IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 1 Ver. IV (Jan. - Feb. 2016), PP 21-40
www.iosrjournals.org

Modeling and Analysis of a Prey-Predator System with Disease in Predator

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Abstract: *In the present paper a prey-predator model with disease that spreads among the predator species only is proposed and investigated. It is assumed that the disease is horizontally transmitted by contact between the infected predator and the susceptible predator. The local and global stability analyses are carried out. The persistence conditions of the model are established. Local bifurcation analyses are performed. Numerical simulation is used extensively to detect the occurrence of Hopf bifurcation and confirm our obtained analytical outcomes.*
Keywords: *Prey-Predator, Stability, Local bifurcation, Hopf bifurcation, Persistence.*

I. Introduction

A prey-predator type of interaction among species can be clearly seen in many ecological systems throughout the world, such as a deer-lion relation. In nature, prey and predator species exhibit fluctuation of abundance or population increase and decrease. The study of this fluctuation that is in apparently stable patterns has long been of interest to animal conservationists and mathematicians. Consequently the dynamics of prey-predator interactions have been studied extensively in the last three decades see for example [1-4] and the references therein.

The evolution of disease in natural populations has always been an important field of both theoretical and experimental studies due to their effects on the existence populations. Although, most of the previous studies have been focused on the interactions between the pathogen and its host [5-7], it is obvious that populations are generally involved in complex trophic interactions with other populations. Therefore, this should be taken in to account when one construct mathematical model of the evolution of diseases in real ecosystems. Indeed it has been shown that predation of infected populations can both increase and decrease the infection prevalence [8-9]. Accordingly, mathematical epidemiology to study the dynamics of diseases spread has become an interesting topic of research study and received much attention from scientists after the pioneering work of Kermack-McKendrick. A number of mathematical models of disease spread have been introduced relevant to the type of diseases, for example SI, SIS, SIR, SEIR, SEIRS [10-14] and references therein.

Eco-epidemiology is a rather new branch of study, merging features of interacting populations among which a transmissible disease spreads. It can be viewed as the coupling of an ecological prey-predator (or competition) model and an epidemiological SI, SIS or SIRS model. Following Anderson and May (1982) who were the first to propose an eco-epidemiological model by merging the ecological prey-predator model introduced by Lotka and Volterra, and the epidemiological SIR model introduced by Kermack and McKendrick, many works have been devoted to the study of the effects of a disease on a prey-predator system [15-18] and references therein.

Keeping the above in view, most of the previous studies focused on the disease in prey-predator system with vertical transmitted of disease. However, in this paper an eco-epidemiological model consisting of prey-predator model with horizontally transmitted of disease within predator population is proposed and studied.

II. The Model Formulation

In this section a mathematical model, which describes the dynamical behavior of a prey-predator system with horizontally transmitted infectious disease in predator, is proposed and analyzed. Consequently, in order to formulates this model the following hypotheses are considered

1. Let $X(T)$ denotes the density of the prey species at time T, $S(T)$ is the population density of the susceptible predator at time T and $I(T)$ represents the population density of the infected predator at time T.
2. It is assumed that in the absence of the predators the prey species reproduces logistically with intrinsic growth rate $r > 0$ and carrying capacity $K > 0$.
3. The predators $S(T)$ and $I(T)$ consume the prey according to the Holling type –II functional response with attack rates $a_1 > 0$ and $a_2 > 0$ respectively and half saturation constant $b > 0$.
4. It is assumed that the disease is not transferred from parent to offspring rather than that it's transferred between the species in the same generation, which is known horizontally transmitted. Therefore all the newborns of the predators, due to feeding process on the prey, are susceptible specie with conversion rates $0 < e_i < 1$ with $i = 1,2$. Furthermore, the disease is transmitted from infected predator to susceptible predator by contact according to saturated incident rate with infected rate $c > 0$ and the disease's inhibitory effect rate $\alpha > 0$.
5. The predators decay exponentially in the absence of the prey species with death rates $d_1 > 0$ and $d_2 > 0$ for the susceptible and infected predator respectively.

Accordingly the dynamics of the prey-predator system with infectious disease in predator that described above can be represented mathematically by the following set of nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1 - \frac{X}{K}\right) - \frac{a_1 XS}{b + X} - \frac{a_2 XI}{b + X} \\
\frac{dS}{dT} &= \left(\frac{e_1 a_1 S + e_2 a_2 I}{b + X}\right)X - \frac{cSI}{1 + \alpha I} - d_1 S \\
\frac{dI}{dT} &= \frac{cSI}{1 + \alpha I} - d_2 I
\end{aligned} \tag{1}$$

With $X(0) \geq 0,\ S(0) \geq 0,\ I(0) \geq 0$. Clearly, due to the biological meaning of the variables given in system (1) the system defines on the following domain $R_+^3 = \{(X,Y,Z) \in R^3 : X \geq 0, S \geq 0, I \geq 0\}$. Now in order to reduce the number of parameters and determine which set of parameters control the behavior of the system, the following dimensionless variables and parameters are used in system (1).

$$t = rT,\ x = \frac{X}{K},\ s = \frac{S}{K},\ i = \frac{I}{K}\ w_1 = \frac{a_1}{r},\ w_2 = \frac{b}{K},\ w_3 = \frac{a_2}{r},\ w_4 = \frac{cK}{r},\ w_5 = \alpha K,\ w_6 = \frac{d_1}{r},\ w_7 = \frac{d_2}{r} \tag{2}$$

Accordingly the dimensionless form of system (1) becomes

$$\begin{aligned}
\frac{dx}{dt} &= x(1 - x) - \frac{w_1 xs}{w_2 + x} - \frac{w_3 xi}{w_2 + x} \\
\frac{ds}{dt} &= \left(\frac{e_1 w_1 s + e_2 w_3 i}{w_2 + x}\right)x - \frac{w_4 si}{1 + w_5 i} - w_6 s \\
\frac{di}{dt} &= \frac{w_4 si}{1 + w_5 i} - w_7 i
\end{aligned} \tag{3}$$

with $x(0) \geq 0, s(0) \geq 0,\ i(0) \geq 0$. Clearly the interaction functions in the right hand side of system (3) are continuous and have continuous partial derivatives, and hence they are Liptchazian functions. Hence system (3) has a unique solution. Moreover all solutions of system (3) are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (3) those initiate in the R_+^3 are uniformly bounded.

Proof: Let $(x(t), s(t), i(t))$ be any solution initiate in R_+^3. Since we have that

$$\frac{dx}{dt} \leq x(1 - x)$$

Then straightforward computation shows that $x \leq 1$ as $t \to \infty$. Now consider the function $W = x + s + i$, then we obtain that

$$\frac{dW}{dt} + \mu W \leq 2$$

here $\mu = \min.\{1, w_6, w_7\}$. So, straightforward computation gives that $W \leq \frac{2}{\mu}$ as $t \to \infty$. Hence all solutions are uniformly bounded. ■

III. Local Stability Analysis And Persistence

In this section the existence conditions of all possible equilibrium points of system (3) are established and then the local stability analyses of them are discussed. There are at most four non-negative equilibrium points of system (3), these are described as follows:

The vanishing equilibrium point that denoted by $E_0 = (0,0,0)$ and the predator free equilibrium point, say $E_1 = (1,0,0)$, which full down on the $x-$axis, are always exist. The disease free equilibrium point $E_2 = (\bar{x}, \bar{s}, 0)$ where

$$\bar{x} = \frac{w_2 w_6}{e_1 w_1 - w_6};\ \bar{s} = \frac{e_1 w_2 [e_1 w_1 - (1 + w_2) w_6]}{w_1 (e_1 w_1 - w_6)^2} \tag{4}$$

exists uniquely in the interior of $xs-$plane if and only if the following condition holds

$$e_1 w_1 > (1 + w_2) w_6 \tag{5}$$

Finally the positive equilibrium point $E_3 = (x^*, s^*, i^*)$ exists uniquely in the interior of R_+^3 provided that there is a positive solution to the following algebraic system of equations.

$$\begin{aligned}
f_1(x,s,i) &= (1-x) - \frac{w_1 s}{w_2 + x} - \frac{w_3 i}{w_2 + x} = 0 \\
f_2(x,s,i) &= \left(\frac{e_1 w_1 s + e_2 w_3 i}{w_2 + x}\right) x - \frac{w_4 s i}{1 + w_5 i} - w_6 s = 0 \\
f_3(x,s,i) &= \frac{w_4 s}{1 + w_5 i} - w_7 = 0
\end{aligned} \tag{6}$$

Straightforward computation shows that system (6) has a unique positive solution given by

$$x^* = \frac{\gamma_1 i^* + \gamma_2}{\gamma_3 i^* + \gamma_4};\ s^* = \frac{w_7 (1 + w_5 i^*)}{w_4} \tag{7a}$$

while i^* represents a positive root of the following equation

$$A i^3 + B i^2 + C i + D = 0 \tag{7b}$$

where $\gamma_1 = w_2 w_7 (w_4 + w_5 w_6) > 0$, $\gamma_2 = w_2 w_6 w_7 > 0$,

$\gamma_3 = w_5 w_7 (e_1 w_1 - w_6) + w_4 (e_2 w_3 - w_7)$, $\gamma_4 = w_7 (e_1 w_1 - w_6)$

$A = -(w_1 w_5 w_7 + w_3 w_4) \gamma_3^{\ 2} < 0$,

$B = (w_2 w_4 - w_1 w_7) \gamma_3^{\ 2} + ((1 - w_2)\gamma_3 - \gamma_1) w_4 \gamma_1 - 2(w_1 w_5 w_7 + w_3 w_4) \gamma_3 \gamma_4$

$C = 2(w_2 w_4 - w_1 w_7) \gamma_3 \gamma_4 + w_4 (1 - w_2)(\gamma_1 \gamma_4 + \gamma_2 \gamma_3) - 2 w_4 \gamma_1 \gamma_2 - (w_1 w_5 w_7 + w_3 w_4) \gamma_4^{\ 2}$

$D = (w_2 w_4 - w_1 w_7) \gamma_4^{\ 2} + w_4 ((1 - w_2) \gamma_4 - \gamma_2) \gamma_2$

Clearly, it is easy to verify that E_3 exists uniquely in the interior of R_+^3 if the following sufficient conditions hold

$$\left.\begin{aligned}
& e_1 w_1 > w_6 \\
& e_2 w_3 > w_7 \\
& B < 0 \text{ or } C > 0 \\
& D > 0
\end{aligned}\right. \tag{7c}$$

Now the general Jacobian matrix $DF = J(x,s,i)$ of system (3), where $F = (F_1, F_2, F_3)^t$ with $F_1 = x f_1, F_2 = f_2$ and $F_3 = i f_3$, can be written as:

$$J(x,s,i)=(a_{ij})_{3\times 3} \tag{8}$$

where $a_{11}=x\left[-1+\dfrac{w_1 s+w_3 i}{(w_2+x)^2}\right]+f_1$, $a_{12}=\dfrac{-w_1 x}{w_2+x}$, $a_{13}=\dfrac{-w_3 x}{w_2+x}$, $a_{21}=\dfrac{(e_1 w_1 s+e_2 w_3 i)w_2}{(w_2+x)^2}$,

$a_{22}=\dfrac{e_1 w_1 x}{w_2+x}-\dfrac{w_4 i}{1+w_5 i}-w_6$, $a_{23}=\dfrac{e_2 w_3 x}{w_2+x}-\dfrac{w_4 s}{(1+w_5 i)^2}$, $a_{31}=0$, $a_{32}=\dfrac{w_4 i}{1+w_5 i}$, $a_{33}=-\dfrac{w_4 w_5 s i}{(1+w_5 i)^2}+f_3$.

Then the Jacobian matrix of system (3) at vanishing equilibrium point $E_0=(0,0,0)$ is

$$J(E_0)=\begin{pmatrix}1 & 0 & 0\\ 0 & -w_6 & 0\\ 0 & 0 & -w_7\end{pmatrix} \tag{9a}$$

Clearly, the $J(E_0)$ has the following eigenvalues

$$\lambda_{0x}=1>0;\ \lambda_{0s}=-w_6<0;\ \lambda_{0i}=-w_7<0 \tag{9b}$$

where $\lambda_{0u};u=x,s,i$ represents the eigenvalue of $J(E_0)$ in the $u-$direction. Therefore the vanishing equilibrium point E_0 is a saddle point with unstable manifold in the $x-$direction and stable manifold in the $si-$ plane.

The Jacobian matrix of system (3) at the predator free equilibrium point $E_1=(1,0,0)$ is given by

$$J(E_1)=\begin{pmatrix}-1 & -\dfrac{w_1}{w_2+1} & -\dfrac{w_3}{w_2+1}\\ 0 & \dfrac{e_1 w_1}{w_2+1}-w_6 & \dfrac{e_2 w_3}{w_2+1}\\ 0 & 0 & -w_7\end{pmatrix}=(b_{ij}) \tag{10a}$$

Therefore the eigenvalues of $J(E_1)$ are

$$\lambda_{1x}=-1<0,\ \lambda_{1s}=\frac{e_1 w_1}{w_2+1}-w_6,\ \lambda_{1i}=-w_7 \tag{10b}$$

It is easy to verify that all the above eigenvalues will be negative and hence E_1 is locally asymptotically stable in the R_+^3, if the following condition holds

$$\frac{e_1 w_1}{w_2+1}<w_6 \tag{10c}$$

Otherwise the predator free equilibrium point E_1 is a saddle point with unstable manifold in the $s-$direction and stable manifold in the $xi-$ plane.

The Jacobian matrix of system (3) at the disease free equilibrium point $E_2=(\bar{x},\bar{s},0)$ can be written

$$J(E_2)=\begin{pmatrix}\bar{x}\left[-1+\dfrac{w_1\bar{s}}{(w_2+\bar{x})^2}\right] & \dfrac{-w_1\bar{x}}{w_2+x} & \dfrac{-w_3\bar{x}}{w_2+\bar{x}}\\ \dfrac{e_1 w_1\bar{s}w_2}{(w_2+\bar{x})^2} & 0 & \dfrac{e_2 w_3\bar{x}}{w_2+\bar{x}}-w_4\bar{s}\\ 0 & 0 & w_4\bar{s}-w_7\end{pmatrix}=(c_{ij}) \tag{11a}$$

Then the characteristic equation $J(E_2)$ is given by

$$(\lambda^2-T_2\lambda+D_2)(c_{33}-\lambda)=0 \tag{11b}$$

here $T_2=c_{11}=\bar{x}\left[-1+\dfrac{w_1\bar{s}}{(w_2+\bar{x})^2}\right]$ and $D_2=-c_{12}c_{21}=\dfrac{e_1 w_1^2 w_2\bar{s}\bar{x}}{(w_2+\bar{x})^3}>0$. Thus, the eigenvalues of $J(E_2)$ are:

$$\lambda_{2x},\lambda_{2s}=\frac{1}{2}\left(T_2\pm\sqrt{T_2^2-4D_2}\right);\lambda_i=w_4\bar{s}-w_7 \tag{11c}$$

Clearly, the eigenvalues λ_{2x} and λ_{2s} have negative real parts provided that

$$w_1\bar{s} < (w_2 + \bar{x})^2 \tag{11d}$$

While the eigenvalue λ_{2i} is negative provided that

$$w_4\bar{s} < w_7 \tag{11e}$$

Consequently, the disease free equilibrium point E_2 is locally asymptotically stable in R_+^3 provided that the conditions (11d) and (11e) are satisfied. Moreover it will be saddle point if we violate at least one of these two conditions.

Finally, The Jacobian matrix of system (3) at the positive equilibrium point $E_3 = (x^*, s^*, i^*)$ is given by

$$J(E_3) = (d_{ij}) \tag{12a}$$

here $d_{11} = x^*\left[-1 + \dfrac{w_1 s^* + w_3 i^*}{(w_2 + x^*)^2}\right]$, $d_{12} = \dfrac{-w_1 x^*}{w_2 + x^*}$, $d_{13} = \dfrac{-w_3 x^*}{w_2 + x^*}$,

$d_{21} = \dfrac{(e_1 w_1 s^* + e_2 w_3 i^*)w_2}{(w_2 + x^*)^2}$, $d_{22} = \dfrac{e_1 w_1 x^*}{w_2 + x^*} - \dfrac{w_4 i^*}{1 + w_5 i^*} - w_6$,

$d_{23} = \dfrac{e_2 w_3 x^*}{w_2 + x^*} - \dfrac{w_4 s^*}{(1 + w_5 i^*)^2}$, $d_{31} = 0$, $d_{32} = \dfrac{w_4 i^*}{1 + w_5 i^*}$, $d_{33} = -\dfrac{w_4 w_5 s^* i^*}{(1 + w_5 i^*)^2}$

Thus the characteristic equation of $J(E_3)$ can be written as:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{12b}$$

with $A_1 = -(d_{11} + d_{22} + d_{33})$, $A_2 = d_{11}d_{22} - d_{12}d_{21} + d_{22}d_{33} - d_{23}d_{32} + d_{11}d_{33}$,

$A_3 = -d_{11}(d_{22}d_{33} - d_{23}d_{32}) - d_{21}(d_{13}d_{32} - d_{12}d_{33})$,

$$\begin{aligned}\Delta = A_1A_2 - A_3 = &-(d_{11} + d_{22})(d_{11}d_{22} - d_{12}d_{21}) + d_{11}d_{33}A_1\\ &-(d_{22} + d_{33})(d_{22}d_{33} - d_{23}d_{32}) - d_{11}d_{22}d_{33} + d_{13}d_{21}d_{32}\end{aligned}$$

Therefore, by using the Routh-Hurwitz criterion the following theorem can be proved directly.

Theorem (2): The positive equilibrium point E_4 of system (3) is locally asymptotically stable in R_+^3 provided that the following sufficient conditions are satisfied

$$w_1 s^* + w_3 i^* < R_1^* \tag{13a}$$

$$\frac{w_4 s^*}{e_2 w_3 R_2^{*2}} < \frac{x^*}{R_1^*} < \frac{1}{e_1 w_1}\left(\frac{w_4 i^*}{R_2^*} + w_6\right) \tag{13b}$$

$$x^*(e_1 w_1 w_5 s^* + e_2 w_3 R_2^*) < (w_5 w_6 + w_4)s^* R_1^* \tag{13c}$$

$$\begin{aligned}&w_5 s^*\left(R_1^{*2} - w_1 s^* - w_3 i^*\right)e_1 w_1 x^* R_2^* + w_2 w_3\left(e_1 w_1 s^* + e_2 w_3 i^*\right)R_2^{*2}\\ &\qquad < w_5 s^*\left(R_1^{*2} - w_1 s^* - w_3 i^*\right)\left(w_4 i^* + w_6 R_2^*\right)R_1^*\end{aligned} \tag{13d}$$

where $R_1^* = w_2 + x^*$ and $R_2^* = 1 + w_5 i^*$.

Proof. Straightforward computation shows that A_1, A_3 and Δ are positive under the sufficient conditions (13a)-(13d) and hence according to Routh-Hurwitz criterion all the eigenvalues of the $J(E_3)$ have negative real parts . Thus E_3 is locally asymptotically and the proof is complete. ■

It is well known that, the persistence of an ecological system means the coexistence of all the species for all positive time. Moreover the coexistence of all the species for all the positive time is satisfied mathematically if the solution of the system doesn't has omega limit set on the boundary planes. Therefore the conditions of the persistence of the system (3) are established in the following theorem.

Theorem (3): Suppose that the is no periodic dynamics in $xs-$plane and that

$$\frac{e_1 w_1}{w_2 + 1} > w_6 \tag{14a}$$

$$w_4\bar{s} > w_7 \tag{14b}$$

Then system (3) is uniformly persistence.

Proof. Let p be any point in the positive octant and let $o(p)$ be the orbit through it. Let $\Omega(p)$ denotes the omega limit set of the orbit through the point p. Clearly $\Omega(p)$ is bounded due to the boundedness of the system (1). We claim that $E_0 \notin \Omega(p)$. If $E_0 \in \Omega(p)$ then according to the Butler-McGehe lemma [19], there is a point $q \in \Omega(p) \cap W^s(E_0)$, where $W^s(E_0)$ represents the stable manifold of E_0. Now since $o(q)$ lies in $\Omega(p)$ and $W^s(E_0)$ is the $si-$plane, then the orbit through q, which denoted by $o(q)$, is unbounded orbit which leads to contradiction.

Now we claim that $E_1 \notin \Omega(p)$, otherwise $E_1 \in \Omega(p)$. Since E_1 is saddle point due to condition (14a) with stable manifold represented by $xi-$plane, hence again by Butler-McGehe lemma there is a point $q \in \Omega(p) \cap W^s(E_1)$, where $W^s(E_1)$ is the stable manifold of E_1. Moreover since $o(q)$ lies in $\Omega(p)$ and $W^s(E_1)$ is the $xi-$plane, then the orbit through q that denoted by $o(q)$ is unbounded orbit which leads to contradiction too. Similarly we get contradiction if we assume that $E_2 \in \Omega(p)$ due to using condition (14b) which guarantees that E_2 is saddle point with stable manifold $xs-$plane.

Therefore $\Omega(p)$ doesn't intersect any of boundary planes of the R_+^3, then system (3) is persistent. In addition to that since system (3) is bounded system then according to theorem of Butler et al [20], system (3) becomes uniformly persistent. ■

IV. Global Stability Analysis

In this section the region of global stability, that known as basin of attraction, of each equilibrium point is determined with the help of Lyapunov method as shown in the following theorems.

Theorem (4): Suppose that the predator free equilibrium point $E_1 = (1,0,0)$ is locally asymptotically stable, then it's a globally asymptotically stable in the R_+^3, Provided that

$$\frac{w_1}{w_2} < w_6 \tag{15a}$$

$$\frac{w_3}{w_2} < w_7 \tag{15b}$$

Proof: Define the function

$$V_1 = \left[x - \tilde{x} - \tilde{x} \ln\left(\frac{x}{\tilde{x}}\right) \right] + s + i$$

where $\tilde{x} = 1$. Clearly the function V_1 is continuously differentiable, positive definite, real valued function with $V_1(\tilde{x},0,0) = 0$ and $V_1(x,s,i) > 0$ for all $(x,s,i) \neq (\tilde{x},0,0) \in R_+^3$. Now by differentiate V_1 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dV_1}{dt} \leq -(x-1)^2 - \left(w_6 - \frac{w_1}{w_2}\right)s - \left(w_7 - \frac{w_3}{w_2}\right)i$$

Thus it is easy to verify that $\frac{dV_1}{dt}$ is negative definite in the R_+^3 under the conditions (15a) and (15b). Therefore, for any initial point in the R_+^3 the solution of system (3) approaches asymptotically to E_1. Thus E_1 is a globally asymptotically stable in the R_+^3, and the proof is complete. ■

Theorem (5): Suppose that the disease free equilibrium point $E_2 = (\bar{x},\bar{s},0)$ is locally asymptotically stable, then it's a globally asymptotically stable in the R_+^3, provided that

$$w_1 \bar{s} < w_2 \bar{R}_1 \tag{16a}$$

$$w_7 > w_4 \bar{s} + \frac{e_1 w_3 \bar{x}}{\bar{R}_1} \tag{16b}$$

$$\frac{e_1 w_2}{\overline{R}_1} > e_2 \tag{16c}$$

here $\overline{R}_1 = w_2 + \overline{x}$.

Proof: Consider the function

$$V_2 = c_1\left[x - \overline{x} - \overline{x}\ln\left(\frac{x}{\overline{x}}\right)\right] + c_2\left[s - \overline{s} - \overline{s}\ln\left(\frac{s}{\overline{s}}\right)\right] + c_3 i$$

where $c_j\,;\, j = 1,2,3$ are positive constants to be determined. Clearly the function V_2 is continuously differentiable, positive definite, real valued function with $V_2(\vec{x},\vec{s},0) = 0$ and $V_2(x,s,i) > 0$ for all $(x,s,i) \neq (\vec{x},\vec{s},0)$ in the R_+^3. Now by differentiate V_2 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dV_2}{dt} = -c_1\left(1 - \frac{w_1\overline{s}}{R_1\overline{R}_1}\right)(x-\overline{x})^2 - \frac{w_1}{R_1}\left(c_1 - c_2\frac{e_1 w_2}{\overline{R}_1}\right)(x-\overline{x})(s-\overline{s})$$
$$-\frac{w_3}{R_1}(c_1 - c_2 e_2)xi + c_1\frac{w_3\overline{x}i}{R_1} - c_2\frac{e_2 w_3\overline{s}xi}{sR_1} - (c_2 - c_3)\frac{w_4 si}{R_2} + c_2\frac{w_4\overline{s}i}{R_2} - c_3 w_7 i$$

where $R_1 = w_2 + x$ and $R_2 = 1 + w_5 i$. Consequently by choosing the positive constants as $c_1 = \frac{e_1 w_2}{\overline{R}_1}$ and $c_2 = c_3 = 1$, then by substituting these values in the above equation and doing some algebraic computation, we obtain that

$$\frac{dV_2}{dt} \leq -\frac{e_1 w_2}{\overline{R}_1}\left(1 - \frac{w_1\overline{s}}{R_1\overline{R}_1}\right)(x-\overline{x})^2 - \frac{w_3}{R_1}\left(\frac{e_1 w_2}{\overline{R}_1} - e_2\right)xi - \left(w_7 - \frac{w_4\overline{s}}{R_2} - \frac{e_1 w_2 w_3\overline{x}}{\overline{R}_1 R_1}\right)i$$

Thus it is easy to verify that $\frac{dV_2}{dt}$ is negative definite in the R_+^3 under the conditions (16a), (16b) and (16c). Therefore, for any initial point in the R_+^3 the solution of system (3) approaches asymptotically to E_2. Thus E_2 is a globally asymptotically stable in the R_+^3, and hence the proof is complete. ■

Theorem (6): Suppose that the positive equilibrium point $E_3 = (x^*, s^*, i^*)$ is locally asymptotically stable, then it's a globally asymptotically stable in the region R_+^3, provided that

$$w_1 s^* + w_3 i^* < w_2 R_1^* \tag{17a}$$
$$q_{12}^{\ 2} < q_{11} q_{22} \tag{17b}$$
$$q_{13}^{\ 2} < q_{11} q_{33} \tag{17c}$$
$$q_{23}^{\ 2} < q_{22} q_{33} \tag{17d}$$

where, $q_{12} = \frac{e_2 w_2 w_3 i^*}{sR_1 R_1^*}$, $q_{13} = \frac{e_1 w_2 w_3}{R_1 R_1^*}$, $q_{22} = \frac{e_2 w_3 x^* i^*}{ss^* R_1^*}$, $q_{23} = \frac{e_2 w_3 x}{ss^* R_1 R_1^*}(w_2 s^* + x^* s)$, $q_{11} = \frac{e_1 w_2}{R_1^*}\left[1 - \frac{w_1 s^*}{R_1 R_1^*} - \frac{w_3 i^*}{R_1 R_1^*}\right]$ and $q_{33} = \frac{w_4 w_5 s^*}{R_2 R_2^*}$ with $R_1^* = w_2 + x^*$, $R_2^* = 1 + w_5 i^*$ while R_1 and R_2 as given in theorem (5).

Proof: Consider the function

$$V_3 = c_1^*\left[x - x^* - x^*\ln\left(\frac{x}{x^*}\right)\right] + c_2^*\left[s - s^* - s^*\ln\left(\frac{s}{s^*}\right)\right] + c_3^*\left[i - i^* - i^*\ln\left(\frac{i}{i^*}\right)\right]$$

where $c_j^*\,;\, j = 1,2,3$ are positive constants to be determined, V_3 is continuously differentiable, positive definite, real valued function with $V_3(x^*, s^*, i^*) = 0$ and $V_3(x,s,i) > 0$ for all $(x,s,i) \neq (x^*, s^*, i^*)$ in the R_+^3. So by differentiate V_3 with respect to the time and then simplifying the resulting terms we obtain that

$$\frac{dV_3}{dt} = -c_1^*\left(1 - \frac{w_1 s^*}{R_1 R_1^*} - \frac{w_3 i^*}{R_1 R_1^*}\right)(x - x^*)^2 - c_2^* \frac{e_2 w_3 x^* i^*}{ss^* R_1^*}(s - s^*)^2 - c_3^* \frac{w_4 w_5 s^*}{R_2 R_2^*}(i - i^*)^2$$
$$- c_1^* \frac{w_3}{R_1}(x - x^*)(i - i^*) + \frac{1}{R_1}\left(c_2^* \frac{w_2}{R_1^*}\left(e_1 w_1 + \frac{e_2 w_3 i^*}{s}\right) - c_1^* w_1\right)(x - x^*)(s - s^*)$$
$$+\left(c_3^* \frac{w_4}{R_2} - c_2^*\left(\frac{w_4}{R_2 R_2^*} - \frac{e_2 w_3 x}{ss^* R_1 R_1^*}\left(w_2 s^* + x^* s\right)\right)\right)(s - s^*)(i - i^*)$$

Consequently by choosing the positive constants as $c_1^* = \frac{e_1 w_2}{R_1^*}, c_2^* = 1$ and $c_3^* = \frac{1}{R_2^*}$, then by substituting these values in the above equation and doing some algebraic computation, we obtain that

$$\frac{dV_3}{dt} = -q_{11}(x - x^*)^2 - q_{22}(s - s^*)^2 - q_{33}(i - i^*)^2$$
$$+ q_{12}(x - x^*)(s - s^*) - q_{13}(x - x^*)(i - i^*) + q_{23}(s - s^*)(i - i^*)$$

Therefore by using the conditions (17a)-(17d), its obtain that

$$\frac{dV_3}{dt} \le -\left[\sqrt{\frac{q_{11}}{2}}(x - x^*) - \sqrt{\frac{q_{22}}{2}}^2 (s - s^*)\right]^2 - \left[\sqrt{\frac{q_{11}}{2}}(x - x^*) + \sqrt{\frac{q_{33}}{2}}^2 (i - i^*)\right]^2$$
$$- \left[\sqrt{\frac{q_{22}}{2}}(s - s^*) - \sqrt{\frac{q_{33}}{2}}^2 (i - i^*)\right]^2$$

Clearly, $\frac{dV_3}{dt}$ is negative definite in the R_+^3. Therefore, for any initial point in the R_+^3 the solution of system (3) approaches asymptotically to E_3. Thus E_3 is a globally asymptotically stable in the R_+^3, and hence the proof is complete. ■

V. Local Bifurcation

It is well known that the bifurcation occurs if and only if there is a qualitative change in the behavior of the solution of a system as variations in the control parameter. Therefore in this section an application to the Sotomoyor's theorem [21] is performed to study the occurrence of local bifurcation near the equilibrium points of system (3). Recall that the fact that the equilibrium point is a non hyperbolic point is a necessary but not sufficient condition for occurrence of local bifurcation in the neighbourhood of that point. Therefore the parameters, which change the equilibrium points from hyperbolic to non hyperbolic equilibrium point, are considered as a candidate bifurcation parameters of system (3) as shown in the next theorems.

Consider the Jacobian matrix of system (3) that is given in Eq.(8), then its easy to verify that for any vector $V = (v_1, v_2, v_3)^T$, we have that

$$D^2 F(s, i, z)(V, V) = (u_{ij})_{3\times 1} \tag{18}$$

here $u_{11} = \left[-2 + \frac{2w_2(w_1 s + w_3 i)}{R_1{}^3}\right] v_1^2 - \frac{2w_1 w_2}{R_1{}^2} v_1 v_2 - \frac{2w_2 w_3}{R_1{}^2} v_1 v_3$,

$$u_{21} = -\frac{2(e_1 w_1 s + e_2 w_3 i) w_2 v_1^2}{R_1{}^3} + \frac{2e_1 w_1 w_2 v_1 v_2}{R_1{}^2} + \frac{2e_2 w_2 w_3 v_1 v_3}{R_1{}^2} - \frac{2w_4 v_2 v_3}{R_2{}^2} + \frac{2w_4 w_5 s v_3^2}{R_2{}^3},$$

$$u_{31} = \frac{2w_4 v_2 v_3}{R_2{}^2} - \frac{2w_4 w_5 s v_3^2}{R_2{}^3},$$

Theorem (7): If the death rate of susceptible predator w_6 passes through the value $w_6^* = \frac{e_1 w_1}{w_2 + 1}$, then system (3) near the predator free equilibrium point E_1 undergoes transcritical bifurcation but neither saddle-node nor pitchfork bifurcation can occur.

Proof: Clearly the Jacobian matrix of system (3) at E_1 with $w_6^* = \frac{e_1 w_1}{w_2 + 1}$ is given by $\widetilde{J}(E_1, w_6^*) = (\widetilde{b}_{ij})$, where $\widetilde{b}_{ij} = b_{ij}$ in $J(E_1)$ and $\widetilde{b}_{22} = 0$, hence E_1 becomes a non-hyperbolic equilibrium point with zero eigenvalue $\widetilde{\lambda}_s = 0$. Let $\widetilde{V} = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3)^T$ be the eigenvector corresponding to the zero eigenvalue $\widetilde{\lambda} = 0$ of the matrix $\widetilde{J}$. Then the system $\widetilde{J}\widetilde{V} = 0$ gives that

$$\widetilde{V} = \left(\frac{w_1}{w_2 + 1} \widetilde{v}_2, \widetilde{v}_2, 0 \right)^T \text{ with } \widetilde{v}_3 \in R \text{ and } \widetilde{v}_3 \neq 0$$

Let $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3)^T$ be the eigenvector corresponding to zero eigenvalue $\widetilde{\lambda} = 0$ of the matrix $\widetilde{J}^T$. Then the system $\widetilde{J}^T \widetilde{\Psi} = 0$ gives that

$$\widetilde{\Psi} = \left(0, \frac{w_7 (w_2 + 1)}{e_2 w_3} \widetilde{\psi}_3, \widetilde{\psi}_3 \right)^T \text{ with } \widetilde{\psi}_3 \in R \text{ and } \widetilde{\psi}_3 \neq 0$$

Now, since $\widetilde{\Psi}^T F_{w_6}(E_1, w_6^*) = 0$, then according to Sotomayor's theorem saddle-node bifurcation can't occur. Further, straightforward computation shows that

$$\widetilde{\Psi}^T [DF_{w_6}(E_1, w_6^*) \widetilde{V}] = -\frac{w_7 (w_2 + 1) \widetilde{v}_2}{e_2 w_3} \widetilde{\psi}_3 \neq 0$$

Also due to Eq. (18) we get that

$$\widetilde{\Psi}^T [D^2 F(E_1, w_6^*)(\widetilde{V}, \widetilde{V})] = \frac{-2 e_1 w_1^2 w_2 w_7}{e_2 w_3 (w_2 + 1)^2} \widetilde{v}_2^2 \widetilde{\psi}_3 \neq 0$$

Therefore transcritical bifurcation takes place but not pitchfork bifurcation and hence the proof is complete. ■

Theorem (8): Assume that

$$\frac{e_2 w_3 \bar{x}}{w_4 \overline{R}_1} < \bar{s} < w_1 \overline{R}_1^{\,2} \tag{19}$$

Thus if the death rate of infected predator w_7 passes through the value $w_7^* = w_4 \bar{s}$, then system (3) near the disease free equilibrium point E_2 undergoes transcritical bifurcation but neither saddle-node nor pitchfork bifurcation can occur.

Proof: Clearly the Jacobian matrix of system (3) at E_2 with $w_7^* = w_4 \bar{s}$ is given by $\overline{J}(E_2, w_7^*) = (\bar{c}_{ij})$ where $\bar{c}_{ij} = c_{ij}$ in $J(E_2)$ with $\bar{a}_{33} = 0$, hence E_2 becomes non-hyperbolic equilibrium point with zero eigenvalue $\bar{\lambda}_z = 0$. Let $\overline{V} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)^T$ be the eigenvector corresponding to $\bar{\lambda}_z = 0$ in $\overline{J}$. Then the system $\overline{J}\overline{V} = 0$ gives that

$$\overline{V} = (\bar{\alpha}_1 \bar{v}_3, \bar{\alpha}_2 \bar{v}_3, \bar{v}_3)^T \text{ with } \bar{v}_3 \in R \text{ and } \bar{v}_3 \neq 0$$

here $\bar{\alpha}_1 = -\frac{\bar{c}_{23}}{\bar{c}_{21}}$ and $\bar{\alpha}_2 = \frac{\bar{c}_{11}\bar{c}_{23} - \bar{c}_{13}\bar{c}_{21}}{\bar{c}_{12}\bar{c}_{21}}$. Clearly condition (19) guarantees that $\bar{\alpha}_1 > 0$ and $\bar{\alpha}_2 < 0$.

Let $\overline{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)^T$ be the eigenvector corresponding to $\bar{\lambda}_z = 0$ in $\overline{J}^T$. Then the system $\overline{J}^T \overline{\Psi} = 0$ gives that

$$\overline{\Psi} = (0, 0, \bar{\psi}_3)^T \text{ with } \bar{\psi}_3 \in R \text{ and } \bar{\psi}_3 \neq 0$$

Now, since $\overline{\Psi}^T F_{w_7}(E_2, w_7^*) = 0$, then according to Sotomayor's theorem saddle-node bifurcation can't occur. Further, straightforward computation shows that

$$\overline{\Psi}^T [DF_{w_7}(E_2, w_7^*) \overline{V}] = -\bar{v}_3 \bar{\psi}_3 \neq 0$$

Also due to Eq. (18) we obtain that

$$\overline{\Psi}^T [D^2 F(E_2, w_7^*)(\overline{V}, \overline{V})] = 2 w_4 (\bar{\alpha}_2 - w_5 \bar{s}) \bar{v}_3^2 \bar{\psi}_3 \neq 0$$

Therefore transcritical bifurcation takes place but not pitchfork bifurcation and hence the proof is complete. ■

Theorem (9): Assume that the conditions (13a) and (13b) hold along with the following conditions

$$(w_5 w_6 + w_4)s^* R_1^* < x^*(e_1 w_1 w_5 s^* + e_2 w_3 R_2^*) \tag{20a}$$

$$w_2\left(e_1 w_1 s^* + e_2 w_2 i^*\right)\left(w_3 R_2^* + w_1 w_5 s^*\right) < x^* i^* \left(R_1^{*2} - (w_1 s^* + w_3 i^*)\right)\left[x^*\left(e_1 w_1 w_5 s^* + e_2 w_3 R_2^*\right) - w_4 s^* R_1^*\right] \tag{20b}$$

Thus as the parameter w_6 passes through the value

$$w_6^* = \frac{\left[x^*\left(e_1 w_1 w_5 s^* + e_2 w_3 R_2^*\right) - w_4 s^* R_1^*\right]}{w_5 s^* R_1^*} + \frac{w_2\left(e_1 w_1 s^* + e_2 w_2 i^*\right)\left(w_3 R_2^* + w_1 w_5 s^*\right)}{w_5 x^* s^* i^* \left((w_1 s^* + w_3 i^*) - R_1^{*2}\right) R_1^*} \tag{20c}$$

system (3) near the positive equilibrium point E_3 undergoes saddle-node bifurcation but neither transcritical nor pitchfork bifurcation can occur.

Proof: Clearly the Jacobian matrix of system (3) at E_3 with $w_6 = w_6^*$, which is positive under the conditions (13a), (20a) and (20b), can be written as $J^*(E_3, w_6^*) = J(E_3)$ with $d_{22}^* = \frac{e_1 w_1 x^*}{R_1^*} - \frac{w_4 i^*}{R_2^*} - w_6^*$, which is negative under the condition (13b). Now straightforward computation shows that A_3 in the characteristic equation (12b) vanishing ($A_3 = 0$) at $w_6 = w_6^*$ and hence the Jacobian matrix $J^*(E_3, w_6^*)$ has zero eigenvalue, say $\lambda^* = 0$. Thus E_3 is a non-hyperbolic point.

Let $V^* = (v_1^*, v_2^*, v_3^*)^T$ be the eigenvector corresponding to $\lambda^* = 0$ in J^*. Then we get that:

$$V^* = (\alpha_1^* v_3^*, \alpha_2^* v_3^*, v_3^*)^T \text{ with } v_3^* \in R \text{ and } v_3^* \neq 0$$

here $\alpha_1^* = \frac{d_{12}^* d_{23}^* - d_{13}^* d_{22}^*}{d_{11}^* d_{22}^* - d_{12}^* d_{21}^*} < 0$ and $\alpha_2^* = \frac{d_{13}^* d_{21}^* - d_{11}^* d_{23}^*}{d_{11}^* d_{22}^* - d_{12}^* d_{21}^*}$. Let $\Psi^* = (\psi_1^*, \psi_2^*, \psi_3^*)^T$ be the eigenvector corresponding to $\lambda^* = 0$ in J^{*T}. Then we obtain that:

$$\Psi^* = (\psi_1^*, \beta_1^* \psi_1^*, \beta_2^* \psi_1^*)^T \text{ with } \psi_1^* \in R \text{ and } \psi_1^* \neq 0$$

here $\beta_1^* = \frac{d_{13}^* d_{32}^* - d_{12}^* d_{33}^*}{d_{22}^* d_{33}^* - d_{23}^* d_{32}^*} > 0$ and $\beta_2^* = \frac{d_{12}^* d_{23}^* - d_{13}^* d_{22}^*}{d_{22}^* d_{33}^* - d_{23}^* d_{32}^*} > 0$. Now since

$$\Psi^{*T} F_{w_6}(E_3, w_6^*) = -\beta_1^* s^* \psi_1^* \neq 0$$

Then according to Sotomayor's theorem the first condition of saddle-node bifurcation is satisfied. Moreover, due to Eq. (18), we have

$$\Psi^{*T}[D^2 F(E_3, w_6^*)(V^*, V^*)] = 2 v_3^{*2} \psi_1^* \left[-\alpha_1^{*2} + \frac{w_1 w_2 \alpha_1^*}{R_1^{*3}}\left(1 - e_1 \beta_1^*\right)\left(\alpha_1^* s^* - \alpha_2^* R_1^*\right) + \frac{w_2 w_3 \alpha_1^*}{R_1^{*3}}\left(1 - e_2 \beta_1^*\right)\left(\alpha_1^* i^* - R_1^*\right) + \frac{w_4}{R_2^{*2}}\left(\beta_2^* - \beta_1^*\right)\left(\alpha_2^* - w_5 s^*\right)\right]$$

Straightforward computation shows that $\Psi^{*T}[D^2 F(E_3, w_6^*)(V^*, V^*)] \neq 0$ under the above conditions. Thus saddle node bifurcation occurs but neither transcritical nor pitchfork bifurcation can occur. ■

VI. Numerical Simulation

In this section, the global dynamics and the possibility of the occurrence of bifurcation of system (3) are investigated numerically. The objective is to confirm our obtained analytical outcomes. Consequently, for the following hypothetical set of biologically feasible data, system (3) is solved numerically starting at different sets of initial conditions and then the obtained trajectories along with their time series are drawn in Fig. (1) and Fig. (2) respectively.

$$w_1 = 0.3, w_2 = 0.2, w_3 = 0.3, w_4 = 0.75, w_5 = 0.5, w_6 = 0.05, w_7 = 0.1, e_1 = 0.5, e_2 = 0.4 \tag{21}$$

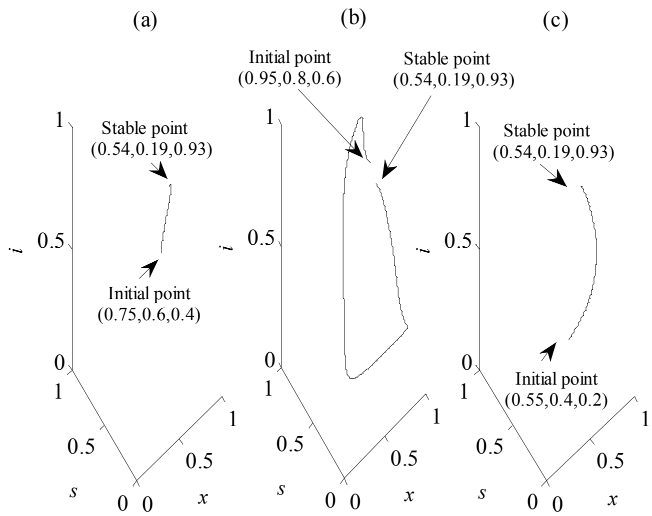

Fig.1: Globally asymptotically stable positive equilibrium point $E_3 = (0.54,0.19,0.93)$ of system (3) at the set of data (21); (a) Starting from $(0.75,0.6,0.4)$. (b) Starting from $(0.95,0.8,0.6)$. (c) Starting from $(0.55,0.4,0.2)$.

According to these figures for the data given by Eq. (21), system (3) has a globally asymptotically stable positive equilibrium point that shows the persistence of the system in the form of point attractor.

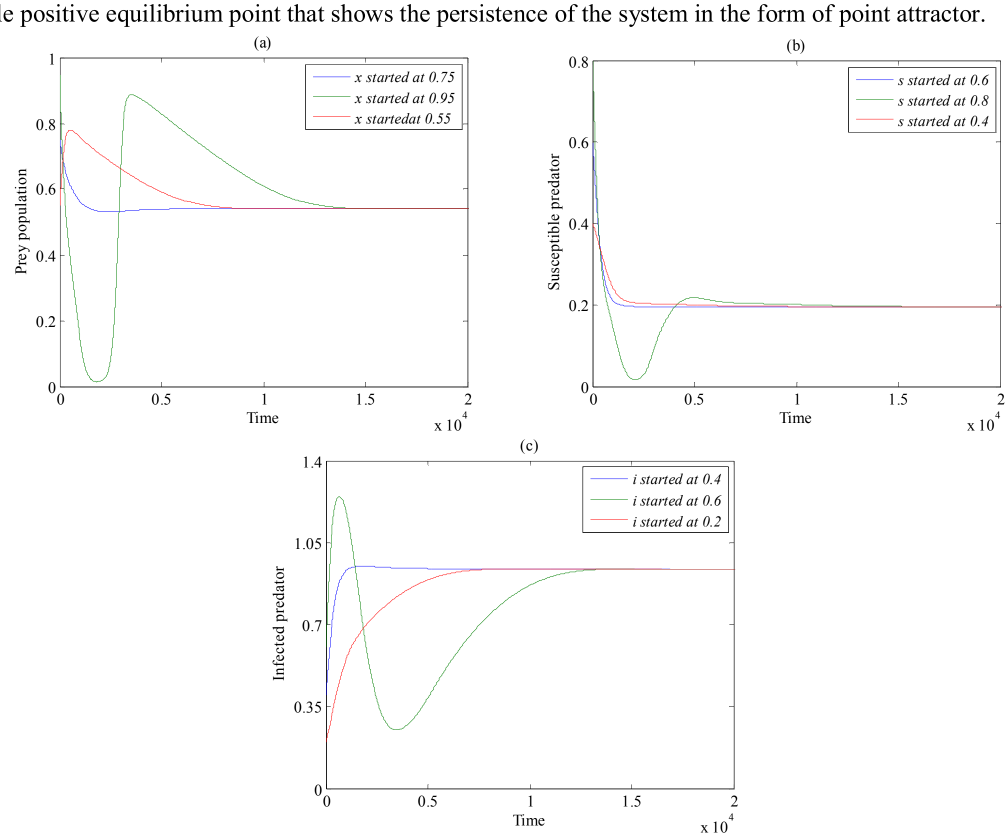

Fig. 2: Time series of the point attractor given in Fig. (1); (a) Trajectories of prey population starting from different points. (b) Trajectories of susceptible predator population starting from different points. (c) Trajectories of infected predator population starting from different points.

Now in order to investigate the occurrence of bifurcation in system (3), the numerical solution of system (3) is determined as a function of one parameter at a time and then the obtained trajectory is drawing in a form of 3D attractor and / or time series as shown below. So by varying the parameter w_1 in the range

$0 < w_1 \le 0.12$, it is observed that the solution of system (3) approaches asymptotically to the predator free equilibrium point E_1 as shown in the typical figure given by Fig. (3). However varying the parameter w_1 in the range $0.46 \le w_1 < 1.55$ leads to approaching to the periodic dynamics in the interior of R_+^3 as shown in the typical figure given by Fig. (4). Moreover varying this parameter in the range $w_1 \ge 1.55$ causes extinction in the infected population and the solution approaches asymptotically to the periodic dynamics in the interior of $xs-$plane as shown in the typical figure given by Fig. (5).

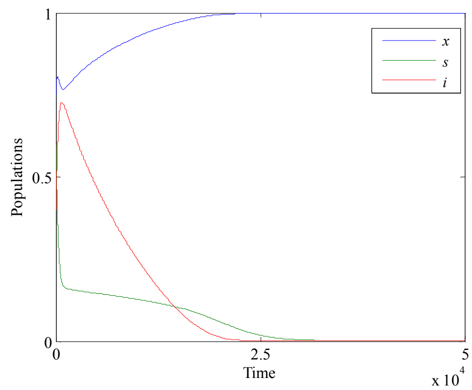

Fig. 3: The time series of the trajectory of system (3) for the data given in Eq. (21) with $w_1 = 0.05$ in which the solution approaches asymptotically to $E_1 = (1,0,0)$.

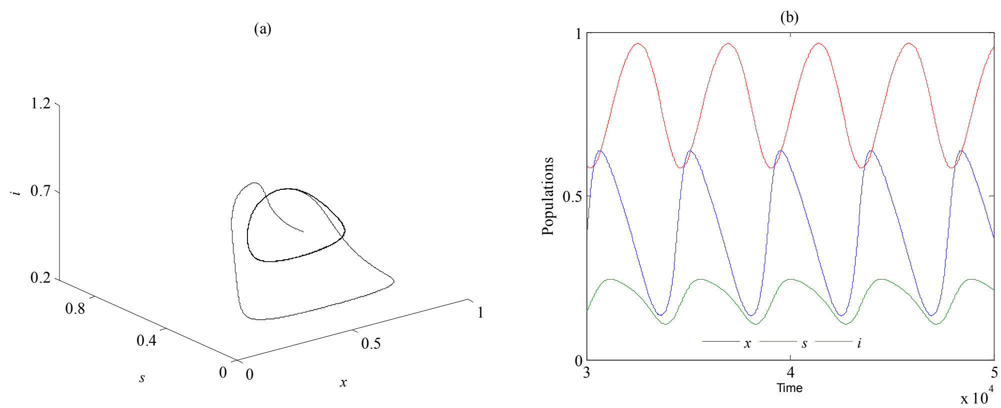

Fig. 4: The trajectory of system (3) for the data given in Eq. (21) with $w_1 = 0.5$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

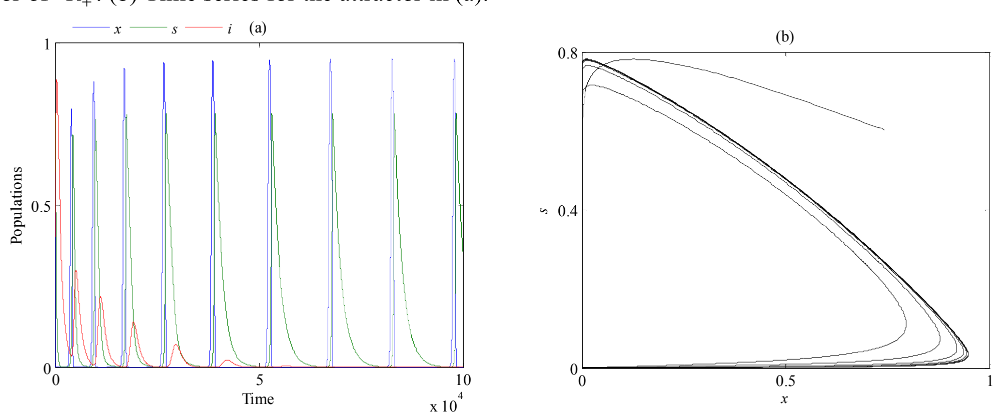

Fig. 5: The trajectory of system (3) for the data given in Eq. (21) with $w_1 = 1.75$. (a) The time series of the periodic attractor. (b) Periodic attractor in the interior of $xs-$plane.

Now, varying the parameter w_2 in the range $0 < w_2 \le 0.15$ leads to destabilizing of system (3) and the solution approaches asymptotically to the periodic dynamics in the interior of R_+^3 as explained in the typical figure given by Fig. (6). However by varying the parameter w_2 in the range $w_2 \ge 1.9$, it is observed that the solution approaches to the predator free equilibrium point E_1.

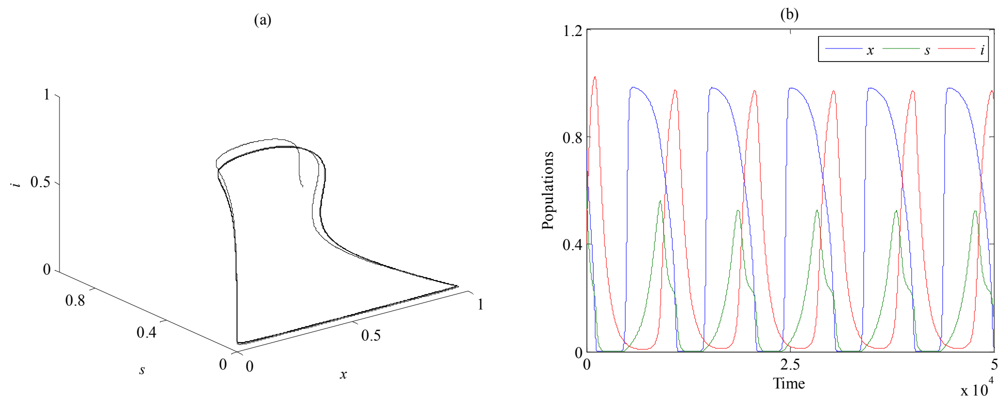

Fig. 6: The trajectory of system (3) for the data given in Eq. (21) with $w_2 = 0.1$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

Moreover, varying the parameter w_3 in the range $w_3 \ge 0.34$ leads to losing the stability at the positive equilibrium point and the solution approaches asymptotically to the periodic dynamics in the interior of R_+^3 as explained in the typical figures given by Fig. (7) and Fig. (8) respectively. Clearly these figures show that increasing the value of w_3 leads to increase in the size of periodic, which indicate to occurrence of Hopf bifurcation.

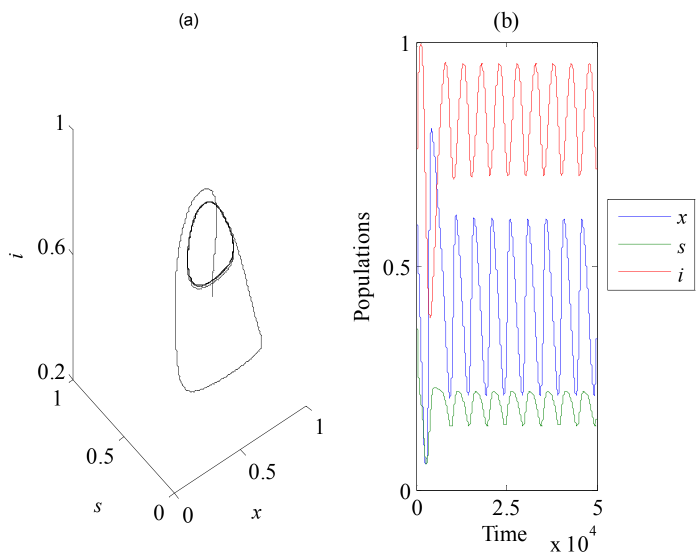

Fig. 7: The trajectory of system (3) for the data given in Eq. (21) with $w_3 = 0.34$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

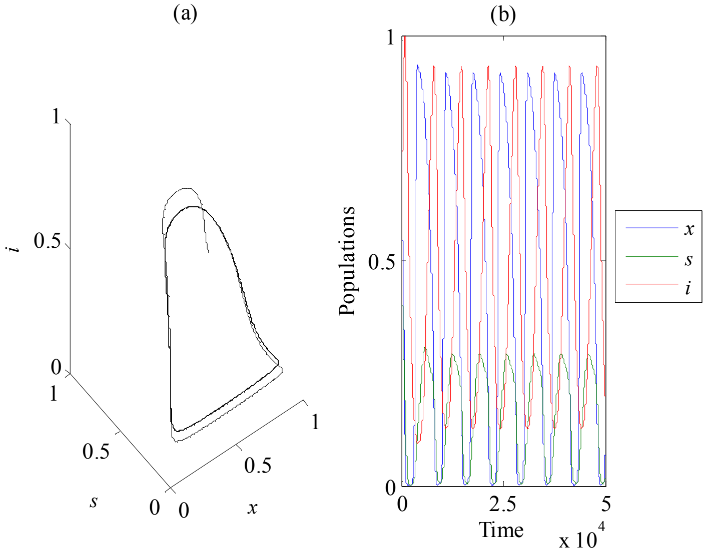

Fig. 8: The trajectory of system (3) for the data given in Eq. (21) with $w_3 = 0.4$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

Now varying the parameter w_4 in the range $0.14 < w_4 \le 0.43$ causes occurrence of Hopf bifurcation too, as shown in the typical figure give by Fig. (9). However varying this parameter in the range $0 < w_4 \le 0.14$ leads to extinction in the infected population and the solution approaches asymptotically to the periodic dynamics in the interior of $xs-$plane as explained in the typical figure given by Fig. (10).

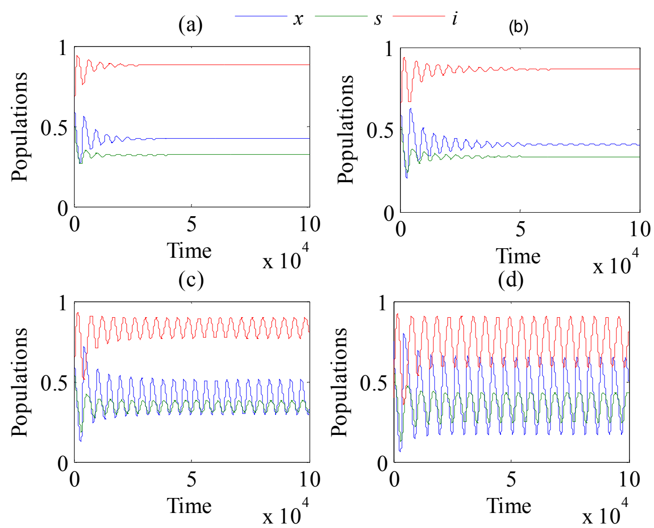

Fig. 9: The time series of the trajectory of system (3) for the data given in Eq. (21) with different values of w_4. (a) Asymptotically stable point for $w_4 = 0.45$. (b) Small periodic attractor for $w_4 = 0.43$. (c) Periodic attractor for $w_4 = 0.41$. (d) Large periodic attractor for $w_4 = 0.39$.

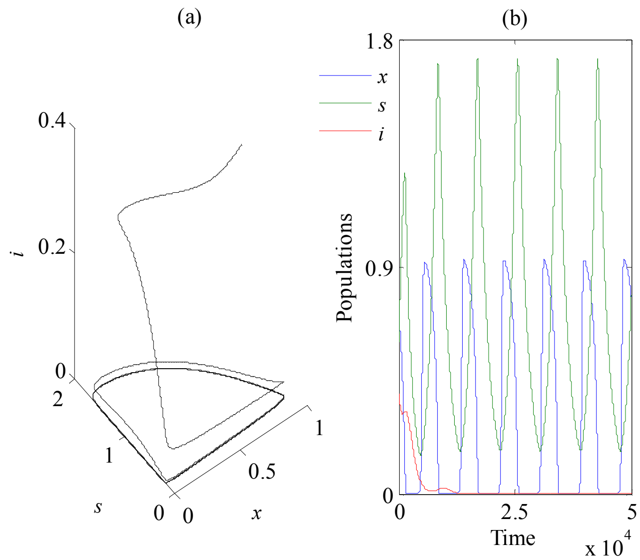

Fig. 10: The trajectory of system (3) for the data given in Eq. (21) with $w_4 = 0.1$. (a) Periodic attractor in the interior of $xs-$plane. (b) Time series for the attractor in (a).

Similarly varying the parameter w_5 in the range $w_5 \geq 1.78$ leads to occurrence of Hopf bifurcation and the solution approaches asymptotically to periodic dynamics in the interior of R_+^3, as shown in the typical figure given by Fig. (11).

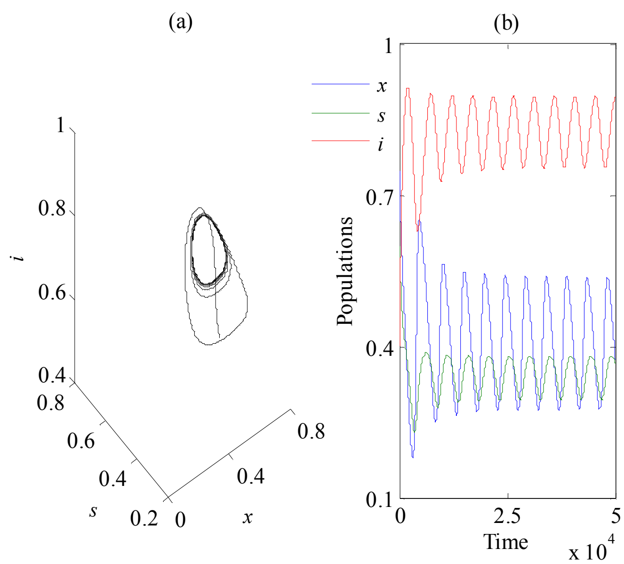

Fig. 11: The trajectory of system (3) for the data given in Eq. (21) with $w_5 = 1.9$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

Now varying the parameter w_6 in the range $w_6 \geq 0.13$ leads to losing the stability of the positive equilibrium point and the solution approaches asymptotically to the predator free equilibrium point E_1 as explained in the typical figure given by Fig. (12).

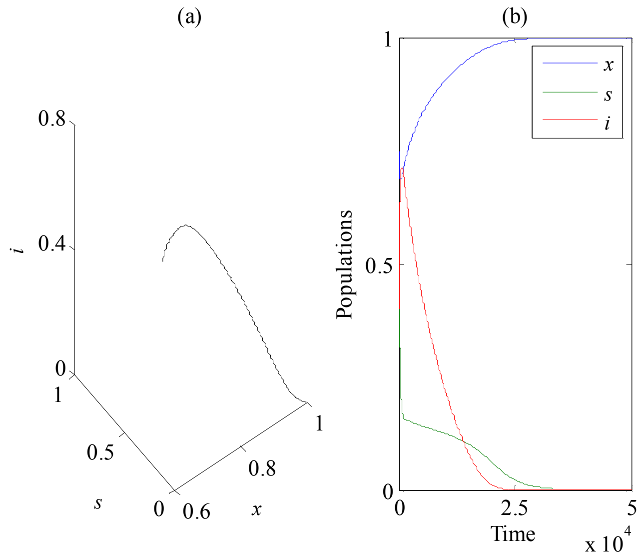

Fig. 12: The trajectory of system (3) for the data given in Eq. (21) with $w_6 = 0.15$. (a) E_1 is asymptotically stable point of system (3). (b) Time series for the attractor in (a).

Moreover varying the parameter w_7 in the range $0 < w_7 \leq 0.08$ leads to occurrence of Hopf bifurcation in the interior of R_+^3 as shown in the typical figure given by Fig. (13). On the other hand varying this parameter in the range $w_7 \geq 0.69$ causes extinction in the infected population and the solution approaches asymptotically to periodic dynamics in the interior of $xs-$plane as shown in the typical figure represented by Fig. (14).

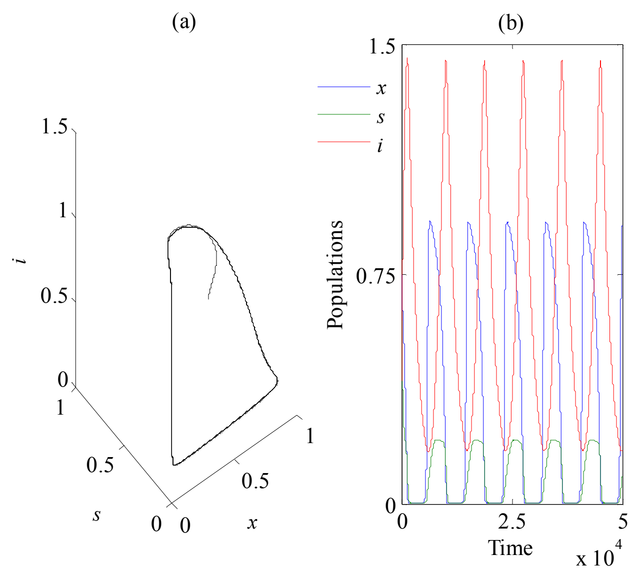

Fig. 13: The trajectory of system (3) for the data given in Eq. (21) with $w_7 = 0.05$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

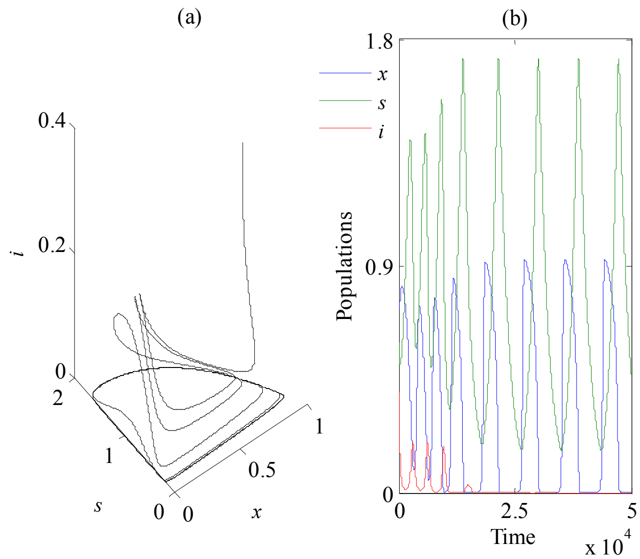

Fig. 14: The trajectory of system (3) for the data given in Eq. (21) with $w_7 = 0.69$. (a) Periodic attractor in the interior of $xs-$plane. (b) Time series for the attractor in (a).

Now varying the parameter e_1 in the range $0 < e_1 \leq 0.2$ leads to extinction in predator species and the solution approaches asymptotically to the predator free equilibrium point E_1 as shown in the typical figure represented by Fig. (15). However varying this parameter in the range $e_1 \geq 0.75$ causing destabilizing of the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in the interior of R^3_+, which indicates to occurrence of Hopf bifurcation, as shown in the typical figure given by Fig. (16).

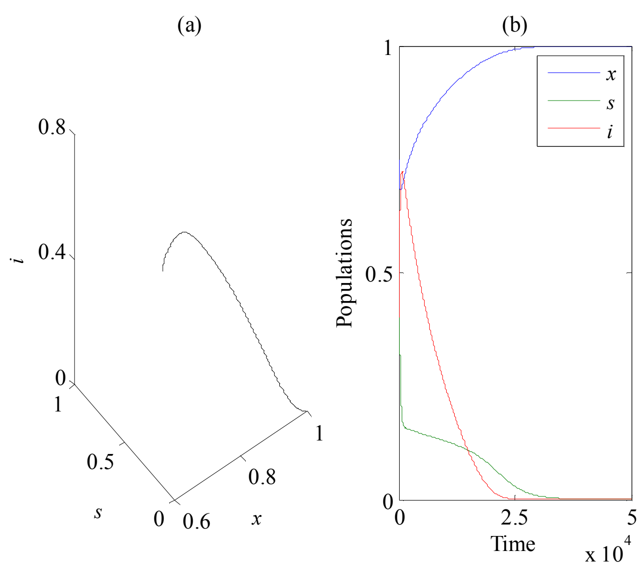

Fig. 15: The trajectory of system (3) for the data given in Eq. (21) with $e_1 = 0.1$. (a) E_1 is asymptotically stable point of system (3). (b) Time series for the attractor in (a).

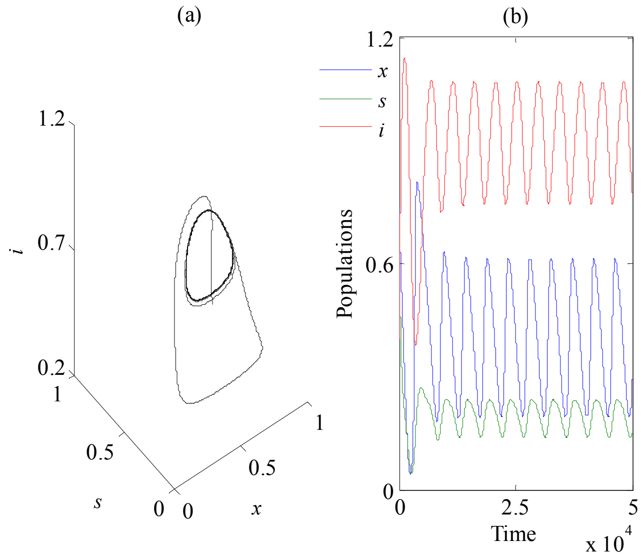

Fig. 16: The trajectory of system (3) for the data given in Eq. (21) with $e_1 = 0.8$. (a) Periodic attractor in the interior of R_+^3. (b) Time series for the attractor in (a).

Finally varying the parameter e_2 in the range $e_2 \geq 0.46$ leads to destabilizing the positive equilibrium point and the solution of system (3) approaches to the periodic dynamics in the interior of R_+^3 as shown in the typical figure represented by Fig. (17).

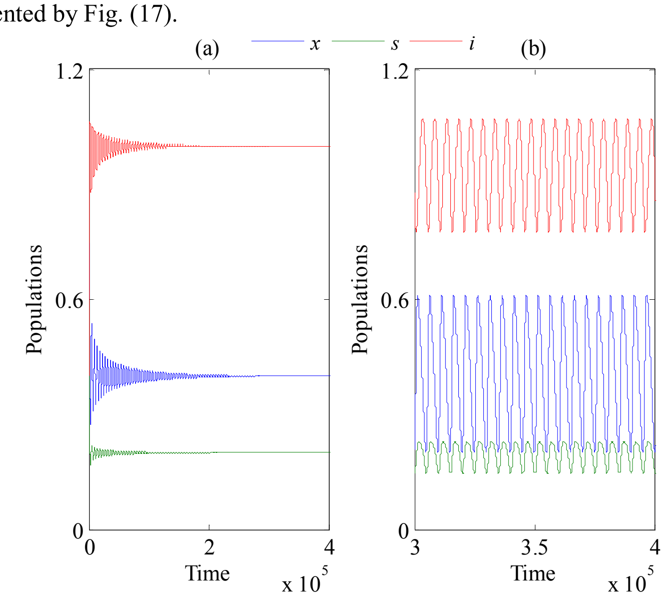

Fig. 17: The time series of the trajectory of system (3) for the data given in Eq. (21) with different values of e_2. (a) Asymptotically stable point for $e_2 = 0.45$. (b) Periodic attractor for $e_2 = 0.46$.

Keeping the above in view, it is observed that for the data given by Eq. (21) with $w_1 = 0.2$, $w_2 = 0.4$, $w_4 = 0.15$ and $w_7 = 0.4$ the solution of system (3) approaches asymptotically to the disease free equilibrium point E_2 as shown in the typical figure given by Fig. (18).

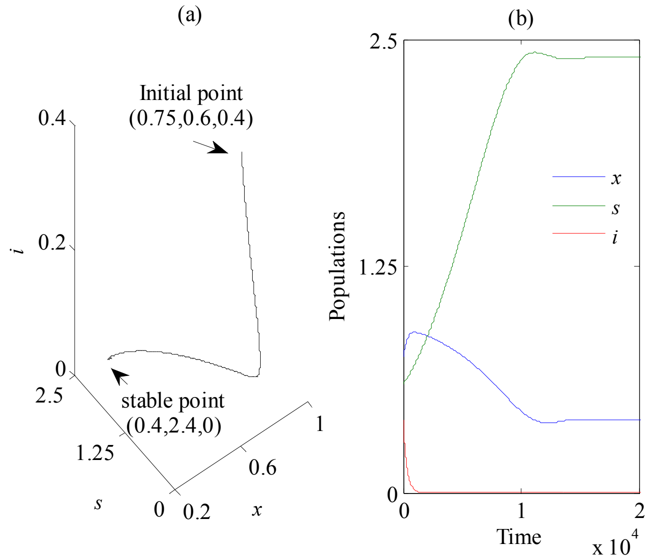

Fig. 18: The trajectory of system (3) for the data given in Eq. (21) with $w_1 = 0.2$, $w_2 = 0.4$, $w_4 = 0.15$ and $w_7 = 0.4$. (a) E_2 is asymptotically stable point of system (3). (b) Time series for the attractor in (a).

VII. Conclusions And Discussion

In this paper an eco-epidemiological model, consisting of a prey – predator system with disease in predator, has been proposed and analyzed analytically as well as numerically. It is assumed that the disease is horizontally transmitted within predator population. It is observed that this model has at most four nonnegative equilibrium points. The local and global stability of all possible equilibrium points are investigated. The conditions that guarantee the persistence of the model are established. The local bifurcation analyses near the equilibrium points are carried out. Finally numerical simulation is used to investigate the global dynamics of the model and specify the set of control parameters in the proposed model in addition to occurrence of Hopf bifurcation. It is observed that, for the chosen hypothetical set of data given by Eq. (21), system (3) is rich in the dynamics and sensitive to varying in their parameters. Indeed the system undergoes different types of bifurcations including Hopf bifurcation. In the following we summarize the obtained results for the data given by Eq. (21):

1. The system (3) has only two types of attractors: asymptotically stable point and periodic dynamics.
2. For the set of data given in Eq. (21), system (3) has a globally asymptotically stable positive equilibrium point and hence the system is persists.
3. Decreasing the susceptible predator attack rate (w_1) under a specific value causes extinction in predator species and the system approaches to predator free equilibrium point. However increasing this parameter above a specific value leads to destabilized the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in interior of R_+^3, which indicates to occurrence of Hopf bifurcation and persistence of the system in the form of periodic attractor. Further increasing the value of this parameter above another specific value causes extinction in the infected population and the system approaches asymptotically to a periodic dynamics in the interior of xs – plane.
4. Decreasing the predator half saturation parameter (w_2) under a specific value leads to destabilized the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in interior of R_+^3, which indicates to occurrence of Hopf bifurcation and persistence of the system in the form of periodic attractor. However increasing this parameter above a specific value causes extinction in predator species and the system approaches to predator free equilibrium point.
5. Decreasing the infected predator attack rate (w_3) do not has effects on the dynamics of system (3) and the solution still approaches to a positive equilibrium point. However increasing this parameter above a specific value leads to destabilized the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in interior of R_+^3, which indicates to occurrence of Hopf bifurcation and persistence of the system in the form of periodic attractor.

6. Decreasing the predator infected rate (w_4) under a specific value leads to destabilized the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in interior of R_+^3, which indicates to occurrence of Hopf bifurcation and persistence of the system in the form of periodic attractor. Further decreasing this parameter under another specific value causes extinction in the infected population and the system approaches asymptotically to a periodic dynamics in the interior of $xs-$plane. However increasing this parameter has no effects on the dynamics of the system and the solution still approaches to positive equilibrium point.
7. It is observed that varying the disease's inhibitory effect rate represented by w_5 and the infected predator conversion rate e_2 have the same effect on the dynamical behavior of system (3) as that occurred by (w_3).
8. Decreasing the susceptible predator death rate (w_6) do not has effects on the dynamics of system (3) and the solution still approaches to a positive equilibrium point. However increasing this parameter above a specific value leads to extinction in predator species and then the system changes its stability from the positive equilibrium point to the predator free equilibrium point, which means a bifurcation in the system take place.
9. Decreasing the infected predator death rate (w_7) under a specific value leads to destabilized the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in interior of R_+^3, which indicates to occurrence of Hopf bifurcation and persistence of the system in the form of periodic attractor. However increasing this parameter above a specific value causes extinction in the infected population and the system approaches asymptotically to a periodic dynamics in the interior of $xs-$plane.
10. Decreasing the susceptible predator conversion rate (e_1) under a specific value causes extinction in predator species and the system approaches to predator free equilibrium point. However increasing this parameter above a specific value leads to destabilized the positive equilibrium point and the solution of system (3) approaches asymptotically to periodic dynamics in interior of R_+^3, which indicates to occurrence of Hopf bifurcation and persistence of the system in the form of periodic attractor.
11. Finally, for the data in Eq. (21) with $w_1 = 0.2$, $w_2 = 0.4$, $w_4 = 0.15$ and $w_7 = 0.4$, the solution of system (3) approaches asymptotically to the disease free equilibrium point, which indicates to losing the persistence of the system too.

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