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Some fixed point theorems in complete fuzzy 2 - metric spaces

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Abstract: *The notion of 2 - metric spaces was introduced by Gähler in the year 1963 and since then researchers are trying to study the concept of 2 - metric spaces in the fuzzy structures. Very recently , Dey and Saha made a very good contribution in the form of a book to study fixed point theory in 2 - metric spaces. In the present paper , we state and prove some fixed point theorems on fuzzy 2 - metric spaces due to Sharma by introducing the notion of ε - chain and (ε, λ) uniformly locally contractive mappings on fuzzy 2 - metric spaces. Our results extend the famous fixed point theorems due to R. Caccioppoli and M. Edelstein on classical metric spaces. We prove an important Lemma and deduce the Banach contraction theorem on fuzzy 2 - metric spaces as a corollary and also illustrate our results with examples.*

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I. Introduction

The concept of fuzzy metric spaces was introduced by Kramosil and Michalek [21] in the year 1975 by generalizing the notion of probabilistic metric spaces , introduced by Menger [22] to fuzzy settings. Deng [8 ,9] introduced the notion of fuzzy pseudo - metric spaces with the metric defined between two fuzzy points and developed its properties in the year 1982. Later in the year 1994 , George and Veeramani [14] modified the notion of fuzzy metric spaces of Kramosil and Michalek with the help of continuous t - norm and obtained Hausdorff topology for this kind of fuzzy metric spaces. It may be recalled that there are several notions of fuzzy metric spaces introduced by various authors such as [8 ,11 ,20 ,21]. In recent years , the study of fixed point theorems satisfying some contractive - type conditions has been given immense interest and a large number of research papers devoted to the development of fixed point theorems and their applications appeared in the literature. To mention a few , we cite [16 , 17, 19 ,23]. In particular , Grabiec [15] extended the Banach contraction theorem and Edelstein contraction theorem on classical metric spaces to fuzzy metric spaces in the year 1988. Cho and Jung [2] proved a common fixed point theorem for four weak compatible mappings of an ε - chainable fuzzy metric space. For examples and elementary properties of fuzzy metric spaces , we refer to [25]. Recent literature on fixed point theory in fuzzy metric spaces may also be viewed in [5, 6, 7, 24].

Gähler investigated 2 - metric spaces in a series of his papers [12 ,13]. It may be recalled that a 2 - metric d [12] on a non - empty set X is a function $d : X^3 \to \mathbb{R}$ satisfying some conditions that are analogous to the area function in Euclidean spaces. Sharma , Sharma and Iseki [28] investigated , for the first time , the contraction type mappings in the 2 - metric spaces. A brief account of evolution of fixed point theory in 2 - metric spaces may also be viewed in [10]. This motivated the idea of 2 - metric spaces in the fuzzy settings. Sharma [27] proved an interesting common fixed point theorem for three mappings in fuzzy 2 - metric spaces. Han [18] extended the results of Cho [1] to fuzzy 2 - metric spaces which are a generalization of the results due to Sharma [27]. Das and Saha [3] proved the Banach contraction Theorem and Edelstein contraction Theorem in fuzzy 2 - metric spaces which are extensions of the results in fuzzy metric spaces due to Grabiec [15].

Motivated by our earlier work [3] , we have extended the Theorems due to Das and Saha [4] to fuzzy 2 - metric spaces. The structure of the paper is as follows. After the preliminaries in section 2 , we prove an important Lemma and then extend the Caccioppoli fixed point theorem and Edelstein fixed point theorem to fuzzy 2 - metric spaces in section 3. We also deduce the Banach contraction theorem in fuzzy 2 - metric spaces [3] as a corollary and construct examples to illustrate our results.

II. Preliminaries

In this section , we recall some definitions and known results which are already in the literature.

Definition 2.1.[29] A **fuzzy set** A in X is a mapping $A: X \to [0,1]$. For $x \in X$, $A(x)$ is called the grade of membership of x.

Definition 2.2. [26] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a **continuous t-norm** , if $([0,1], *)$ is an abelian topological monoid with unity 1 such that $a * b \leq c * d$, whenever $a \leq c$, $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.3. [27] The 3 - tuple $(X, M,*)$ is called a **fuzzy 2 - metric space** , if X is an arbitrary set , $*$ is a continuous t - norm and M is a fuzzy set in $X^3 \times [0,\infty[$, satisfying the following conditions:
For all $x, y, z, u \in X$, and $t,\ t_1, t_2, t_3 > 0$, $M(x, y, z, 0) = 0$, (2.3.1) $M(x, y, z, t) = 1$, for all $t > 0$, if and only if at least two of x, y, z are equal , (2.3.2) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) = M(y, x, z, t) = M(z, x, y, t) = M(z, y, x, t)$,(2.3.3) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$, (2.3.4) $M(x, y, z, .): [0,\infty[\to [0,1]$ is left continuous , and (2.3.5) $\lim_{t\to\infty} M(x, y, z, t) = 1$. (2.3.6)

Example 2.4. [27] Let (X, d) be a 2 - metric space and $a * b = ab$, for every $a, b \in [0,1]$. Let M_d be a fuzzy set in $X^3 \times [0,\infty[$, given by $M_d(x, y, z, t) = \frac{t}{t+d(x,y,z)}$, if $t > 0$ and $M_d(x, y, z, 0) = 0$. Then $(X, M_d ,*)$ is a fuzzy 2 - metric space and M_d is called the **standard fuzzy 2 - metric** induced by the 2 - metric d. Thus every 2 - metric d induces a fuzzy 2 - metric M_d on X.

Definition 2.5. [27]A sequence $\{x_n\}$ in a fuzzy 2 - metric space $(X, M,*)$ is said to be
(a) a **Cauchy sequence** , if $\lim_{n\to\infty} M\left(x_n, x_{n+p}, z, t\right) = 1$, for all $z \in X$, $t > 0$, $p > 0$.
(b) **convergent** to $x \in X$ (in symbols , $\lim_{n\to\infty} x_n = x$ or $x_n \to x$) , if
$\lim_{n\to\infty} M(x_n, x, z, t) = 1$, for all $z \in X$, $t > 0$.

Definition 2.6. [27] A fuzzy 2 - metric space $(X, M,*)$ is said to be **complete** , if every Cauchy sequence in X is convergent.

Lemma 2.7.(a) [18] $M(x, y, z, .)$ is non - decreasing for all $x, y, z \in X$.
(b) [18] Let $(X, M,*)$ be a fuzzy 2 - metric space. Let there exists $q \in]0,1[$ such that $M(x, y, z, qt) \geq M(x, y, z, t)$, for all $x, y, z \in X$, $t > 0$. If $z \neq x,\ z \neq y$, then $x = y$.

Theorem 2.8. (**Caccioppoli**) Let (X, ρ) be a complete metric space and $T: X \to X$ be a mapping. Suppose for each positive integer n , $\rho(T^n x, T^n y) \leq a_n \rho(x, y)$, for all $x, y \in X$ and $a_n > 0$ is independent of x , y. If the series $\sum a_n$ is convergent , then T has a unique fixed point.

Theorem 2.9. Let (X, ρ) be a complete metric space and $T: X \to X$ be a continuous mapping. If for some positive integer m , T^m is a contraction mapping , i.e., $\rho(T^m x, T^m y) \leq \alpha\rho(x, y)$, for all x , $y \in X$ and some $0 < \alpha < 1$, then T has a unique fixed point.

Theorem 2.10. (**Edelstein**) Let (X, ρ) be a complete , ε - chainable metric space and $T: X \to X$ be (ε, λ) uniformly locally contractive. Then there exists a unique fixed point of T.

Theorem 2.11. [3] Let $(X, M,*)$ be a complete fuzzy 2 - metric space and $T: X \to X$ be a contraction , i.e., a mapping satisfying $M(Tx, Ty, z, kt) \geq M(x, y, z, t)$, (2.11.1)
for all x , y , $z \in X$, $t > 0$ and some $0 < k < 1$. Then T has a unique fixed point.

III. Main Results

In this section , we state and prove the main results of our paper. We extend the Theorems 2.8 , 2.9 and 2.10 to fuzzy 2 - metric spaces and deduce the Theorem 2.11 as a Corollary. We also illustrate our results with suitable examples. Before we proceed , let us state and prove an important Lemma which is used in all of our theorems.

Lemma 3.1. If $\{x_n\}$ is a sequence in a fuzzy 2 - metric space $(X, M,*)$, then for all $z \in X$, $t > 0$, $p > 0$,

$$M(x_n , x_{n+p}, z, t) \geq M(x_n, x_{n+1}, x_{n+p} , \tfrac{t}{2(p-1)+1}) * M(x_{n+1}, x_{n+2}, x_{n+p}, \tfrac{t}{2(p-1)+1}) * \ldots * M(x_{n+p-2}, x_{n+p-1}, x_{n+p}, \tfrac{t}{2(p-1)+1}) * M(x_n, x_{n+1}, z , \tfrac{t}{2(p-1)+1}) * M(x_{n+1}, x_{n+2}, z , \tfrac{t}{2(p-1)+1}) * \ldots * M(x_{n+p-1}, x_{n+p}, z , \tfrac{t}{2(p-1)+1}) * M(x_{n+p-1}, x_{n+p}, z , \tfrac{t}{2(p-1)+1}) \quad (3.1.1)$$

Proof. We shall prove the result (3.1.1) by induction on p. As $M(x, y, z, .)$ is non - decreasing and $a * a \leq a$, we get , $M(x_n , x_{n+1}, z, t) \geq M(x_n , x_{n+1}, z, \frac{t}{2})$

$$\geq M\left(x_n , x_{n+1}, z, \tfrac{t}{2}\right) * M(x_n , x_{n+1}, z, \tfrac{t}{2}).$$

Therefore , (3.1.1) holds for $p = 1$. To apply Induction , let $p > 1$ and the result (3.1.1) hold for every $j < p$. We get ,

$M(x_n , x_{n+p}, z, t) \geq M(x_n, x_{n+1}, x_{n+p} , t_0) * M(x_n, x_{n+1}, z, t_0) * M\left(x_{n+1}, x_{n+p}, z, t_1\right)$,
where $t_0 = \frac{t}{2p-1}$, $t_1 = t\frac{2p-3}{2p-1}$.

$\geq M(x_n, x_{n+1}, x_{n+p} , t_0) * M(x_n, x_{n+1}, z, t_0) * M\left(x_{n+1}, x_{n+2}, x_{n+p}, t_0\right) * M\left(x_{n+2}, x_{n+3}, x_{n+p}, t_0\right) *\ldots \ldots * M(x_{n+p-2}, x_{n+p-1}, x_{n+p}, t_0) * M(x_{n+1}, x_{n+2}, z , t_0) * M(x_{n+2}, x_{n+3}, z , t_0) *\ldots \ldots * M(x_{n+p-1}, x_{n+p}, z , t_0) * M(x_{n+p-1}, x_{n+p}, z , t_0)$, (by Induction hypothesis).

$= M(x_n, x_{n+1}, x_{n+p} , t_0) * M(x_{n+1}, x_{n+2}, x_{n+p}, t_0) * \ldots * M(x_{n+p-2}, x_{n+p-1}, x_{n+p}, t_0) * * M(x_n, x_{n+1}, z , t_0) * M(x_{n+1}, x_{n+2}, z , t_0) * \ldots * M(x_{n+p-1}, x_{n+p}, z , t_0) * M(x_{n+p-1}, x_{n+p}, z , t_0)$.

Therefore , the result (3.1.1) holds for p and hence by Induction , the result follows. □

3.2. Extension of R. Caccioppoli′s fixed point Theorem to fuzzy 2 - metric spaces

We now state and prove the fuzzy analogue of R. Caccioppoli′s fixed point Theorem (Theorem 2.8) in fuzzy 2 - metric spaces. Here we deduce the fuzzy Banach Contraction Theorem in fuzzy 2 - metric spaces [3] as a Corollary and also illustrate our results with an Example.

Theorem 3.3. *If $(X, M,*)$ is a complete fuzzy 2- metric space and $T: X \to X$ is a mapping satisfying the followings:*

For every positive integer n , $M(T^n x, T^n y, z, k_n t) \geq M(x, y, z, t)$, (3.3.1)

for all $x, y, z \in X$, $t > 0$ and $k_n > 0$ being independent of x, y and *if $k_n \to 0$, then T has a unique fixed point.*

Proof. Let $x \in X$, $x_n = T^n x$, $n \in \mathbb{N}$. Now $\{x_n\}$ is a sequence of points of X such that $x_1 = Tx$, $x_2 = Tx_1$, … … , $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. We get by Lemma (3.1):

For all $z \in X$, $t > 0$, $p > 0$,

$1 \geq M(x_n, x_{n+p}, z, t) \geq M(x_n, x_{n+1}, x_{n+p}, t_0) * M(x_{n+1}, x_{n+2}, x_{n+p}, t_0) * \ldots \ldots *$
$M(x_{n+p-2}, x_{n+p-1}, x_{n+p}, t_0) * M(x_n, x_{n+1}, z, t_0) * M(x_{n+1}, x_{n+2}, z, t_0) * \ldots \ldots$
$* M(x_{n+p-1}, x_{n+p}, z, t_0) * M(x_{n+p-1}, x_{n+p}, z, t_0)$, where $t_0 = \frac{t}{2(p-1)+1}$.

$\geq M(x, x_1, x_{n+p}, \frac{t_0}{k_n}) * M(x, x_1, x_{n+p}, \frac{t_0}{k_{n+1}}) * \ldots \ldots * M(x, x_1, x_{n+p}, \frac{t_0}{k_{n+p-2}}) * M(x, x_1, z, \frac{t_0}{k_n}) *$
$M(x, x_1, z, \frac{t_0}{k_{n+1}}) * \ldots \ldots * M(x, x_1, z, \frac{t_0}{k_{n+p-1}}) * M(x, x_1, z, \frac{t_0}{k_{n+p-1}})$, by (3.3.1).

$\to 1 * 1 * \ldots \ldots * 1 * 1 * 1 * \ldots \ldots * 1 * 1 = 1$, as $n \to \infty$, by (2.3.6).

Therefore $\lim_{n \to \infty} M(x_n, x_{n+p}, z, t) = 1$, for all $z \in X$, $t > 0$, $p > 0$ and so $\{x_n\}$ is a Cauchy sequence in X. As X is complete , $\exists y \in X$ such that $x_n \to y$ as $n \to \infty$. We have for all $z \in X$, $t > 0$,

$1 \geq M(y, Ty, z, t) \geq M(y, Ty, x_{n+1}, \frac{t}{3}) * M(y, x_{n+1}, z, \frac{t}{3}) * M(x_{n+1}, Ty, z, \frac{t}{3})$
$\geq M(x_{n+1}, y, Ty, \frac{t}{3}) * M(x_{n+1}, y, z, \frac{t}{3}) * M(x_n, y, z, \frac{t}{3k_1})$, by (3.3.1).
$\to 1 * 1 * 1 = 1$, as $n \to \infty$.

This gives $M(y, Ty, z, t) = 1$, for every $z \in X$, $t > 0$ and thus $Ty = y$, a fixed point of T. To show uniqueness , let $Tw = w$, for some $w \in X$. We get $T^n y = y$, $T^n w = w$, for every positive integer n. Therefore , for all $z \in X$, $t > 0$, we get ,

$1 \geq M(y, w, z, t) = M(T^n y, T^n w, z, t) \geq M(y, w, z, \frac{t}{k_n}) \to 1$ as $n \to \infty$. Therefore , $M(y, w, z, t) = 1$, for all $z \in X$, $t > 0$. Hence $y = w$ and so the fixed point is unique. □

Remark 3.4. It may be noted that for $n = 1$, the condition (3.3.1) does not reduce to the condition (2.11.1) as $k_1 < 1$ and $k_1 \to 0$ are not assured. But we may deduce the Theorem (2.11) as a Corollary as follows.

Corollary 3.5. *If $(X, M,*)$ is a complete fuzzy 2- metric space and $T: X \to X$ is a contraction , i.e., a mapping satisfying $M(Tx, Ty, z, kt) \geq M(x, y, z, t)$,* (3.5.1)

for all $x, y, z \in X$, $t > 0$ and some $0 < k < 1$, then T has a unique fixed point.

Proof. For any positive integer n , we have, $M(T^n x, T^n y, z, k^n t) \geq M(T^{n-1} x, T^{n-1} y, z, k^{n-1} t)$, by (3.5.1).
$\geq M(T^{n-2} x, T^{n-2} y, z, k^{n-2} t) \geq \ldots \ldots \geq M(x, y, z, t)$, for all $x, y, z \in X$, $t > 0$ and some $0 < k < 1$. Also $k^n \to 0$, as $n \to \infty$. Therefore , by the Theorem (3.3) , T has a unique fixed point. □

The following Example illustrates the Theorem (3.3).

Example 3.6. Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and $d(x, y, z) = 1$, if x, y, z are distinct and $\left\{\frac{1}{n}, \frac{1}{n+1}\right\} \subseteq \{x, y, z\}$; $d(x, y, z) = 0$, otherwise. Now , (X, d) is a complete 2 - metric space. Let M be the standard fuzzy 2 - metric on X induced by d. Then $(X, M,*)$ is a complete fuzzy 2 - metric space. Let $T: X \to X$ be given by $Tx = 1$. Now $M(T^n x, T^n y, z, k^n t) = 1$, for every $x, y, z \in X$, $t > 0$, $0 < k < 1$. Also $k^n \to 0$. Therefore , T satisfies all the conditions of the Theorem (3.3) and has a unique fixed point $1 \in X$. □

3.7. Unique fixed point of a map which is not a Contraction

In the following Theorem , we establish a unique fixed point of a self mapping of a fuzzy 2 - metric space $(X, M,*)$ which is not necessarily a Contraction. This Theorem , in fact , is the extension to fuzzy 2 - metric space of our Theorem [4, Theorem 3.4].

Theorem 3.8. *Let $(X, M,*)$ be a complete fuzzy 2- metric space and $T: X \to X$ be a continuous mapping. Let there exist a positive integer m such that T^m is a contraction , i.e. ,*

$M(T^m x, T^m y, z, kt) \geq M(x, y, z, t)$, *for all $x, y, z \in X$, $t > 0$ and some $0 < k < 1$.* (3.8.1)

If $x_n \to x$, $y_n \to y$ implies $M(x_n, y_n, z, t) \to M(x, y, z, t)$, for all $z \in X$, $t > 0$, (3.8.2)

then T has a unique fixed point.

Proof. We put $B = T^m$. Then for any $x_0 \in X$ and any positive integer n , we have
$M(B^n T x_0, B^n x_0, z, k^n t) \geq M(B^{n-1} T x_0, B^{n-1} x_0, z, k^{n-1} t)$, by (3.8.1).

$$\geq M(B^{n-2}Tx_0, B^{n-2}x_0, z, k^{n-2}t)$$
$$\geq \dots \dots$$
$$\geq M(Tx_0, x_0, z, t) \text{ , for all } z \in X, t > 0. \tag{3.8.3}$$

As B is a contraction , B has a unique fixed point $x \in X$. By the proof of the fuzzy Banach contraction Theorem , we get $B^n x_0 \to x$ as $n \to \infty$. As T is continuous, $TB^n x_0 = B^n Tx_0 \to Tx$. By (3.8.2) , we have

$$M(B^n Tx_0, B^n x_0, z, t) \to M(Tx, x, z, t) \text{ , as } n \to \infty \text{ for all } z \in X, t > 0. \tag{3.8.4}$$

Using (3.8.3) , we get $1 \geq M(B^n Tx_0, B^n x_0, z, t) \geq M(Tx_0, x_0, z, \frac{t}{k^n}) \to 1$, as $n \to \infty$, for all $z \in X, t > 0$.
Therefore , $\lim_{n\to\infty} M(B^n Tx_0, B^n x_0, z, t) = 1$, for all $z \in X, t > 0$. Then by (3.8.4) , we obtain , $M(Tx, x, z, t) = 1$, for all $z \in X, t > 0$. Hence $Tx = x$, a fixed point of T. If $Ty = y$, for some $y \in X$, then $By = T^m y = T^{m-1} y = \dots \quad \dots = y$, and so y is a fixed point of B. Hence $x = y$ and so the fixed point of T is unique. □

3.9. ε - chain , (ε , λ) uniformly locally contractive mapping in a fuzzy 2 - metric space

We now introduce the notions of **ε - chain** and (ε , λ) **uniformly locally contractive mapping** in fuzzy 2 - metric spaces as follows:

Definition 3.10. Let $(X, M,*)$ be a fuzzy 2 - metric space and $\varepsilon > 0$. A finite sequence $x = x_0 , x_1 , \dots \dots , x_{n-1}, x_n = y$ is called an **ε - chain** from x to y if $M(x_{i-1}, x_i, z, t) > 1 - \varepsilon$, for all $z \in X$, $t > 0$, $i = 1 , 2 , \dots \dots , n$. A fuzzy 2 - metric space $(X, M,*)$ is said to be **ε - chainable**, if for every $x, y \in X$, there is an ε - chain from x to y.

Definition 3.11. Let $(X, M,*)$ be a fuzzy 2 - metric space and $T: X \to X$ be a mapping.
(a) T is said to be **continuous** , if for every $x \in X$, $x_n \to x$ implies $Tx_n \to Tx$.
(b) For $\varepsilon > 0$, $0 < \lambda < 1$, T is called (ε , λ) **uniformly locally contractive** if

$$M(x, y, z, t) > 1 - \varepsilon \text{ implies } M(Tx, Ty, z, t) \geq M\left(x, y, z, \frac{t}{\lambda}\right), \tag{3.11.1}$$

for all x , y , $z \in X$, $t > 0$. Clearly a uniformly locally contractive mapping T is continuous.

Example 3.12. Let $(X, M,*)$ be the complete fuzzy 2 - metric space of the Example (3.6) and $T: X \to X$ be given by $Tx = 1$. Let $\varepsilon > 0$, $0 < \lambda < 1$. Now $M(x, y, z, t) > 1 - \varepsilon$ gives x, y, z are distinct and $\left\{\frac{1}{n} , \frac{1}{n+1}\right\} \subseteq \{x, y, z\}$, for some positive integer n. Therefore , $1 = M(Tx, Ty, z, t) \geq M\left(x, y, z, \frac{t}{\lambda}\right)$, for all x , y , $z \in X$, $t > 0$.
Therefore , T is an (ε , λ) uniformly locally contractive map. □

3.13. Extension of M. Edelstein′s fixed point Theorem to fuzzy 2 - metric spaces

Here we shall state and prove a fixed point Theorem in fuzzy 2 - metric spaces using the notions of ε - chain and (ε , λ) uniformly locally contractive mappings .This Theorem extends the fixed point Theorem due to M. Edelstein (Theorem 2.10) to fuzzy 2 - metric spaces.

Theorem 3.14. *If $(X, M,*)$ is a complete , ε - chainable fuzzy 2- metric space and $T: X \to X$ is an (ε, λ) uniformly locally contractive mapping , then T has a unique fixed point.*

Proof. Let $x \in X$ be arbitrarily fixed. Let $Tx \neq x$ (otherwise a fixed point is assured). Let $x = x_0 , x_1 , \dots \quad \dots , x_{n-1}$, $x_n = Tx$ be an ε - chain from x to Tx. We get , $M(x_{i-1}, x_i, z, t) > 1 - \varepsilon$, for all $z \in X, t > 0, i = 1 , 2 , \dots , n$.
Let us first prove the result:

$$M(T^m x_{i-1}, T^m x_i, z, t) \geq M\left(x_{i-1}, x_i, z, \frac{t}{\lambda^m}\right), \tag{3.14.1}$$

for every positive integer m and every $z \in X, t > 0, i = 1 , 2 , \dots \quad \dots , n$. By (3.11.1) we get , $M(Tx_{i-1}, Tx_i, z, t) \geq M(x_{i-1}, x_i, z, \frac{t}{\lambda})$ for all $z \in X$, $t > 0, i = 1 , 2 , \dots \quad \dots , n$. So , (3.14.1) holds for $m = 1$. To apply induction , let $m > 1$ and assume (3.14.1) for all $j < m$. We get ,

$$1 - \varepsilon < M\left(x_{i-1}, x_i, z, \frac{t}{\lambda^m}\right) \leq M\left(T^{m-1}x_{i-1}, T^{m-1}x_i, z, \frac{t}{\lambda}\right) \text{, by induction hypothesis.}$$
$$\leq M(T^m x_{i-1}, T^m x_i, z, t) \text{ , by (3.11.1) ,}$$

for all $z \in X, t > 0, i = 1 , 2 , \dots \quad \dots , n$. Therefore , (3.14.1) holds for m. Hence by induction (3.14.1) holds for all $m \in \mathbb{N}$. Using Lemma (3.1) , we get , for all $z \in X$, $t > 0$,

$$1 \geq M(T^m x, T^{m+1}x, z, t) = M(T^m x_0, T^m x_n, z, t) = M(Bx_0, Bx_n, z, t) \text{ , writing } B = T^m.$$
$$\geq M(Bx_0, Bx_1, Bx_n, t_0) * M(Bx_1, Bx_2, Bx_n, t_0) * \dots \quad \dots * M(Bx_{n-2}, Bx_{n-1}, Bx_n, t_0) *$$
$$M(Bx_0, Bx_1, z, t_0) * M(Bx_1, Bx_2, z, t_0) * \dots \quad \dots * M(Bx_{n-1}, Bx_n, z, t_0)$$
$$* M(Bx_{n-1}, Bx_n, z, t_0) \text{ , where } t_0 = \frac{t}{2(n-1)+1} .$$
$$\geq M(x_0, x_1, Bx_n, \frac{t_0}{\lambda^m}) * M(x_1, x_2, Bx_n, \frac{t_0}{\lambda^m}) * \dots$$
$$\dots * M(x_{n-2}, x_{n-1}, Bx_n, \frac{t_0}{\lambda^m}) * M(x_0, x_1, z, \frac{t_0}{\lambda^m}) * M(x_1, x_2, z, \frac{t_0}{\lambda^m}) * \dots$$

$\ldots * \ M\left(x_{n-1}, x_n, z, \frac{t_0}{\lambda^m}\right) * \ M\left(x_{n-1}, x_n, z, \frac{t_0}{\lambda^m}\right)$, by (3.14.1).
$\rightarrow 1 * 1 * \ldots \ \ldots \ * 1 * 1 * 1 * \ldots \ \ldots \ * 1 * 1 = 1$, as $m \rightarrow \infty$. Therefore,

$$\lim_{m\to\infty} M(T^m x, T^{m+1}x, z, t) = 1 \text{, for all } z \in X, t > 0. \tag{3.14.2}$$

We now get, for all $z \in X$, $t > 0$, $p > 0$, $1 \geq M(T^m x, T^{m+p}x, z, t) \geq M(T^m x, T^{m+1}x, T^{m+p}x, r_0) * M(T^{m+1}x, T^{m+2}x, T^{m+p}x, r_0) * \ldots \ \ldots * M(T^{m+p-2}x, T^{m+p-1}x, T^{m+p}x, r_0) * M(T^m x, T^{m+1}x, z, r_0) * M(T^{m+1}x, T^{m+2}x, z, r_0) * \ldots \ldots * \ M(T^{m+p-1}x, T^{m+p}x, z, r_0) * \ M(T^{m+p-1}x, T^{m+p}x, z, r_0)$, where $r_0 = \frac{t}{2(p-1)+1}$.
$\rightarrow 1 * 1 * \ldots \ \ldots * 1 * 1 * 1 * \ldots \ \ldots * 1 * 1 = 1$, as $m \rightarrow \infty$, by (3.14.2). Hence $\lim_{m\to\infty} M(T^m x, T^{m+p}x, z, t) = 1$, for all $z \in X$, $t > 0$, $p > 0$ and so $\{T^m x\}$ is a Cauchy sequence in X. As X is complete, $\exists y \in X$ such that $\lim_{m\to\infty} T^m x = y$. As T is obviously continuous, we get $\lim_{m\to\infty} T^{m+1}x = Ty$. Hence $Ty = y$, a fixed point of T. To show uniqueness, let $Tw = w$, for some $w \in X$. Let $y = w_0, w_1, \ldots \ \ldots, w_{k-1}$, $w_k = w$ be an ε - chain. Now for any positive integer l, we get,
$1 \geq M(y, w, z, t) = M(T^l y, T^l w, z, t) = M(T^l w_0, T^l w_k, z, t) = M(Sw_0, Sw_k, z, t)$, where, $S = T^l$.
$\geq M(Sw_0, Sw_1, Sw_k, s_0) * \ M(Sw_1, Sw_2, Sw_k, s_0) * \ldots \ \ldots$
$* \ M(Sw_{k-2}, Sw_{k-1}, Sw_k, s_0) * \ M(Sw_0, Sw_1, z, s_0) * \ M(Sw_1, Sw_2, z, s_0) * \ldots \ \ldots$
$* \ M(Sw_{k-1}, Sw_k, z, s_0) \ * \ M(Sw_{k-1}, Sw_k, z, s_0)$, where $s_0 = \frac{t}{2(k-1)+1}$.
$\geq M(w_0, w_1, Sw_k, \frac{s_0}{\lambda^l}) * \ M(w_1, w_2, Sw_k, \frac{s_0}{\lambda^l}) * \ldots \ \ldots * M\left(w_{k-2}, w_{k-1}, Sw_k, \frac{s_0}{\lambda^l}\right) *$
$* M\left(w_0, w_1, z, \frac{s_0}{\lambda^l}\right) * M(w_1, w_2, z, \frac{s_0}{\lambda^l}) * \ldots \ \ldots * M\left(w_{k-1}, w_k, z, \frac{s_0}{\lambda^l}\right) * \ M\left(w_{k-1}, w_k, z, \frac{s_0}{\lambda^l}\right)$, by (3.14.1).
$\rightarrow 1 * 1 * \ldots \ \ldots * 1 * 1 * 1 * \ldots \ \ldots * 1 * 1 = 1$, as $l \rightarrow \infty$, for all $z \in X$, $t > 0$. Therefore,
$M(y, w, z, t) = 1$, for all $z \in X$, $t > 0$ and so $y = w$. Hence the fixed point is unique. □

3.15. Remarks. A huge amount of work is done on the development and application of fuzzy metric spaces and considerably a rich literature is available by now for the fixed point theory in the fuzzy structures. But the theory of fuzzy 2 - metric spaces is still in the embryonic stage and is in the developing process and there is a huge scope of works in this field. We sincerely hope that the works incorporated in the paper and in [3] be useful in the development of the fixed point theory in fuzzy 2 - metric spaces.

References

[1]. S. H. Cho, *On common fixed points in fuzzy metric spaces*, Int. Math. Forum 1(10) (2006) 471 - 479.
[2]. S. H. Cho and J. H. Jung, *On common fixed point theorems in fuzzy metric spaces*, Int. Math. Forum 1(29) (2006) 1441 - 1451.
[3]. N. R. Das and M. L. Saha, *Some fixed point theorems in fuzzy 2- metric spaces*, J. Fuzzy Math. 21(4) (2013) 753 - 758.
[4]. N. R. Das and M. L. Saha, *On fixed points in fuzzy metric spaces*, Ann. Fuzzy Math. Inform. 7 (2) (2014) 313 - 318.
[5]. N. R. Das and M. L. Saha, *Compatible mappings and common fixed points in fuzzy metric spaces*, J. Fuzzy Math. 23(1) (2015) 199 - 208.
[6]. N. R. Das and M. L. Saha, *On fixed points in complete fuzzy normed linear spaces*, Ann. Fuzzy Math. Inform. 10 (4) (2015) 515 – 524.
[7]. N. R. Das and M. L. Saha, *Non - commuting mappings and common fixed points in fuzzy metric spaces*, Journal of Fuzzy Math. ,24 (1) (2016).
[8]. Zi - ke Deng, *Fuzzy pseudo- metric spaces*, J. Math. Anal. Appl. 86 (1982) 74 - 95.
[9]. Zi - ke Deng, *Separation axioms for completeness and total boundedness in fuzzy pseudo- metric spaces*, J. Math. Anal. Appl. 192 (1985) 141 - 150.
[10]. D. Dey and M. Saha, Fixed point theorems acting on 2 - metric spaces: A brief Study, Scholars Press.
[11]. M. A. Erceg, *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl. 69 (1979) 205 - 230.
[12]. S. Gähler, *2-metrische raume and ihre topologische structure*, Math. Nachr. 26 (1963) 115 - 148.
[13]. S. Gähler, *Linear 2 - normierte raume*, Math. Nachr. 28 (1964) 1 - 43.
[14]. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems 64 (1994) 395 - 399.
[15]. M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems 27 (1988) 385 - 389.
[16]. V. Gregori and S. Romaguera, *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems 115 (2000) 485 - 489.
[17]. V. Gregori and S. Romaguera, *On completion of fuzzy metric spaces*, Fuzzy Sets and Systems 130 (3) (2002) 399 - 404.
[18]. J. Han, *A common fixed point theorem on fuzzy 2- metric spaces*, J. Chungcheong Math. Soc. 23(4) (2010) 645 - 655.
[19]. N. - H. Hsu, *On the completion of fuzzy metric spaces*, Bull. Nat. Taiwan Nor. Univ. 37 (1992) 385 - 393.
[20]. O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems 12 (1984) 215 - 229.
[21]. I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika 11(5) (1975) 336 - 344.
[22]. K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. USA, 28 (1942) 535 – 537
[23]. D. Mihet, *A class of contractions in fuzzy metric spaces*, Fuzzy Sets and Systems161(8) (2010) 1131 - 1137.
[24]. M. L. Saha and S. Manro, *Common fixed points of faintly compatible mappings*, Amer. Research J. Math. 2(1) (2015) 38 - 43.
[25]. A. Sapena, *A contribution to the study of fuzzy metric spaces*, Appl. Gen. Topol. 2(1) (2001) 63 - 75.
[26]. B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., 10 (1960) 313 - 334.
[27]. S. Sharma, *On fuzzy metric spaces*, Southeast Asian Bull. Math. 26 (2002) 33 - 145.
[28]. P. L. Sharma, B. K. Sharma and K. Iseki, *Contractive type mapping on 2-metric spaces*, Math. Japonica 21(1976) 67 – 70
[29]. L. A. Zadeh, *Fuzzy sets*, Inform. and Control 8 (1965) 338 - 353.