IOSR Journal of Mathematics (IOSR-JM)
e-ISSN:2278-5728, p-ISSN: 2319-765X.Volume12,Issue 1 Ver. III (Jan.-Feb.2016)PP 75-77
www.iosrjournals.org

On the Non Homogeneous Ternary Quadratic Equation $x^2 + xy + y^2 = 12z^2$

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ABSTRACT: *The Ternary Quadratic Diophantine Equation given by* $x^2 + xy + y^2 = 12z^2$ *is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.*

Keyword- *Quadratic equation with three unknowns, Integral Solutions.*

I. Introduction

The Theory of Diophantine equation offers a rich variety of fascinating problems. In particulars, quadratic equations, homogeneous and non-homogeneous have aroused the interest of numerous Mathematicians. Since ambiguity [1-3]. This paper concerns with the problem of determining non-trivial solutions of the non-homogeneous quadratic equation with three unknowns given by $x^2 + xy + y^2 = 7z^2$. A few relations among the solutions are presented.

NOTATIONS USED

- $T_{m,n}$ -Polygonal number of rank n with size m.
- P_m^n - Pyramidal number of rank n with size m.
- Pr_n - Pronic number of rank n.
- SO_n -Stella Octangular number of rank n.
- Obl_n -Oblong number of rank n.
- OH_n - Octahedral number of rank n.
- Tet_n -Tetrahedral number of rank n.
- PP_n - Pentagonal Pyramidal number of rank n

II. Method Of Analysis

The quadratic Diophantine Equation with three Unknowns to be solved for its non-Zero distinct Integral solution is

$$\mathbf{x^2 + xy + y^2 = 7z^2} \quad (1)$$

Introduction of the linear transformations

$$x = u + v \text{ and } y = u - v \quad (2)$$

In (1) leads to $3u^2 + v^2 = 12z^2$ (3)

Different patterns of solutions of (3) and hence that of (1) using (2) are given below.

PATTERN-I

$$x = u + 3v \text{ and } y = u - 3v \quad (4)$$

in (1) leads to $u^2 + 3v^2 = 4z^2$ (5)

Let z = z(a, b)= $a^2 + 3b^2$ (6)

Write 4 = $(1 + i\sqrt{3})(1 - i\sqrt{3})$ (7)

Using (6) and (7) in (5) and applying the method of factorization

$$u + i\sqrt{3}\, v = (1 + i\sqrt{3})\,(a + i\sqrt{3}b)^2$$

Equating the real and imaginary parts, we have

$$u = u(a, b) = a^2 - 6ab - 3b^2$$
$$v = v(a, b) = a^2 - 3b^2 + 2ab$$

Substituting the above values of u and v in (4) we get

$$x = x(a, b) = 4a^2 - 12b^2$$
$$y = y(a, b) = -2a^2 - 12ab + 6b^2$$
$$z = z(a, b) = a^2 + 3b^2$$

Properties

1. $y(a,b)+2z(a,b)-108t_{3,b}+t_{126,b}\equiv 0(mod\ 107)$
2. $x(a,a+1)+2y(a,a+1)=42Pr_a$
3. $x(a,3)+y(a,3)+2z(a,3)-188t_{3,b}+t_{180,a}\equiv 0(mod\ 219)$
4. $y(\mathrm{a},7\mathrm{a}^2-4)+2z(\mathrm{a},7\mathrm{a}^2-4)-t_{26,b}+36CP_a^{14}\equiv 0(mod\ 11)$
5. $x(a,a+1)-2y(a,a+1)-t_{18,a}-24Pr_a\equiv 0(mod\ 7)$
6. $2z(b(b+1),b)+y(b(b+1),b)+24P_b^5-116t_{3,b}+t_{94,b}\equiv 0(mod\ 103)$
7. $x(a,(a+1)(a+2))+2y(a,(a+1)(a+2))=144P_a^3$
8. $3\{x(a,a)+z(a,a)\} and\ y(a,a)+z(3a,a)$represents a nasty number.

PATTERN-II

Rewrite (5) as $\frac{3(v+z)}{z+u}=\frac{z-u}{v-z}=\frac{A}{B}\ (B\neq 0)$

The above equation is equivalent to the system of equations,

$$-Au+3Bv+(3B-A)z=0$$
$$-Bu+Av+(A+B)z=0$$

This is satisfied by

$$u=3\mathrm{B}^2+6AB-\mathrm{A}^2$$
$$v=\mathrm{A}^2+2AB-3\mathrm{B}^2$$
$$z=\mathrm{A}^2+3\mathrm{B}^2$$

Hence in view of (4), the corresponding solutions of (1) are given by

$$x=x(A,B)=2\mathrm{A}^2-6\mathrm{B}^2+12\mathrm{AB}$$
$$y=y(A,B)=12\mathrm{B}^2-4\mathrm{A}^2$$
$$z=z(A,B)=\mathrm{A}^2+3\mathrm{B}^2$$

Properties

1. $x(A,3)+2z(A,3)-128t_{3,A}+t_{122,A}\equiv 0(mod\ 87)$
2. $x(B(B+1),B)-2z(B(B+1),B)-24P_B^5+88t_{3,B}-t_{66,B}\equiv 0(mod\ 75)$
3. $x(A,2\mathrm{A}^2+1)+2z(A,2\mathrm{A}^2+1)-36\ OH_A-t_{10,A}\equiv 0(mod\ 3)$
4. $2z(A(A+1),B-x(A(A+1),B)-t_{26,B}+24Ct_{A,B}\equiv 24(mod\ 11)$
5. $y(A,A+1)+2x(A,A+1)=24Pr_A$
6. $x(A,1)-S_A+t_{10,A}\equiv 7(mod\ 15)$
7. $4\{x(b,b)+y(b,b)\}\ and\ 12\{x(a,-a)+y(a,a)+z(a,a)\}$ represents a nasty number.

PATTERN-III

Equation (5) is Equivalent to $\mathrm{u}^2+3\mathrm{v}^2=(2\mathrm{z})^2$ which is satisfied by

$$u=3\mathrm{p}^2-\mathrm{q}^2$$
$$v=2\mathrm{pq}$$
$$z=\frac{1}{2}(3\mathrm{p}^2+\mathrm{q}^2)$$

Put P=2A, q=2B

In view (4), the non zero distinct integral solutions of (1) are

$$x(A,B)=12\mathrm{A}^2-4\mathrm{B}^2+24\mathrm{AB}$$
$$y(A,B)=12\mathrm{A}^2-4\mathrm{B}^2-24\mathrm{AB}$$
$$z(A,B)=6\mathrm{A}^2+2\mathrm{B}^2$$

Properties

1. $y(A,(A+1)(A+2)(A+3))-y(A,(A+1)(A+2)(A+3))=1152pt_A$
2. $y(A,2A^2+1)+2z(A,2A^2+1)-t_{50,A}+72\ OH_A\equiv 0(mod\ 23)$
3. $x(A,(A+1),B)-2z(A(A+1)B)+t_{18,B}-48Ct_{A,B}\equiv 48(mod\ 7)$
4. $x(A,2)+2z(A,2)-168\ t_3\,,A_3+t_{122,A}\equiv 0(mod95)$
5. $x(B(B+1),B)-2z(B(B+1),B)-48P_B^5+88\ t_{3,B}-t_{74,B}\equiv 0(mod79)$
6. $x(A,3)-88\ t_{3,A}+t_{66,A}\equiv 36(mod3)$
7. $x(a,a)-z(a,a)$ represents a nasty number.

PATTERN-IV

Rewrite (5) as $\mathrm{u}^2+3\mathrm{v}^2=4\mathrm{z}^2*1$ (8)

Write (4) as $4=(1+i\sqrt{3})\,(1-i\sqrt{3})$ (9)

and $1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$ (10)

Assume $z = a + 3b$

Using (9) and (10) in (8), we get the system of equation,

$$\left(u + i\sqrt{3}\, v\right) = \left(1 + i\sqrt{3}\right)\frac{(1+i\sqrt{3})}{2} \cdot (\mathrm{a} + \mathrm{i}\sqrt{3}\mathrm{b})^2$$

Equating the real and imaginary parts in the above equation, we get

$$u = u(a,b) = -a^2 + 3b^2 - 6ab$$
$$v = v(a,b) = a^2 - 3b^2 - 2ab$$

In view of (4), the non-Zero distinct integral solution of (1) are

$$x = x(a,b) = 2a^2 - 6b^2 - 12ab$$
$$y = y(a,b) = -4a^2 + 12b^2$$
$$z = z(a,b) = a^2 + 3b^2$$

Properties

1. $x(b+1,b) - 2z(b+1,b) + 12(obl)_b + t_{26,b} \equiv 0(mod\ 11)$
2. $y(a,(a+1)(a+2)(a+3)) + 2x(a,(a+1)(a+2)(a+3)) = -576Pt_a$
3. $2z(a,(a+1)) + x(a,(a+1)) + 12Pr_a - t_{10,a} \equiv 0(mod\ 3)$
4. $x(1,b) - 2z(1,b) + 128t_{3,b} - t_{106,b} \equiv 0(mod\ 103)$
5. $y(a,a(a+1) + 2x(a,a(a+1) = -48P_a^5$
6. $2z(a,2) + x(a,2) - 188t_{2,a} + t_{182,a} \equiv 0(mod\ 207)$
7. $2z(b(b+1),b) - x(b(b+1),b) - 24P_b^5 - 84t_{3,b} + t_{62,b} \equiv 0(mod\ 71)$
8. $x(a,a) + y(a,a)\ and\ x(a,a) + z(a,a)$ represents a nasty numbers.

PATTERN-V

Also write 1 as

$$1 = 1/49\left(1 + 4i\sqrt{3}\right)(1 - 4i\sqrt{3})$$

In equation (8),

$$(u + i\sqrt{3}v) = (1 + i\sqrt{3})\ \frac{(1+i\sqrt{3})}{7}\ (a + i\sqrt{3}b)^2$$

Equating Real and Imaginary parts in the above relation, we get

$$u = u(a,b) = \frac{1}{7}(33b^2 - 11a^2 - 30ab)$$
$$v = v(a,b) = \frac{1}{7}(5a^2 - 12b^2 - 22ab)$$

Put $a = 7A, b = 7B$ then

$$u = u(A,B) = 231B^2 - 77A^2 - 210\mathrm{AB}$$
$$v = v(A,B) = 35A^2 - 105B^2 - 154AB)$$

In view (4) the non-zero distinct integral solutions of (1) are

$$x = x(A,B) = 28A^2 - 84B^2 - 672AB$$
$$y = y(A,B) = 546B^2 - 182A^2 - 252AB$$
$$z = z(A,B) = 49A^2 + 147B^2$$

Properties

1. $49x(B+1,B) - 28z(B+1,B) + 32928(obl)_B + t_{164666,B} \equiv 0(mod\ 8281)$
2. $182x((A,(A+1)(A+2)(A+3)) + 28y(A,(A+1)(A+2)(A+3)) = -3104640Pt_A$
3. $x(A,1) - S_A - t_{46,A} \equiv 85(mod\ 645)$
4. $84z\left(A, A(A+1)\right) + 147x(A,A(A+1) - t_{16466,A} + 98784PP_A \equiv 0(mod\ 8231)$
5. $x(B^2+1,B) - x(B^2-1,B) - 124Pr_B + t_{26,B} \equiv 0(mod\ 1468)$
6. $546z(A,2A^2-1) - 147y(A,2A^2-1) - 53508Obl_A - 37044So_A \equiv 0(mod\ 53508)$

III. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the quadratic equations with three unknowns.

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