Is Smooth If There Are Atlases (U_α, h_α) On M , (V_B, g_B) On N , Such That The Maps $(g_B f h_\alpha^{-1})$ Are Smooth Wherever They Are Defined f Is A Homeomorphism If Is Smooth And A Smooth Inverse. A Differentiable Structures Is Topological Is A Manifold It An Open Covering U_α Where Each Set U_α Is Homeomorphism, Via Some Homeomorphism h_α To An Open Subset Of Euclidean Space R^n, Let M Be A Topological Space , A Chart In M Consists Of An Open Subset $U \subset M$ And A Homeomorphism h Of U Onto An Open Subset Of R^m, A C^r Atlas On M Is A Collection (U_α, h_α) Of Charts Such That The U_α Cover M And (h_B, h_α^{-1}) The Differentiable .

Definition 3.2.1

Let M Be A Metric Space We Now Define What Is Meant By The Statement That M Is An N-Dimensional C^∞ Manifold. (I) A Chart On M Is A Pair (U, φ) With U An Open Subset Of M And φ A Homeomorphism A (1-1) Onto, Continuous Function With Continuous Inverse From U To An Open Subset Of R^n, Think Of φ As Assigning Coordinates To Each Point Of U . (Ii) Two Charts (U, φ) And (V, ψ) Are Said To Be Compatible If The Transition Functions.

$$\begin{Bmatrix} (\psi \circ \varphi)^{-1} : \varphi(U \cap V) \subset R^n \to \psi(U \cap V) \subset R^n \\ (\varphi \circ \psi)^{-1} : \psi(U \cap V) \subset R^n \to \varphi(U \cap V) \subset R^n \end{Bmatrix} \tag{23}$$

Are C^∞ That Is All Partial Derivatives Of All Orders Of $(\psi \circ \varphi^{-1})$ And $(\varphi \circ \psi^{-1})$ Exist And Are Continuous.(Iii) An Atlas For M Is A Family $A = \{(U_i, \varphi_i) : i \in I\}$ Of Charts On M Such That $\{U_i\}_{i \in I}$ Is An Open Cover Of M And Such That Every Pair Of Charts In A Are Compatible. The Index Set I Is Completely Arbitrary. It Could Consist Of Just A Single Index. It Could Consist Of Uncountable Many Indices. An Atlas A Is Called Maximal If Every Chart (U, φ) On M That Is Compatible With Every Chat Of A .

Example 3.2.2 : Surfaces An N-Dimensional

Any Smooth N-Dimensional R^{n+1} Is An N-Dimensional Manifold. Roughly Speaking A Subset Of R^{n+m} A An N-Dimensional Surface If , Locally m Of The $m+n$ Coordinates Of Points On The Surface Are Determined By The Other n Coordinates In A C^∞ Way , For Example , The Unit Circle S^1 Is A One Dimensional Surface In R^2 . Near (0.1) A Point $(x, y) \in R^2$ Is On S^1 If And Only If $y = \sqrt{1-x^2}$ And Near (-1.0) , (x, y) Is On S^1 If And Only If $y = -\sqrt{1-x^2}$. The Precise Definition Is That M Is An N-Dimensional Surface In R^{n+m} If M Is A Subset Of R^{n+m} With The Property That For Each $z = (z_1, \ldots, z_{n+m}) \in M$ There Are A Neighborhood U_z Of z In R^{n+m}, And n Integers. $1 \leq J_1 \leq j_2 \leq \ldots \leq j_{n+m}$ C^∞ Function $f_k(x_{j1}, \ldots, x_{jn})$, $k \in \{1, \ldots, n+m\} / \{j_1, \ldots, j_n\}$ Such That The Point $x = (x_1, \ldots, x_{n+m}) \in U_z$. That Is We May Express The Part Of M That Is Near z As $x_{i1} = f_{i1}(x_{j1}, x_{j2}, \ldots, x_{jn})$, $x_{i2} = f_{i2}(x_{j1}, x_{j2}, \ldots, x_{jn})$, $x_{im} = f_{im}(x_{j1}, x_{j2}, \ldots, x_{jn})$. Where There For Some C^∞ Function $f_1, \ldots, f_m$. We Many Use $x_{j1}, x_{j2}, \ldots, x_{jn}$ As Coordinates For R^2 In $M \cap U_z$ .Of Course An

Atlas Is With $\varphi_z(x) = (x_{j1},..., x_{jn})$ Equivalently, M Is An N-Dimensional Surface In R^{n+m} If For Each $z \in M$, There Are A Neighborhood U_z Of z In R^{n+m} , And $m\, C^{\infty}$ Functions $g_k : U_z \to R$ With The Vector $\{ \nabla_{gz}(z) \big| ,1 \le k \le m \}$ Linearly Independent Such That The Point $x \in U_z$ Is In M If And Only If $g_k(x) = 0$ For All $1 \le k \le m$ .To Get From The Implicit Equations For M Given By The g_k To The Explicit Equations For M Given By The f_k One Need Only Invoke (Possible After Renumbering Of x) . A Topological Space M Is Called An M-Dimensional Topological Manifold With Boundary $\partial M \subset M$ If The Following Conditions.(I) M Is Hausdorff Space .(Ii) For Any Point $p \in M$ There Exists A Neighborhood U Of p Which Is Homeomorphism To An Open Subset $V \subset H^m$ (Iii) M Has A Countable Basis Of Open Sets, Can Be Rephrased As Follows Any Point $p \in U$ Is Contained In Neighborhood U To $D^m \cap H^m$ The Set M Is A Locally Homeomorphism To R^m Or H^m The Boundary $\partial M \subset M$ Is Subset Of M Which Consists Of Points p .

Definition 3.2.3

Let X Be A Set A Topology U For X Is Collection Of X Satisfying :(I) ϕ And X Are In U .(Ii) The Intersection Of Two Members Of U Is In U .(Iii) The Union Of Any Number Of Members U Is In U . The Set X With U Is Called A Topological Space The Members $U \in u$ Are Called The Open Sets. Let X Be A Topological Space A Subset $N \subseteq X$ With $x \in N$ Is Called A Neighborhood Of x If There Is An Open Set U With $x \in U \subseteq N$, For Example If X A Metric Space Then The Closed Ball $D_\varepsilon(x)$ And The Open Ball $D_\varepsilon(x)$ Are Neighborhoods Of x A Subset C Is Said To Closed If $X \setminus C$ Is Open

Definition 3.2.4

A Function $f : X \to Y$ Between Two Topological Spaces Is Said To Be Continuous If For Every Open Set U Of Y The Pre-Image $f^{-1}(U)$ Is Open In X .

Definition 3.2.5

Let X And Y Be Topological Spaces We Say That X And Y Are Homeomorphisms If There Exist Continuous Function $f :(X \to Y) , g :(Y \to X)$ Such That $(f \circ g) = id_y$ And $(g \circ f) = id_X$ We Write $X \cong Y$ And Say That f And g Are Homeomorphisms Between X And Y , By The Definition A Function $f :(X \to Y)$ Is A Homeomorphisms If And Only If (I) f Is A Objective (Ii) f Is Continuous (Iii) f^{-1} Is Also Continuous.

3 .3 Differentiable Manifolds

A Differentiable Manifolds Is Necessary For Extending The Methods Of Differential Calculus To Spaces More General R^n A Subset $S \subset R^3$ Is Regular Surface If For Every Point $p \in S$ The A Neighborhood V Of P Is R^3 And Mapping $x : u \subset R^2 \to V \cap S$ Open Set $U \subset R^2$ Such That: (I) x Is Differentiable Homomorphism (Ii) The Differentiable $(dx)_q : R^2 \to R^3$, The Mapping x Is Called Aparametnzation Of S At P The Important Consequence Of Differentiable Of Regular Surface Is The Fact That The Transition Also Example Below If $x_\alpha : U_\alpha \to S^1$ And $x_\beta : U_\beta \to S^1$ Are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \ne \phi$ The Mappings

$$\left(x_\beta^{-1} \circ x_\alpha\right) : x^{-1}(w) \to R^2 , \left(x_\alpha^{-1} \circ x_\beta\right) = x_\beta^{-1}(w) \to R$$

A Differentiable Manifold Is Locally Homeomorphism To R^n The Fundamental Theorem On Existence, Uniqueness And Dependence On Initial Conditions Of Ordinary Differential Equations Which Is A Local Theorem Extends Naturally To Differentiable Manifolds. For Familiar With Differential Equations Can Assume The Statement Below Which Is All That We Need For Example X Be A Differentiable On A Differentiable Manifold M And $p \in M$ Then There Exist A Neighborhood $p \in M$ And $U_p \subset M$ An Inter $(-\delta, \delta)$, $\delta \ge 0$, And A Differentiable Mapping $\varphi :(-\delta, \delta) \times U \to M$ Such That Curve $t \to \varphi(t, q)$ And $\varphi(0, q) = q$ Acurve $\alpha :(-\delta, \delta) \to M$ Which Satisfies The Conditions $\alpha^{-1}(t) = X(\alpha(t))$ And $\alpha(0) = q$ Is Called A Trajectory Of The Field X That Passes Through q For $t = 0$. A Differentiable Manifold Of Dimension N Is A Set M And A Family Of Injective Mapping $x_\alpha \subset R^n \to M$ Of Open Sets $u_\alpha \in R^n$ Into M Such That: (I) $u_\alpha x_\alpha(u_\alpha) = M$ (Ii) For Any α, β With $x_\alpha(u_\alpha) \cap x_\beta(u_\beta)$ (Iii) The Family (u_α, x_α) Is Maximal Relative To Conditions (I),(Ii) The Pair (u_α, x_α) Or The Mapping x_α With $p \in x_\alpha(u_\alpha)$ Is Called A Parameterization , Or System Of Coordinates Of M , $u_\alpha x_\alpha(u_\alpha) = M$ The Coordinate Charts (U, φ) Where U Are Coordinate Neighborhoods Or Charts , And φ Are Coordinate Homeomorphisms Transitions Are Between Different Choices Of Coordinates Are Called Transitions Maps $\varphi_{i,j} :(\varphi_j \circ \varphi_i^{-1})$.

Which Are Anise Homeomorphisms By Definition, We Usually Write $x = \varphi(p), \varphi : U \to V \subset R^n$ Collection U And $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ For Coordinate Charts With Is $M = \cup U_i$ Called An Atlas For M Of Topological Manifolds. A Topological Manifold M For Which The Transition Maps $\varphi_{i,j} = (\varphi_j \circ \varphi_i)$ For All Pairs φ_i, φ_j In The Atlas Are Homeomorphisms Is Called A Differentiable , Or Smooth Manifold , The Transition Maps Are Mapping Between Open Subset Of R^m , Homeomorphisms Between Open Subsets Of R^m Are C^∞ Maps Whose Inverses Are Also C^∞ Maps , For Two Charts U_i And U_j The Transitions Mapping

(24) $$(\varphi_{i,j}) = (\varphi_j \circ \varphi_i^{-1}) : [\varphi_i(U_i \cap U_j)] \to [\varphi_j(U_i \cap U_j)]$$

Since $(\psi' \circ \psi^{-1})$ And $(\varphi' \circ \varphi^{-1})$ Are Homeomorphisms It Easily Follows That Which Show That Our Notion Of Rank Is Well Defined $(J\ f'')_{x'} = J(\psi' \circ \psi^{-1})_y \ \ J\ f'(\varphi' \circ \varphi^{-1})^{-1}$, If A Map Has Constant Rank For All $p \in N$ We Simply Write $rk(f)$, These Are Called Constant Rank Mapping. The Product Two Manifolds M_1 And M_2 Be Two C^k -Manifolds Of Dimension n_1 And n_2 Respectively The Topological Space $M_1 \times M_2$ Are Arbitral Unions Of Sets Of The Form $U \times V$ Where U Is Open In M_1 And V Is Open In M_2 , Can Be Given The Structure C^k Manifolds Of Dimension n_1, n_2 By Defining Charts As Follows For Any Charts M_1 On (V_j, ψ_j) , M_2 We Declare That $(U_i \times V_j\ ,\ \varphi_i \times \psi_j)$ Is Chart On $M_1 \times M_2$ Where $\varphi_i \times \psi_j : U_i \times V_j \to R^{(n_1+n_2)}$ Is Defined So That $\varphi_i \times \psi_j(p,q) = (\varphi_i(p)\ ,\ \psi_j(q))$ For All $(p,q) \in U_i \times V_j$. A Given A C^k N-Atlas, A On M For Any Other Chart (U,φ) We Say That (U,φ) Is Compatible With The Atlas A If Every Map $(\varphi_i \circ \varphi^{-1})$ And $(\varphi \circ \varphi_i^{-1})$ Is C^k Whenever $U \cap U_i \neq 0$ The Two Atlases A And $\tilde{A}$ Is Compatible If Every Chart Of One Is Compatible With Other Atlas A Sub Manifolds Of Others Of R^n For Instance S^2 Is Sub Manifolds Of R^3 It Can Be Obtained As The Image Of Map Into R^3 Or As The Level Set Of Function With Domain R^3 We Shall Examine Both Methods Below First To Develop The Basic Concepts Of The Theory Of Riemannian Sub Manifolds And Then To Use These Concepts To Derive A Equantitive Interpretation Of Curvature Tensor , Some Basic Definitions And Terminology Concerning Sub Manifolds, We Define A Tensor Field Called The Second Fundamental Form Which Measures The Way A Sub Manifold Curves With The Ambient Manifold , For Example X Be A Sub Manifold Of Y Of $\pi : E \to X$ And $g : E_1 \to Y$ Be Two Vector Brindled And Assume That E Is Compressible , Let $f : E \to Y$ And $g : E_1 \to Y$ Be Two Tubular Neighborhoods Of X In Y Then There Exists .

Theorem 3.3.1

Let $m, n \in N$ And Let $U \subset R^{n+m}$ Be An Open Set, Let $g : U \to R^m$ Be C^∞ With $g(x_0, y_0) = 0$ For Some $x_0 \in R^n\ , y_0 \in R^m$ With $(x_0, y_0) \in U$. Assume That $\det\left[\frac{\partial g_i}{\partial y_j}(x_0, y_0)\right]_{1 \le i,j \le m} \neq 0$ Then There Exist Open Sets $V \subset R^{n+m}$ And $W \subset R^n$ With $(x_0, y_0) \in V$ Such That , For Each $x \in W$ There Is A Unique $(x\ ,\ y\ ) \in V$ With $g(x\ ,\ y\ ) = 0$ If The Y Above Is Denoted $f(x_0) = y_0$ And $g\ (x,\ f(x)\ ) = 0$ For All $x \in W$ The N-Sphere S^n Is The N-Dimensional Surface R^{n+1} Given Implicitly By Equation $g(x_1, \ldots, x_{n+1}) = (x_1^2 + \ldots + x_{n+1}^2) = 0$ In A Neighborhood Of , For Example The Northern Hemisphere S^n Is Given Explicitly By The Equation $x_{n+1} = \sqrt{x_1^2 + \ldots + x_n^2}$ If You Think Of The Set Of All 3×3 Real Matrices As R^9 (Because A 3×3 Matrix Has 9 Matrix Elements) Then $.\ S\ O(3) = \{\ 3 \times 3\ real\ \ matrices\ \ \ R, R^t R = 1, \det\ R = 1\ \}$

Example 3.3.2

The Torus T^2 Is The Two Dimensional Surface $T^2 = \left\{(x,y,z) \in R^3, (\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1/4\right\}$ In R^3 In Cylindrical Coordinates $x = r\cos\theta\ ,\ y = r\sin\theta\ , z = 0$ The Equation Of The Torus Is $(r-1)^2 + z^2 = 1/4$ Fix Any $\theta\ , say\ \ \theta_0$. Recall That The Set Of All Points In R^n That Have $\theta = \theta_0$ Is An Open Book, It Is A Hall-Plane That Starts At The z Axis. The Intersection Of The Tours With That Half Plane Is Circle Of Radius 1/2 Centered On $r = 1, z = 0$ As φ Runs Form $0\ to\ 2\pi$, The Point $r = 1 + 1/2\cos\ \varphi$ And $\theta = \theta_0$ Runs Over That Circle. If We Now Run θ From $0\ to\ 2\pi$ The Point $(x,y,z) = ((1 + 1/2\cos\ \varphi)\cos\ \theta, (1 + 1/2\sin\ \varphi)$ Runs Over The Whole Torus. So We May Build Coordinate Patches For T^2 Using θ And φ With Ranges $(0, 2\pi)$ Or $(-\pi, \pi)$ As Coordinates)

Definition 3.3.3

(I) A Function f From A Manifold M To Manifold N (It Is Traditional To Omit The Atlas From The Notation) Is Said To Be C^{∞} At $m \in M$ If There Exists A Chart $\{U, \varphi\}$ For M And Chart $\{V, \psi\}$ For N Such That $m \in U$, $f(m) \in v$ And $(\psi \circ f \circ \varphi^{-1})$ Is C^{∞} At $\varphi(m)$. (Ii) Tow Manifold M And N Are Diffeomorphic If There Exists A Function $f : M \to N$ That Is (1-1) And Onto With N And f^{-1} On C^{∞} Everywhere. Then You Should Think Of M And N As The Same Manifold With m And $f(m)$ Being Two Names For Same Point, For Each $m \in M$.

IV. Conclusion

The Basic Notions On Applications Geometry Riemannian Knowledge Of Calculus Manifolds, Including The Geometric Formulation Of The Notion Of The Differential And The Inverse Function Theorem. The Differential Geometry Of Surfaces With The Basic Definition Of Differentiable Manifolds, Starting With Properties Of Covering Spaces And Of The Fundamental Group And Its Relation To Covering Spaces.

References

[1]. Osman.Mohamed M,Basic Integration On Smooth Manifolds And Application Maps With Stokes Theorem ,Http//Www.Ijsrp.Org-6-Januarly2016.

[2]. Osman.Mohamed M, Fundamental Metric Tensor Fields On Riemannian Geometry With Application To Tangent And Cotangent ,Http//Www.Ijsrp.Org- 6januarly2016.

[3]. Osman.Mohamed M, Operate Theory Riemannian Differentiable Manifolds ,Http//Www.Ijsrp.Org- 6januarly2016.

[4]. J.Glover,Z,Pop-Stojanovic, M.Rao, H.Sikic, R.Song And Z.Vondracek, Harmonic Functions Of Subordinate Killed Brownian Motion,Journal Of Functional Analysis 215(2004)399-426.

[5]. M.Dimitri, P.Patrizia , Maximum Principles For Inhomogeneous Ellipti Inequalities On Complete Riemannian Manifolds , Univ. Degli Studi Di Perugia,Via Vanvitelli 1,06129 Perugia,Italy E-Mails : Mugnai@Unipg,It , 24July2008.

[6]. Noel.J.Hicks . Differential Geometry , Van Nostrand Reinhold Company450 West By Van N.Y10001.

[7]. L.Jin,L.Zhiqin , Bounds Of Eigenvalues On Riemannian Manifolds, Higher Eduationpress And International Press Beijing-Boston ,ALM10,Pp. 241-264.

[8]. S.Robert. Strichaiz, Analysis Of The Laplacian On The Complete Riemannian Manifolds. Journal Of Functional Analysis52,48,79(1983).

[9]. P.Hariujulehto.P.Hasto, V.Latvala,O.Toivanen , The Strong Minmum Principle For Quasisuperminimizers Of Non-Standard Growth, Preprint Submitted To Elsevier , June 16,2011.-Gomez,F,Rniz Del Potal 2004 .

[10]. Cristian , D.Paual, The Classial Maximum Principles Some Of ITS Extensions And Applicationd , Series On Math.And It Applications , Number 2-2011.

[11]. H.Amann ,Maximum Principles And Principlal Eigenvalues , J.Ferrera .J.Lopez

[12]. J.Ansgar Staudingerweg 9.55099 Mainz,Germany E-Mail:Juengel@Mathematik.Uni-Mainz.De ,U.Andreas Institute Fiir Mathematic,MA6-3,TU Berlin Strabe Des 17.Juni. 136,10623 Berlin ,Germany, E-Mail: Unterreiter@Math.Tu-Berlin..

[13]. R.J.Duffin,The Maximum Principle And Biharmoni Functions, Journal Of Math. Analysis And Applications 3.399-405(1961).