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The Dynamics of a Prey-Predator Model Incorporating *Svis*-Type of Disease in Prey

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Abstract: *The present work deals with proposed and analyzed an eco-epidemiological model consisting of prey-predator system with $SVIS$ – type of disease in prey species only. The boundedness of the solution is discussed. The local and global stability of the system is carried out. The local bifurcation conditions near the predator free equilibrium point are established. Finally the numerical simulation is used to complete our global analysis of the system.*
Keywords: *prey-predator, SVIS disease, local and global stability, local bifurcation.*

I. Introduction

The study of mathematical models that combine the prey-predator systems and the spread of infectious diseases are greatly important to many of the animal populations as well as fishing operations. These types of studies are now constructing a new field of study known as eco-epidemiology. In addition, infectious diseases became an important regulating factor for human and animal population sizes. In particular, for prey-predator ecosystems, infectious diseases coupled with prey-predator interaction to produce a complex combined effect as regulators of predator and prey sizes. There are many ecological studies of prey-predator systems with disease. This factor (Disease) therefore, was invited to the attention of veterinary medicine and the provision of vaccines for these diseases. In subsequent years, many authors studied the environmental models with infected prey and the papers that relate focusing on subject [1-7]. Also, the incidence rate of the disease, predation rate and the type of disease represent a major factors affecting the dynamics of eco-epidemiology systems.

In 1986 Anderson and May were the first who merged the above two fields, ecological system and epidemiology system, they formulated a prey-predator model with infectious disease spread among prey by contact between them [8]. Haque [9] proposed a prey-predator model includes a Susceptible-Infected-Susceptible (SIS) parasitic infection in the predator population with linear functional response and nonlinear disease incidence rate. Haque and Venturino [10] considered a prey-predator model with SI epidemic disease spread in predators involving linear functional response. Das [11] studied a prey-predator model with SI epidemic disease in predators included Holling type-II as a functional response. Venturino [12] proposed and analyzed prey-predator model with SIS disease in predators included linear functional response and linear disease incidence. Haque and Venturino [13] considered a prey-predator model with SI epidemic disease spread in predators included ratio-dependent functional response and linear disease's incidence rate. Dahlia [14] studied a prey-predator model with SIS epidemic disease in prey. Ahmed and Israa [15] studied a prey-predator model with SIS epidemic disease in predator involving Holling type-II as a functional response. Hadeler and Freedman [16]; Venturino [17], have been devoted to observe the dynamics of such system when prey population is infected with some transmissible diseases. Temple [18]; and Van Dobben [19] observed that the predator take a disproportionately high number of parasite infected prey. In this paper we proposed and analyzed a mathematical model describing prey-predator model having SIS epidemic disease in the prey population involving vaccination and top predator species.

II. Model formulation

In this section an eco-epidemiological system consisting of prey-predator incorporating infections disease in prey species is proposed. In order to formulate the dynamics of such system the following hypotheses are considered.

1. The existence of disease in prey - spacies divides the prey population into three classes, namely susceptible prey population that denotes by $S(T)$, vaccinated prey population that denoted by $V(T)$ and infected prey population denoted by $I(T)$. It is assumed that in the absence of predator the susceptible prey reproduces logistically with intrinsic growth rate $r > 0$ and carrying capacity $k > 0$, while the other classes of prey have the capability to compete for resources. Further the disease is not genetically inherited.

2. The susceptible prey becomes infected either by contact with infected prey at a rate $a_1 > 0$ or due to an external resources at rate $a_2 > 0$. Further the infected prey returns back to be susceptible again at a recover rate $\beta > 0$.
3. Portion of susceptible population, say $a_3 S$; takes vaccine against the disease where $0 < a_3 < 1$ denotes to the vaccine rate. It is assumed that the vaccine may be failed with probability $\alpha \in (0,1)$ and the prey returns back to be susceptible with rate $b_1 \in (0,1)$. This is left $(1-b_1)\alpha V$ from prey individuals become infected either by contact with infected prey at a contact rate $b_2 > 0$ or through an external resources at a rate $b_3 > 0$.
4. The predator which denoted by $P(T)$ consumes the prey according to Lotka-Volttera functional response with positive attack rates a_4, b_4 and c for susceptible, vaccinated and infected prey respectively, while $e_i \in (0,1), i = 1,2,3$ are the conversion factors that denoting the number of newly born predators for each captured of susceptible, vaccinated and infected prey respectively. Finally, in the absence of the prey the predator decays exponentially with natural death rate $d > 0$.
5. According to the above hypotheses the dynamics of the above eco-epidemiological real system can be represented mathematically by the following set of nonlinear differential equations:-

a.
$$\begin{aligned}
\frac{dS}{dT} &= rS\left(1 - \frac{S+V+I}{k}\right) - a_1 SI - a_2 S - a_3 S - a_4 SP + b_1 \alpha V + \beta I \\
\frac{dV}{dT} &= a_3 S - b_1 \alpha V - b_2(1-b_1)\alpha VI - b_3(1-b_1)\alpha V - b_4 VP \\
\frac{dI}{dT} &= a_1 SI + a_2 S + b_2(1-b_1)\alpha VI + b_3(1-b_1)\alpha V - cIP - \beta I \\
\frac{dP}{dT} &= e_1 a_4 SP + e_2 b_4 VP + e_3 cIp - dP
\end{aligned} \tag{1}$$

6. with the initial conditions $S(0) \geq 0, V(0) \geq 0, I(0) \geq 0$ and $P(0) \geq 0$.

a. Note that system (1) contains 17 parameters in all, which make the analysis of the system difficult, so in order to reduce the number of parameters and specify which combination controls the system, the following non dimensional variables and parameters are used in system (1) to get the next dimensionless system.

b.
$$\begin{aligned}
& t = rT,\ x = \frac{S}{k},\ y = \frac{V}{k},\ z = \frac{I}{k},\ w = \frac{a_4}{r}P, \\
& u_1 = \frac{a_1}{r}k,\ u_2 = \frac{a_2}{r},\ u_3 = \frac{a_3}{r}, u_4 = \frac{b_1}{r}\alpha, u_5 = \frac{\beta}{r}, \\
& u_6 = kb_2(1-b_1)\frac{\alpha}{r},\ u_7 = b_3(1-b_1)\frac{\alpha}{r}, u_8 = \frac{b_4}{a_4}, \\
& u_9 = \frac{c}{a_4},\ u_{10} = \frac{e_1 a_4}{r}k,\ u_{11} = \frac{e_2 b_4}{r}k,\ u_{12} = \frac{e_3 c}{r}k,\ u_{13} = \frac{d}{r}
\end{aligned} \tag{2}$$

7. The dimensionless form of system (1) becomes

a.
$$\begin{aligned}
\frac{dx}{dt} &= x(1-x-y-z) - u_1 xz - (u_2+u_3)x - xw + u_4 y + u_5 z = f_1(x,y,z,w) \\
\frac{dy}{dt} &= u_3 x - (u_4+u_7)y - u_6 yz - u_8 yw = f_2(x,y,z,w) \\
\frac{dz}{dt} &= u_1 xz + u_2 x + u_6 yz + u_7 y - u_9 zw - u_5 z = f_3(x,y,z,w) \\
\frac{dw}{dt} &= u_{10} xw + u_{11} yw + u_{12} zw - u_{13} w = f_4(x,y,z,w)
\end{aligned} \tag{3}$$

8. here the initial conditions are given by $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$ and $w(0) \geq 0$.
9. Clearly, system (3) has 13 non dimensional parameters and that means the number of parameters in system (1) by 4. Moreover the interaction functions $f_i(x,y,z,w), i = 1,2,3,4$ are continuous and have continuous partial derivatives on the positive cone.

10. $R_+^4 = \left\{(x, y, z, w) \in R^4 ; x \geq 0, y \geq 0, z \geq 0, w \geq 0\right\}$

11. Therefore these functions are Lipschitzion and hence system (3) has a unique solution, which is bounded and still in R_+^4 for all the positive time as shown in the following theorem.

Theorem (1): All solutions of system (3) that initial in R_+^4 are uniformly bounded.

Proof: Since the prey species consisting of three compartments, namely susceptible, vaccinated and infected population respectively. Then the total prey population is given by $N = x + y + z$, which is growing logistically in the absent of predation. Therefore, it easy to verify that

$$\frac{dN}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \leq N(1 - N)$$

Straightforward computation gives that

$$\lim_{t \to \infty} Sup .N(t) \leq 1 \Rightarrow N(t) = x(t) + y(t) + z(t) \leq 1;\ t > 0$$

Let $M(t) = x(t) + y(t) + z(t) + w(t)$, then from system (3) we obtain that

$$\frac{dM}{dt} \leq 2 - \mu M$$

where $\mu = \min .\{1, u_{13}\}$. Then we get that

$$M(t) < \frac{2}{\mu} + \left(M_0 - \frac{2}{\mu}\right) e^{-\mu t}$$

Thus $M(t) < \frac{2}{\mu}$, $\forall t > 0$, and hence the proof is complete. ■

III. Existence And Stability Of Equilibrium Points

It is easy to verify that the system (3) has at most three biologically feasible equilibrium points. The existence conditions of each of them along with their local stability analyses are discussed as follows

The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exists.

The predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, \bar{z}, 0)$, where

$$\bar{x} = \frac{u_5 \bar{z}[u_4 + (u_6 \bar{z} + u_7)]}{u_3(u_6 \bar{z} + u_7) + (u_1 \bar{z} + u_2)[u_4 + (u_6 \bar{z} + u_7)]}$$
$$\bar{y} = \frac{u_3 u_5 \bar{z}}{u_3(u_6 \bar{z} + u_7) + (u_1 \bar{z} + u_2)[u_4 + (u_6 \bar{z} + u_7)]} \tag{4a}$$

while $\bar{z}$ represents a positive root of the following fourth order polynomial equation

$$A_4 z^4 + A_3 z^3 + A_2 z^2 + A_1 z = 0 \tag{4b}$$

here $A_1 = u_5(u_4 + u_7)[u_3 u_7 + u_2(u_4 + u_7)]^3 > 0$

$$\begin{aligned} A_2 = \Big[& u_5[u_3 u_7 + u_2(u_4 + u_7)]^2 [u_3 u_4(u_5 + u_6 + u_7) + u_3 u_7(u_5 + u_7) + u_2 u_3 u_7(u_4 + u_7) \\ & + (u_2^2 + u_2 + u_1)(u_4 + u_7)^2 - (u_1 u_2 + u_5)(u_4 + u_7)^2 - u_1 u_3 u_7(u_4 + u_7)]\Big] \end{aligned}$$

$$\begin{aligned} A_3 = [& u_5(u_3 u_7 + u_2(u_4 + u_7))(-u_1^2(u_4 + u_7)^2((-1 + u_2)u_4 + (-1 + u_2 + u_3)u_7) \\ & + u_3 u_6(u_2^2 u_4(u_4 + u_7) - 2u_4 u_5(u_4 + u_7) + u_3(u_5 u_7 + u_4(2u_5 + u_6 + u_7)) \\ & + u_2(u_4^2 + u_5 u_7 + u_4(u_5 + u_6 + (1 + u_3)u_7))) + u_1(-u_3^2 u_4 u_6 u_7 + u_2^2(u_4 + u_7)^3 \\ & - 2u_5(u_4 + u_7)^3 + u_3(u_4 + u_7)(u_7(2u_5 + u_7) + u_4(2u_5 + 2u_6 + u_7)) \\ & + u_2(u_4 + u_7)(u_4^2 + (1 + u_3)u_7^2 + u_4(2u_7 + u_3(-u_6 + u_7)))))] \end{aligned}$$

$$A_4 = [u_5(u_1^3(u_4+u_7)^3((-1+u_2)u_4+(-1+u_2+u_3)u_7)-u_1u_3u_6(-u_3^2u_4u_6u_7$$
$$-6u_4u_5(u_4+u_7)^2+u_2^2u_4(u_4+u_7)(2u_4-u_6+2u_7)+u_3(2u_5u_7^2$$
$$+2u_4u_7(4u_5+u_6+u_7)+u_4^2(6u_5+3u_6+2u_7))+u_2(2u_4^3+2u_5u_7^2$$
$$+2u_4u_7(2u_5-(-1+u_3)u_6+(1+u_3)u_7)+u_4^2(2u_5-(-2+u_3)u_6$$
$$+2(2+u_3)u_7)))-u_1^2(u_4+u_7)(-2u_3^2u_4u_6u_7+u_2^2(u_4+u_7)^3-3u_5(u_4+u_7)^3$$
$$+u_3(u_4+u_7)(u_7(3u_5+u_7)+u_4(3u_5+3u_6+u_7))+u_2(u_4+u_7)(u_4^2+(1+u_3)u_7^2$$
$$+u_4(2u_7+u_3(-2u_6+u_7))))-u_3u_6^2(u_2^3u_4(u_4+u_7)+u_2^2((1+u_3)u_4^2+u_5u_7$$
$$+u_4(u_5+u_6+u_7+2u_3u_7))+u_2(u_3^2u_4u_7-2u_4u_5(u_4+u_7)+u_3(u_4^2+2u_5u_7$$
$$+2u_4(2u_5+u_6+u_7)))+u_3(-u_4u_5(3u_4+2u_7)+u_3(u_5u_7+u_4(3u_5+u_6+u_7)))))]$$

Clearly, E_1 exists uniquely in interior of R_+^3 of the xyz – space, provided that the following conditions hold

$$\left.\begin{array}{c} A_2>0 \text{ and } A_4<0 \\ \text{or} \\ A_3<0 \text{ and } A_4<0 \end{array}\right\} \qquad (4c)$$

The positive equilibrium point $E_2=(\hat{x},\hat{y},\hat{z},\hat{w})$ of system (3) can be determined by equating the right hand side of system (3) to the zero and solve the resulting algebraic system. Straightforward computation gives that:

$$\hat{x}=\frac{u_{13}-u_{11}\hat{y}-u_{12}\hat{z}}{u_{10}}$$
$$\hat{w}=\frac{1}{u_8u_{10}}\left[u_3u_{13}-\left(u_4+u_7+\frac{u_3u_{11}}{u_{10}}\right)u_{10}\hat{y}-u_3u_{12}\hat{z}-u_6u_{10}\hat{y}\hat{z}\right] \qquad (5a)$$

while $(\hat{y},\hat{z})$ represents a positive intersection point of the following two isoclines:

$$f(y,z)=r_1y^3+r_2y^2+r_3y+r_4yz+r_5y^2z+r_6yz^2+r_7z+r_8z^2-r_9=0 \qquad (5b)$$
$$g(y,z)=s_1z^2+s_2z+s_3yz+s_4yz^2+s_5y^2z+s_6y^2+s_7y=0 \qquad (5c)$$

Here $r_1=u_8u_{10}u_{11}+u_8u_{11}^2>0$

$$r_2=(u_4u_{10}+u_2u_{11}+u_3u_{11})u_8u_{10}-(u_3+u_{10})u_4u_{11}$$
$$-(u_7+u_8)u_{10}u_{11}-(u_{10}+2u_{11})u_8u_{13}$$
$$r_3=u_3u_4u_{13}+u_4u_{10}u_{13}+u_7u_{10}u_{13}+u_8u_{10}u_{13}$$
$$-u_2u_8u_{10}u_{13}-u_3u_8u_{10}u_{13}+u_3u_{11}u_{13}+u_8u_{13}^2$$
$$r_4=u_5u_8u_{10}^2-u_3u_4u_{12}-u_4u_{10}u_{12}-u_7u_{10}u_{12}-u_8u_{10}u_{12}+u_2u_8u_{10}u_{12}+u_3u_8u_{10}u_{12}$$
$$-u_3u_{11}u_{12}+u_6u_{10}u_{13}-u_8u_{10}u_{13}-u_1u_8u_{10}u_{13}-2u_8u_{12}u_{13}$$
$$r_5=u_8u_{10}u_{11}-u_6u_{10}u_{11}+u_1u_8u_{10}u_{11}+u_8u_{10}u_{12}+2u_8u_{11}u_{12}$$
$$r_6=u_8u_{10}u_{12}-u_6u_{10}u_{12}+u_1u_8u_{10}u_{12}+u_8u_{12}^2$$
$$r_7=2u_3u_{12}u_{13}>0\,,\; r_8=-u_3u_{12}^2<0\,,\; r_9=u_3u_{13}^2>0$$
$$s_1=u_3u_9u_{12}>0\,,\; s_2=u_3u_9u_{13}>0\,,$$
$$s_3=u_4u_9u_{10}+u_7u_9u_{10}+u_3u_4u_9-u_5u_8u_{10}-u_2u_8u_{12}+u_1u_8u_{13}$$
$$s_4=u_6u_9u_{19}-u_1u_8u_{12}\,,\; s_5=u_6u_8u_{10}-u_1u_8u_{11}$$
$$s_6=u_7u_8u_{10}-u_2u_8u_{11}\,, s_7=u_2u_8u_{13}>0$$

Clearly as $z\to 0$ the first isocline (5b) intersects the y – axis at a unique positive point, say $y_1>0$, provided that

$$r_2>0 \text{ or } r_3<0$$
(6a)

However when $z \to 0$ the second isocline (5c) will intersect the $y-$axis at zero or a point $y = y_2$, which is positive provided that

$$u_7 u_{10} < u_2 u_{11} \tag{6b}$$

Consequently, these two isoclines (5b) and (5c) have an intersection point in the interior of the positive quadrant of $yz-$plane, namely $(\hat{y}, \hat{z})$, provided that the following conditions are satisfied.

$$y_2 < y_1 \tag{6c}$$

$$\left.\begin{array}{c} \dfrac{\partial f}{\partial y} > 0 \ \text{ and } \ \dfrac{\partial f}{\partial z} > 0 \\ \text{or} \\ \dfrac{\partial f}{\partial y} < 0 \ \text{ and } \ \dfrac{\partial f}{\partial z} < 0 \end{array}\right\} \tag{6d}$$

$$\left.\begin{array}{c} \dfrac{\partial g}{\partial y} > 0 \ \text{ and } \ \dfrac{\partial g}{\partial z} < 0 \\ \text{or} \\ \dfrac{\partial g}{\partial y} < 0 \ \text{ and } \ \dfrac{\partial g}{\partial z} > 0 \end{array}\right\} \tag{6e}$$

Therefore the positive equilibrium point $E_2 = (\hat{x}, \hat{y}, \hat{z}, \hat{w})$ exists uniquely in the interior of R_+^4 if in addition to above conditions (6a)-(6e) the following conditions are satisfied too.

$$u_{13} > u_{11}\hat{y} + u_{12}\hat{z} \tag{6e}$$

$$u_3\hat{x} > (u_4 + u_7)\hat{y} + u_6\hat{y}\hat{z} \tag{6f}$$

In the following the local stability of each equilibrium points of system (3) is investigated. The Jacobian matrix of system (3) at (x, y, z, w) is given by

$$J = (a_{ij})_{4\times 4} \tag{7}$$

where

$a_{11} = -2x + (1 - u_2 - u_3) - y - (1 + u_1)z - w$, $a_{12} = -x + u_4$, $a_{13} = -(1 + u_1)x + u_5$

$a_{14} = -x$, $a_{21} = u_3$, $a_{22} = -(u_4 + u_7) - u_6 z - u_8 w$, $a_{23} = -u_6 y$, $a_{24} = -u_8 y$

$a_{31} = u_1 z + u_2$, $a_{32} = u_6 z + u_7$, $a_{33} = u_1 x + u_6 y - u_9 w - u_5$, $a_{34} = -u_9 z$

$a_{41} = u_{10} w$, $a_{42} = u_{11} w$, $a_{43} = u_{12} w$, $a_{44} = u_{10} x + u_{11} y + u_{12} z - u_{13}$

Accordingly, the local stability conditions for each of the above equilibrium points are established in the following theorems.

Theorem (2): The vanishing equilibrium point E_0 of system (3) is a saddle point in R_+^4.

Proof: Clearly the Jacobian matrix of system (3) at E_0 can be written as

$$J_0 = \begin{pmatrix} 1 - u_2 - u_3 & u_4 & u_5 & 0 \\ u_3 & -(u_4 + u_7) & 0 & 0 \\ u_2 & u_7 & -u_5 & 0 \\ 0 & 0 & 0 & -u_{13} \end{pmatrix}$$

(8a)

Therefore, the characteristic equation of J_0 is given by

$$[-u_{13} - \lambda]\left[\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3\right] = 0$$

(8b)

here $A_1 = u_2 + u_3 + u_4 + u_5 + u_7 - 1$, $A_2 = (u_4 + u_7)u_5 - (1 - u_2)u_4 - (1 - u_2 - u_3)u_7$ and $A_3 = -u_5(u_4 + u_7) < 0$. Now, according to the Routh-Hawirtiz Criterion all the eigenvalues of J_0 have roots with negative real parts if and only if $A_i (i = 1,3) > 0$ and $\Delta = A_1 A_2 - A_3 > 0$. Since we have $A_3 < 0$ always and the eigenvalue in the $w-$direction, $\lambda_w = -u_{13} < 0$, hence E_0 is a saddle point.

■

Theorem (3): Assume that the predator free equilibrium point $E_1 = (\bar{x}, \bar{y}, \bar{z}, 0)$ exists, then it is locally asymptotically stable provided that

$$\max\left\{u_4, \frac{u_5}{1+u_1}\right\} < \bar{x} < \frac{u_5 - u_6\bar{y}}{u_1} \tag{9a}$$

$$\max\left\{\frac{-2\bar{x} + (1-u_2-u_3) - \bar{y}}{1+u_1}, \frac{-(u_4+u_7)}{u_6}\right\} < \bar{z}$$

(9b)

$$\bar{x} < \frac{u_{13} - u_{11}\bar{y} - u_{12}\bar{z}}{u_{10}}$$

(9c)

$$(b_{22} + b_{33})b_{23} + b_{13}b_{21} > 0 \tag{9d}$$

where b_{ij} represent the Jacobian elements and are given in the proof.

Proof: Since the Jacobian matrix of system (3) at E_1 can be written as

$$J_1 = (b_{ij})_{4\times 4}$$

(10a)

where $b_{11} = -2\bar{x} + (1-u_2-u_3) - \bar{y} - (1+u_1)\bar{z}$, $b_{12} = -\bar{x} + u_4$, $b_{13} = -(1+u_1)\bar{x} + u_5$, $b_{14} = -\bar{x}$, $b_{21} = u_3$, $b_{22} = -(u_4+u_7) - u_6\bar{z}$, $b_{23} = -u_6\bar{y}$, $b_{24} = -u_8\bar{y}$, $b_{31} = u_1\bar{z} + u_2$, $b_{32} = u_6\bar{z} + u_7$, $b_{33} = u_1\bar{x} + u_6\bar{y} - u_5$, $b_{34} = -u_9\bar{z}$, $b_{41} = b_{42} = b_{43} = 0$ and $b_{44} = u_{10}\bar{x} + u_{11}\bar{y} + u_{12}\bar{z} - u_{13}$. Then the characteristic equation of J_1 can be written as

$$(\bar{\lambda}^3 + B_1\bar{\lambda}^2 + B_2\bar{\lambda} + B_3)(b_{44} - \bar{\lambda}) = 0$$

(10b)

with $B_1 = -(b_{11} + b_{22} + b_{33})$, $B_2 = b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33} - b_{23}b_{32}$ and $B_3 = -b_{11}b_{22}b_{33} - b_{12}b_{23}b_{31} - b_{13}b_{21}b_{32} + b_{13}b_{22}b_{31} + b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33}$. So either $(b_{44} - \bar{\lambda}) = 0$, which gives the eigenvalue in the $w-$ direction by $\bar{\lambda}_w = b_{44}$, or $\bar{\lambda}^3 + B_1\bar{\lambda}^2 + B_2\bar{\lambda} + B_3 = 0$.

Now, straightforward computation shows that $\bar{\lambda}_w = b_{44} < 0$ under condition (9a); $B_1 > 0$ and $B_3 > 0$ under the conditions (9a), (9b) and (9c); while $\Delta = B_1B_2 - B_3 > 0$ under the conditions (9a)-(9d). Consequently, according to Routh-Hawirtiz criterion all the eigenvalues of J_1 have negative real parts and hence E_1 is locally asymptotically stable. ■

In the following theorem, the basin of attraction of the predator free equilibrium point of system (3), is established.

Theorem (4): Assume that the predator free equilibrium point E_1 is locally asymptotically stable, then it is a globally asymptotically stable in the sub region $\Omega_1 \subseteq R_+^4$ that satisfy the following sufficient conditions

$$\bar{x} + u_{10} < x$$

(11a)

$$\bar{y} + \frac{u_{11}}{u_8} < y$$

(11b)

$$\bar{z} + \frac{u_{12}}{u_9} < z \tag{11c}$$

$$u_2 + u_3 > 1 \tag{11d}$$

$$u_1x + u_6y < u_5$$

(11e)

$$q_{12}^2 < q_{11}q_{22} \tag{11f}$$

$$q_{13}^2 < q_{11}q_{33}$$

(11g)

$$q_{23}^2 < q_{22}q_{33} \tag{11h}$$

where $q_{11} = (x+\bar{x}) - (1-u_2-u_3) + \bar{y} + (1+u_1)\bar{z}$, $q_{12} = x - u_3 - u_4$, $q_{22} = u_4 + u_7 + u_6\bar{z}$, $q_{13} = (1+u_1)x - u_5 - u_1\bar{z} - u_2$, $q_{23} = u_6 y - u_6\bar{z} - u_7$ and $q_{33} = u_5 - u_1 x - u_6 y$.

Proof: Consider the following function

$$L_1(x,y,z,w) = \frac{1}{2}(x-\bar{x})^2 + \frac{1}{2}(y-\bar{y})^2 + \frac{1}{2}(z-\bar{z})^2 + w$$

It is easy to see that $L_1(x,y,z,w) \in C^1\left(R_+^4, R\right)$, in addition $L_1(\bar{x},\bar{y},\bar{z},0) = 0$ while $L_1(x,y,z,w) > 0, \forall (x,y,z,w) \in R_+^4$ and $(x,y,z,w) \neq (\bar{x},\bar{y},\bar{z},0)$. Furthermore by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}\frac{dL_1}{dt} = &-\left[\frac{q_{11}}{2}(x-\bar{x})^2 + q_{12}(x-\bar{x})(y-\bar{y}) + \frac{q_{22}}{2}(y-\bar{y})^2\right]\\
&-\left[\frac{q_{11}}{2}(x-\bar{x})^2 + q_{13}(x-\bar{x})(z-\bar{z}) + \frac{q_{33}}{2}(z-\bar{z})^2\right]\\
&-\left[\frac{q_{22}}{2}(y-\bar{y})^2 + q_{23}(y-\bar{y})(z-\bar{z}) + \frac{q_{33}}{2}(z-\bar{z})^2\right]\\
&-\left[x-\bar{x}-u_{10}\right]xw - \left[u_8(y-\bar{y}) - u_{11}\right]yw - \left[u_9(z-\bar{z}) - u_{12}\right]zw - u_{13}w\end{aligned}$$

Clearly q_{11} and q_{33} are positive under conditions (11d) and (11e) respectively. Consequently by using the above sufficient conditions (11a)-(11h), it is obtained that

$$\begin{aligned}\frac{dL_1}{dt} < &-\left[\sqrt{\frac{q_{11}}{2}}(x-\bar{x}) + \sqrt{\frac{q_{22}}{2}}(y-\bar{y})\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}(x-\bar{x}) + \sqrt{\frac{q_{33}}{2}}(z-\bar{z})\right]^2\\
&-\left[\sqrt{\frac{q_{22}}{2}}(y-\bar{y}) + \sqrt{\frac{q_{33}}{2}}(z-\bar{z})\right]^2 - u_{13}w\end{aligned}$$

Thus, $\frac{dL_1}{dt}$ is negative definite and hence L_1 is Lyapunov function with respect to E_1 in the sub region Ω_1. So E_1 is a globally asymptotically stable. ■

The next theorem deals with the stability of the positive equilibrium point using the Lyapunov function.

Theorem (5): Assume that the positive equilibrium point $E_2 = (\hat{x},\hat{y},\hat{z},\hat{w})$ exists then it is a asymptotically stable in the sub region $\Omega_2 \subseteq R_+^4$ that satisfy the following sufficient conditions

$$u_2 + u_3 > 1 \quad (12a)$$

$$u_1 x + u_6 y < u_9\hat{w} + u_5 \quad (12b)$$

$$u_{10}x + u_{11}y + u_{12}z < u_{13} \quad (12c)$$

$$p_{12}^2 < \frac{4}{9}p_{11}p_{22} \quad (12d)$$

$$p_{13}^2 < \frac{4}{9}p_{11}p_{33} \quad (12e)$$

$$p_{14}^2 < \frac{4}{9}p_{11}p_{44} \quad (12f)$$

$$p_{23}^2 < \frac{4}{9}p_{22}p_{33} \quad (12g)$$

$$p_{24}^2 < \frac{4}{9}p_{22}p_{44} \quad (12h)$$

$$p_{34}^2 < \frac{4}{9} p_{33} p_{44}$$

(12k)

where $p_{11} = x + \hat{x} - (1 - u_2 - u_3) + \hat{y} + \hat{z}(1 + u_1) + \hat{w}$, $p_{22} = u_4 + u_7 + u_6 \hat{z} + u_8 \hat{w}$,

$p_{33} = u_9 \hat{w} + u_5 - u_1 x - u_6 y$, $p_{44} = u_{13} - u_{10} x - u_{11} y - u_{12} z$, $p_{12} = x - u_4 - u_3$,

$p_{13} = x(1 + u_1) - u_{15} - u_1 \hat{z} - u_2$, $p_{14} = x - u_{10} \hat{w}$, $p_{23} = u_6 y - u_6 \hat{z} - u_7$,

$p_{24} = u_8 y - u_{11} \hat{w}$, and $p_{34} = u_9 z - u_{12} \hat{w}$

Proof: Consider the following function

$$L_2(x, y, z, w) = \frac{1}{2}(x - \hat{x})^2 + \frac{1}{2}(y - \hat{y})^2 + \frac{1}{2}(z - \hat{z})^2 + \frac{1}{2}(w - \hat{w})^2$$

Clearly $L_2 : R_+^4 \to R$ and it is a continuously differentiable function, in addition, $L_2(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = 0$ while $L_2(x, y, z, w) > 0, \forall (x, y, z, w) \in R_+^4$ and $(x, y, z, w) \neq (\hat{x}, \hat{y}, \hat{z}, \hat{w})$. Further by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\begin{aligned}
\frac{dL_2}{dt} = &-\left[\frac{p_{11}}{3}(x - \hat{x})^2 + p_{12}(x - \hat{x})(y - \hat{y}) + \frac{p_{22}}{3}(y - \hat{y})^2 \right] \\
&-\left[\frac{p_{11}}{3}(x - \hat{x})^2 + p_{13}(x - \hat{x})(z - \hat{z}) + \frac{p_{33}}{3}(z - \hat{z})^2 \right] \\
&-\left[\frac{p_{11}}{3}(x - \hat{x})^2 + p_{14}(x - \hat{x})(w - \hat{w}) + \frac{p_{44}}{3}(w - \hat{w})^2 \right] \\
&-\left[\frac{p_{22}}{3}(y - \hat{y})^2 + p_{23}(y - \hat{y})(z - \hat{z}) + \frac{p_{33}}{3}(z - \hat{z})^2 \right] \\
&-\left[\frac{p_{22}}{3}(y - \hat{y})^2 + p_{24}(y - \hat{y})(w - \hat{w}) + \frac{p_{44}}{3}(w - \hat{w})^2 \right] \\
&-\left[\frac{p_{33}}{3}(z - \hat{z})^2 + p_{34}(z - \hat{z})(w - \hat{w}) + \frac{p_{44}}{3}(w - \hat{w})^2 \right]
\end{aligned}$$

It is easy to verify that, p_{11}, p_{33} and p_{44} are positive provided that conditions (12a)-(12c) are satisfied respectively. Consequently, due to conditions (12d)-(12k), we have

$$\begin{aligned}
\frac{dL_2}{dt} = &-\left[\sqrt{\frac{p_{11}}{3}}(x - \hat{x}) + \sqrt{\frac{p_{22}}{3}}(y - \hat{y}) \right]^2 - \left[\sqrt{\frac{p_{11}}{3}}(x - \hat{x}) + \sqrt{\frac{p_{33}}{3}}(z - \hat{z}) \right]^2 \\
&-\left[\sqrt{\frac{p_{11}}{3}}(x - \hat{x}) + \sqrt{\frac{p_{44}}{3}}(w - \hat{w}) \right]^2 - \left[\sqrt{\frac{p_{22}}{3}}(y - \hat{y}) + \sqrt{\frac{p_{33}}{3}}(z - \hat{z}) \right]^2 \\
&-\left[\sqrt{\frac{p_{22}}{3}}(y - \hat{y}) + \sqrt{\frac{p_{44}}{3}}(w - \hat{w}) \right]^2 - \left[\sqrt{\frac{p_{33}}{3}}(z - \hat{z}) + \sqrt{\frac{p_{44}}{3}}(w - \hat{w}) \right]^2
\end{aligned}$$

Therefore, $\frac{dL_2}{dt}$ is negative definite and hence L_2 is a Lyapunov function with respect to E_2 in the sub region Ω_2 . So E_2 is a asymptotically stable.

■

Note that the function L_2 is approaching to infinity as any of its components do the same and its positive definite on R_+^3 , however its derivative is negative definite on the sub region Ω_2 due to the given sufficient conditions. Therefore E_2 is a globally asymptotically stable within Ω_2 .

IV. The local bifurcation analysis

In this section, the effect of parameter values on the dynamical behavior of system (3) near the equilibrium points is studied. It is well known that the existence of non-hyperbolic equilibrium point of the system is a necessary but not sufficient condition for bifurcation to occur. Therefore in the following the parameter that makes the equilibrium point of system (3) as a non-hyperbolic equilibrium point is considered as a candidate bifurcation parameter for the system.

Now consider the Jacobian matrix of system (3) given by equation (7). It is easy to verify that straightforward computation gives that:

$$D^2F(X).(V,V)=\begin{pmatrix} -2v_1\left(v_1+v_2+(1+u_1)v_3+v_4\right) \\ -2v_2(u_6v_3+u_8v_4) \\ 2v_3(u_1v_1+u_6v_2-u_9v_4) \\ 2v_4(u_{10}v_1+u_{11}v_2+u_{12}v_3) \end{pmatrix}$$

(13)

where $X=(x,y,z,w)^T$ *and* $V=(v_1,v_2,v_3,v_4)^T$. Further, $D^3F(X).(V,V,V)=\mathbf{0}$, hence pitchfork bifurcation can't occur.

Now, since the Jacobian matrix of system (3) near the vanishing equilibrium point E_0 can't has zero real part eigenvalue. Therefore, there is no possibility to have bifurcation at E_0 . Moreover in the following theorem the local bifurcation conditions near the other equilibrium point are established.

Theorem (6): Suppose that the conditions (9a) and (9b) together with the following conditions are satisfied

$$(b_{13}b_{21}-b_{11}b_{23})\alpha_3 \neq (b_{14}b_{21}-b_{11}b_{24}) \tag{14}$$

$$b_{12}\alpha_2+b_{13}\alpha_3 \neq -b_{14}$$

(15)

Then for the parameter value $u_{13}^*=u_{10}\bar{x}+u_{11}\bar{y}+u_{12}\bar{z}$ system (3) at the equilibrium E_1 has a transcritical bifurcation, but not saddle-nod bifurcation.

where $\alpha_2=\dfrac{(b_{13}b_{21}-b_{11}b_{23})\alpha_3+(b_{14}b_{21}-b_{11}b_{24})}{(b_{11}b_{22}-b_{12}b_{21})}$

$$\alpha_3=\frac{(b_{11}b_{32}-b_{12}b_{31})(b_{11}b_{24}-b_{14}b_{21})-(b_{11}b_{22}-b_{12}b_{21})(b_{11}b_{34}-b_{14}b_{31})}{(b_{11}b_{22}-b_{12}b_{21})(b_{11}b_{33}-b_{13}b_{31})-(b_{11}b_{32}-b_{12}b_{31})(b_{11}b_{23}-b_{13}b_{21})}$$

Proof: According to the Jacobian matrix at E_1 that given by Eq.(10a), system (3) has zero eigenvalue (say $\lambda_{1w}=0$) at $u_{13}=u_{13}^*$, so the Jacobian matrix $J(E_1)$ with $u_{13}=u_{13}^*$ becomes:

$$J_1^*=J_1\left(E_1,u_{13}^*\right)=(b_{ij}^*)_{4\times 4}$$

where $b_{ij}^*=b_{ij}$; $\forall\, i,j=1,2,3,4$ except b_{44}^* , which is becomes zero.

Now, let $K=(k_1,k_2,k_3,k_4)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1w}=0$ of the matrix J_1^* . Thus $\left(J_1^*-\lambda_{1w}I\right)K=0$, which gives

$$k_1=\alpha_1k_4\,,\,k_2=\alpha_2k_4\,,\,k_3=\alpha_3k_4 \quad \text{and} \quad 0\neq k_4\in R$$

where $\alpha_1=\dfrac{(-b_{12}\alpha_2-b_{13}\alpha_3-b_{14})}{b_{11}}$

Let $L=(l_1,l_2,l_3,l_4)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1w}=0$ of the matrix J_1^{*T} . Then $\left(J_1^{*T}-\lambda_{1w}I\right)L=0$, which gives that l_4 be any nonzero real number while $l_1=l_2=l_3=0$.

Now, consider

$$\frac{\partial F}{\partial u_{13}}=F_{u_{13}}(\mathrm{X},u_{13})=\left(\frac{\partial f_1}{\partial u_{13}},\frac{\partial f_2}{\partial u_{13}},\frac{\partial f_3}{\partial u_{13}},\frac{\partial f_4}{\partial u_{13}}\right)^T=(0,0,0,-w)^T$$

So, $F_{u_{13}}\left(E_1,u_{13}^*\right)=(0,0,0,0)^T$, and hence $L^TF_{u_{13}}\left(E_1,u_{13}^*\right)=0$, thus according to Sotomayor's theorem saddle-node bifurcation can't occur, while the first condition of transcritical bifurcation is satisfied. Also, we have

$$L^T\left[DF_{u_{13}}\left(E_1,u_{13}^*\right)K\right]=-k_4l_4\neq 0$$

Further more according to Eq. (13) we get

$$L^T\left[D^2F\left(E_1,u_{13}^*\right)(K,K)\right]=(2u_{10}\alpha_1+2u_{11}\alpha_2+2u_{12}\alpha_3)k_4^2 l_4$$

Straightforward computation, using the conditions (14)-(15), shows that $L^T\left[D^2F\left(E_1,u_{13}^*\right)(K,K)\right]\neq 0$. Hence, system (3) has transcritical bifurcation at E_1 with the parameter $u_{13}=u_{13}^*$ and the proof is complete. ■

V. Numerical Simulation

In this section, the global dynamics of system (3) is studied numerically. The objectives of this study are confirming our obtained analytical results and detected the set of control parameters that affect the dynamics of the system. Consequently, system (3) is solved numerically for different sets of initial conditions and for different sets of parameters. It is observed that, for the following set of hypothetical parameters the system (3) has a globally asymptotically stable positive equilibrium point as shown in following figure.

$$\begin{array}{l} u_1=0.5, u_2=0.1, u_3=0.25, u_4=0.05, u_5=0.1, u_6=0.1, u_7=0.05, \\ u_8=0.4, u_9=0.4, u_{10}=0.3, u_{11}=0.1, u_{12}=0.2, u_{13}=0.1 \end{array} \tag{16}$$

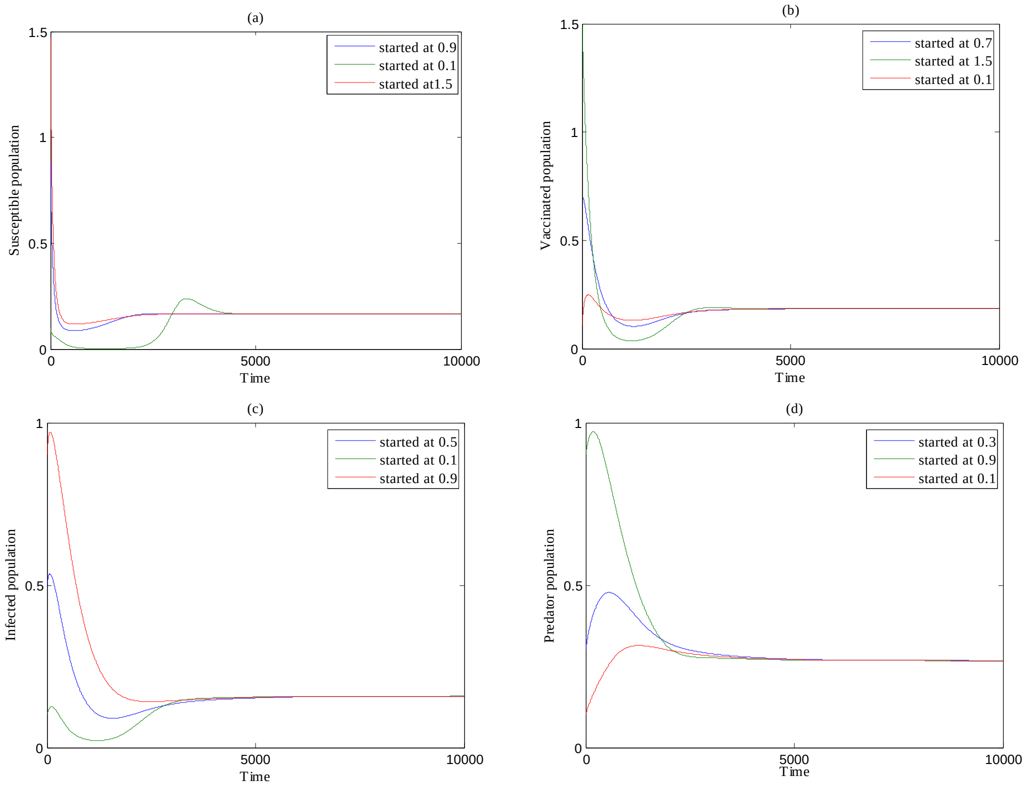

Fig. 1: Time series of trajectories of system (3) for the data (16) started at different initial points. (a) The trajectories of susceptible prey as a function of time. (b) The trajectories of vaccinated prey as a function of time. (c) The trajectories of infected prey as a function of time. (d) The trajectories of predator as a function of time.

Obviously, Fig. (1) shows the existence of a globally asymptotically stable positive equilibrium point $E_2=(0.14,\ 0.16,\ 0.2,\ 0.23)$ for system (3) and this is clear due to convergent from three different initial data.

Note that since the parameters $u_1,u_2,\cdots u_7$ describe the relationships among the compartments of the prey species (x,y and z) and the parameters $u_8,u_9,\cdots u_{12}$ describe the relationships between the predator on one side and one of the prey's compartments on the other side. Therefore varying these parameters don't have qualitative effects on the dynamics of system (3) rather than that they have quantitative effects on the value of positive equilibrium point.

However, for the data given by equation (16) with varying the parameter u_{13} in the range $u_{13} \geq 0.2$, then the trajectory of system (3), starting from different sets of initial data, is approaching asymptotically to the predator free equilibrium point as shown in the typical figures represented by Fig. (2) and Fig. (3).

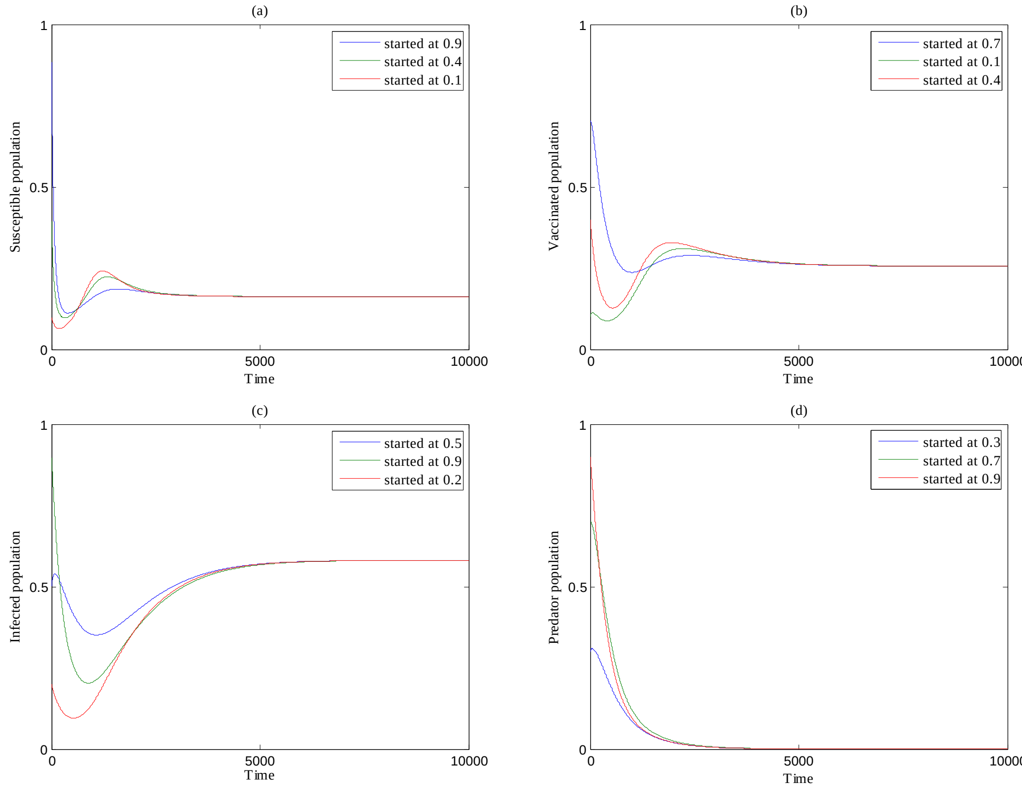

Fig. 2: The trajectory of system (3), for the data (16) with $u_{13} = 0.3$ started at different initial points, approaches to $E_1 = (0.162, 0.256, 0.581, 0)$. (a) The trajectories of susceptible prey as a function of time. (b) The trajectories of vaccinated prey as a function of time. (c) The trajectories of infected prey as a function of time. (d) The trajectories of predator as a function of time.

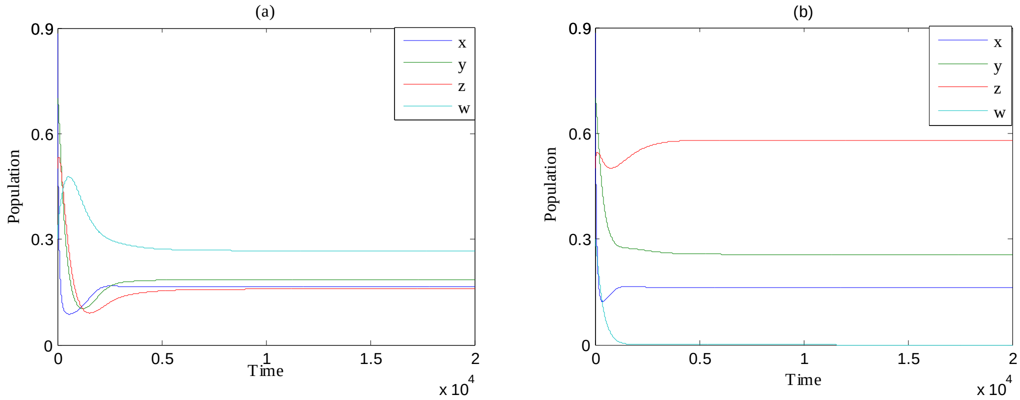

Fig. 3: Time series of the solution of system (3) for the data (16) with different values of u_{13}. (a) Globally asymptotically stable positive equilibrium point for $u_{13} = 0.1$. (b) Globally asymptotically stable predator free equilibrium point E_1 for $u_{13} = 0.25$.

According to these two figures, it's clear that the solution of system (3) approaches asymptotically to the predator free equilibrium point.

VI. Conclusions and discussion

In this paper an eco-epidemiological model consisting of prey-predator system having $SVIS$ – type of disease in prey is proposed and analyzed analytically as well as numerically. It is observed that the system has at most three nonnegative equilibrium point. The local and global stability of these equilibrium points are discussed and it is observed that the vanishing equilibrium point is a saddle point while the predator free equilibrium point and the positive equilibrium point are asymptotically stable under certain conditions. The local bifurcation of the equilibrium points E_0 and E_1 is discussed analytically according to Sotomayor's theorem while that of the positive point is discussed numerically. Furthermore numerical simulation is used to verify our obtained results and specify the set of parameters that control the dynamics of the system. Finally according to the numerical outcomes, it is observed that the system (3) for the data given by (16) has a globally asymptotically stable positive equilibrium point. However increasing the predator death rate above a specific value causes extinction in predator species and the solution approaches asymptotically to the predator free equilibrium point. Consequently the system undergoes a bifurcation around the positive point by varying the predator death rate and the solution of the system change its stability from the positive equilibrium point to the predator free equilibrium point. Finally all the other parameters have quantitative change but note qualitative change on the stability of the positive equilibrium point.

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