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Some Recent Developments On Statistical Convergence ("A New Type Of Convergence") And Convergence In Statistics

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Abstract: *One of the most recent generalizations of concept of classical convergence of sequences ("a new type of convergence") is statistical convergence defined by Fast. Recently, it became the centre of attraction for many researchers. The objective of this review is to discuss fundamental concepts and results in statistical convergence along with various generalizations which have been subsequently formulated. And finally expose the statistics in statistical convergence.*

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I. Introduction

The classical notion of Cauchy Convergence has been generalized in various ways through summability and sequence-to-function methods. The first generalization was that of almost convergence which was initiated as early as 1932 by Banach (1932). It was studied in details by Lorentz (1948).

One of the most recent generalizations of the concept of classical convergence of sequences ("new type of convergence") has been introduced by Fast (1951). He introduced concepts such as statistical convergence, Lacunary statistical convergence and λ-statistical convergence. It was defined almost fifty years ago but recently it became the centre of attraction for many researchers. The objective of this study is to discuss fundamental concepts and results in this field along with various generalizations which have been subsequently formulated. With a view to expose the profound link between the classical theory and statistical convergence theory. We will also look at some construction from the theory of statistical convergence to consider relations between statistical convergence and concepts of convergence in statistics such as the mean (average) and standard deviation.

Before starting to discuss on statistical convergence, we consider a very brief summary of concepts of convergence of sequences of real numbers and some other important definitions related to sequences.

II. Notions and Notations

Definition 2.1: A sequence is a function whose domain of definition is the set $\mathbb{N}$ of all natural numbers. Sequences obtain different names with respect to their range. If the range of the sequence is $\mathbb{R}$, then we call this sequence a real number sequence (or real sequence). If the terms are all rational numbers, then, we called this sequence rational number sequence (or rational sequence). Generally, we use the notation

$$x = \{x_k\}_{k=1}^{\infty} \quad \forall\ k \in \mathbb{N}$$

To represent sequences. For each value of k, the term x_k is known as k^{th} term of *x*. The space of all sequences denoted by ω.

The above definition can be re-written informally as follows:

A sequence $\{x_k\}_{k=1}^{\infty}$ of real numbers is a function x:

$D(x) \subset \mathbb{N} \to \mathbb{R}$ of an infinite subset $D(x)$ of the natural numbers $\mathbb{N}$ into the real numbers $\mathbb{R}$ defined by

$$x(k) = x_k \in \mathbb{R} \qquad \forall\ k \in \mathbb{N}$$

Example 2.1: The sequence $\{x_k\}_{k=1}^{\infty}$ where $x_k = \frac{1}{k}$ is an infinite sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots\right\}$. Formally, of course this is the function with domain $\mathbb{N}$, whose value at each k is $\frac{1}{k}$. The set of values is: $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$.

Example 2.2: Consider the sequence given by $\{x_k\}_{k=1}^{\infty} = \{(-1)^k\}_{k=1}^{\infty}$. This is also the infinite sequence $\{-1, 1, -1, 1, \dots\}$. This sequence represents a function whose domain is $\mathbb{N}$ and its set of values is $\{-1, 1\}$. This is an alternating sequence (a sequence in which the consecutive terms have opposite signs).

Example 2.3: The sequence $\{x_k\}_{k=1}^{\infty} = \{3\}_{k=1}^{\infty}$ is the constant sequence $\{3, 3, 3, \dots\}$ whose set of values is the singleton $\{3\}$.

Definition 2.2: Let $\{x_k\}$ be a sequence and let $\{k_n\}$ be a strictly increasing sequence of natural numbers. The sequence $\{x_{k_n}\}$ is called a subsequence of $\{x_k\}$.

Example 2.4: The sequence $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$ has subsequences $\left\{\frac{1}{2k}\right\}_{k=1}^{\infty}$, $\left\{\frac{1}{k^2}\right\}_{k=1}^{\infty}$.
Example 2.5: The set of prime numbers $2, 3, 5, 7, 11, 13, \ldots$ is a subsequence of the sequence $\{k\}_{k=1}^{\infty}$ of all positive integers.
Definition 2.3: A sequence $\{x_k\}$ is called bounded from above if there exists $M \in \mathbb{R}$, which satisfies the inequality $x_k \leq M$ for all $k \in \mathbb{N}$. In this case, we say M is an upper bound for *x*.
Definition 2.4: we say $\{x_k\}$ is bounded from below if there exists $m \in \mathbb{R}$ which satisfies the inequality $m \leq x_k$ for all $k \in \mathbb{N}$. In this case we say that *m* is a lower bound of *x*.
Definition 2.5: We say that a sequence $\{x_k\}$ is bounded if there exists a real constant $U > 0$, which satisfies the inequality

$$|x_k| \leq U \text{ for all } k \in \mathbb{N}$$

Lemma 2.1: we say $\{x_k\}$ is bounded if and only if it is bounded from below and bounded from above.
Recall that, $l_\infty = \{x \in \omega : x_k \text{ is bounded }\}$.
Example 2.6: Sequences defined by $x_k = \frac{k-1}{k}$ and $y_k = \frac{1}{2k}$ are both bounded.
Analogous to the idea of ordinary bounded sequence, statistically bounded sequence is defined.
Definition 2.6 [Fridy and Orhan, (1997)]: A sequence $\{x_k\}$ is said to be statistically bounded if there exists a number *B* such that

$$\delta\{k : |x_k| > B\} = 0$$

Example 2.7: Let $x = \{x_k\}$ be given by

$$x_k = \begin{cases} k, & \text{if } k \text{ is prime} \\ 1, & \text{if } k \text{ is odd but not prime} \\ 0, & \text{if } k \text{ is even but not prime} \end{cases}$$

It is easy to see that, x is not statistically convergent and unbounded above but it is statistically bounded.

$$\delta\{k : |x_k| > 1\} = 0$$

Definition 2.7: A sequence $\{x_k\}_{k=1}^{\infty}$ of real numbers is said to converge to a real number l if and only if for each $\varepsilon > 0$, there exists a natural number $n(\varepsilon)$ such that

$$|x_k - l| < \varepsilon \quad \forall\ k \geq n(\varepsilon)$$

In this we write:

$$\lim_{k \to \infty} x_k = l \quad or \quad \lim x_k = l \quad or \quad x_k \to l \quad as\ k \to \infty$$

The number l is called the limit of x_k. A sequence which does not converge to some real number is said to diverge. We use the notation c, to represent the space of convergent sequences,

$$c = \{x \in \omega : x_k \text{ is convergent}\}$$

Analogous to the idea of space of ordinary convergence sequence, the space of all statistically convergent sequences is denoted by $\bar{c}$.
Definition 2.8: if a sequence $\{x_k\}$ of real numbers converges to zero, it is called a null sequence. The spaces of all null sequences is denoted by $c_0, i.e.$

$$c_0 = \{x \in \omega : x_k \to 0\}.$$

Example 2.8: The sequence $\{x_k\}$ defined by $x_k = \frac{k+3}{k^2-1}$ is a null sequence.
Remark 2.1: Obviously $c_0 \subset c \subset \omega$.
Analogous to the idea of space of null convergence sequence, the space of statistically null convergent sequence is denoted by $\bar{c}_{0.}$
Theorem 2.1: Every convergence sequence is bounded, however the converse may not be true.
Remark 2.2: In general, a bounded sequence need not be convergent. In fact the sequence $x_k = (-1)^k$ is bounded but not convergent.
Theorem 2.2: A sequence $\{x_k\}_{k=1}^{\infty}$ is convergent to the limit l if and only if all of its subsequences converges to the same limit l.
Theorem 2.3: (Balzano-Weierstrass). Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
Remark 2.3: For statistical analogue of Bolzano-Weierstrass [see Fridy (1993)]
Theorem 2.4: If a sequence $\{x_k\}$ converges to a limit, then the limit is unique.
Proof: Suppose, for contradiction, that the sequence $\{x_k\}$ converges to two limits x and y, $x \neq y$. Then given any $\varepsilon > 0$, there exists a natural number $n(\varepsilon)$ such that

$$|x_k - x| < \frac{\varepsilon}{2}\ \forall\ k \geq n(\varepsilon).$$

Also, there exists $\bar{n}(\varepsilon) \in \mathbb{N}$ such that

$$|x_k - y| < \frac{\varepsilon}{2}\ \forall\ k \geq \bar{n}(\varepsilon).$$

Hence, for all $k \geq max\{n(\varepsilon),\ \bar{n}(\varepsilon)\}$, both inequalities implies

$|x_k - x| < \frac{\varepsilon}{2}$ and $|x_k - y| < \frac{\varepsilon}{2}$ hold. Then

$$|x - y| = |x - x_k + x_k - y|$$
$$\leq |x - x_k| + |x_k - y|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

For all $n \geq max\{n(\varepsilon),\ \bar{n}(\varepsilon)\}$.
This implies that $x = y$. Contradiction. Hence the theorem is proved.
Definition 2.9: for each $\varepsilon > 0$, and $l \in \mathbb{R}$. The set

$$K_\varepsilon(l) = \{x \in \mathbb{R}: |x - l| < \varepsilon\}$$

Is called the ε −neighbourhood of l.
Lemma 2.2: Assume that $x_k \to l$. Then for every $\varepsilon > 0$, except finitely many terms of x_k, all other terms lie in $K_\varepsilon(l)$. in other word, $\{n \in \mathbb{N}: |x_k - l| \geq \varepsilon\}$ is finite.
Definition 2.10: A sequence $\{x_k\}$ is said to be (monotone) increasing if $x_k \leq x_{k+1}$ for all values of k and (monotone) decreasing if $x_k \geq x_{k+1}$ for all k. If $x_k < x_{k+1}$ for all k, $\{x_k\}$ is said to be strictly increasing and if $x_k > x_{k+1}$ for all k, $\{x_k\}$ is said to be strictly decreasing.
A sequence satisfying any of the above conditions is said to be monotone.
Example 2.9: The sequence $1, 2, 3, \ldots\ k, \ldots$ is a strictly increasing unbounded sequence.
Example 2.10: The sequence $\left\{\frac{1}{k}\right\}$ is strictly decreasing sequence which converges to zero.
Theorem 2.5: (monotone convergence theorem): A monotone sequence of real numbers is convergent if and only if it is bounded.
Definition 2.11: Given a bounded sequence $x = \{x_k\}$ of real numbers. The upper limit or limit superior of $\{x_k\}$ denoted by lim sup x_k (or $\overline{\lim}_{k\to\infty} x_k$) and is defined as

$$\limsup x_k = \overline{\lim}_{k\to\infty} x_k$$
$$= \inf_{n\in\mathbb{N}} \sup\{x_k: k \geq n\}$$

Analogous to the idea of ordinary limit superior, statistical limit superior of x were defined as follows [Fridy and Orhan, (1997)]:

$$S - \limsup x = \begin{cases} \sup B_x, & if\ B_x \neq \emptyset \\ -\infty, & if\ B_x = \emptyset \end{cases}$$

Definition 2.12: Given a bounded sequence $x = \{x_k\}$ of real numbers. The lower limit or limit inferior of $\{x_k\}$ denoted by lim inf x_k (or $\underline{\lim}_{k\to\infty} x_k$) and is defined as

$$\liminf x_k = \underline{\lim}_{k\to\infty} x_k)$$
$$= \sup_{n\in\mathbb{N}} \inf\{x_k: k \geq n\}$$

Analogous to the idea of ordinary limit inferior, statistically limit inferior were defined as follows:
Definition 2.13: [Fridy and Orhan, (1997)] The statistical limit inferior of x is defined as

$$S - \liminf x = \begin{cases} \inf A_x, & if\ A_x \neq \emptyset \\ +\infty, & if\ A_x = \emptyset \end{cases}$$

Analogous to the idea of ordinary convergence, statistically convergent sequences were defined as follows:
As it is well known, the theory of statistical convergence and other types of convergences are all based on a density function. This is why we need to start with the definition of denstity function.
Definition 2.14 [Niven and Zuckerman, (1980)]: Let A be a subset of ℕ. The natural density $\delta(A)$ is defined as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |A_n|,$$

Where $A_n = \{k \leq n: k \in A\}$, provided the limit exists. The vertical bars denote the cardinality (or order) of the enclosed set.

III. Statistical Convergence Sequence

Definition 3.1: [Fast, (1951)] A sequence $x = (x_k)$ is said to be statistically convergent to l if for every $\varepsilon > 0$, we have

$$\delta(\{k \in N: |x_k - l| \geq \varepsilon\}) = 0.$$

In this case, we write $S - limx = l$, which is necessarily unique and S denotes the set of all statistically convergent sequences. By an index set, we mean a finite subset $\{k_i\}$ of N with $k_i \leq k_{i+1}$. Thus, an infinite set $\{k_i\}$ is precisely the sequence $\{k_i\}_{i\in N}$ of indices. Note that a sequence x, is statistically convergent to l if and only if there is infinite index set $K = \{k_i\}$ so that $\delta(K) = 1$ and $\lim_k x_k = l$, Mursaleen and Edely (2002).
Analogous to the following classical convergence notions;
Definition 3.2: A sequence $\{x_k\}_{k=1}^{\infty}$ of real numbers is called a Cauchy sequence (a fundamental sequence) if it satisfies the following: $\forall\ \varepsilon > 0,\ \exists\ n(\varepsilon) \in \mathbb{N}$ such that

$$|x_k - x_n| < \varepsilon \quad \forall\ k, n \geq n(\varepsilon)$$

That is equivalently

$$|x_k - x_n| \to 0, \quad as\ k, n \to \infty.$$

Theorem 3.1: Every real valued Cauchy sequence is bounded.

Theorem 3.2: A sequence of real number is convergent if and only if it is a Cauchy sequence.

We have;

Definition 3.3 [Fridy, (1985)]: A sequence $x = (x_k)$ is called statistically Cauchy sequence if for each $\varepsilon > 0$, there exists a natural number $n = n(\varepsilon)$ such that

$$\delta\{k \in N: |x_k - x_n| \geq \varepsilon\} = 0,$$

That is,

$$\lim_n \left(\frac{1}{n}\right) |\{k \leq n: |x_k - x_n| \geq \varepsilon\}| = 0$$

Theorem 3.3: It is also shown that a real sequence is statistically Cauchy if and only if it is statistically convergent.

Proof: Suppose that $x = \{x_k\}$ is statistically convergent to l. By definition there exists a subset K of natural numbers with $\delta(K) = 1$ and $x \to l$ on K in the ordinary sense. This means that x is a Cauchy sequence on K or equivalently x is statistical Cauchy sequence.

Conversely, suppose that $x = \{x_k\}$ is statistical Cauchy sequence. Since every sequence is statistically Cauchy if and only if there exists $K \subset \mathbb{N}$ with $\delta(K) = 1$ and so x is Cauchy on K. Therefore it is convergent on K, which shows that x is statistical Cauchy sequence.

Definition 3.4: We say that the sequence $\{x_k\}$ of real numbers tends to the limit $l \in \mathbb{R}$ if and only if for each $\mathcal{E} > 0$, there exists $n(\varepsilon)$ such that

$$|x_k - l| < \varepsilon \quad \forall\ k \geq l$$

Analogous to the definition of a limit point, statistical limit points were defined as follows:

Definition 3.5 [Fridy, (1993)]: A subsequence $\left(x_{k(n)}\right)$ of sequence x is called a thin subsequence if $K = \{k(n): n \in \mathbb{N}\}$ has natural density zero. If K does not have zero natural density, then the subsequence is said to be nonthin.

We note that $\left(x_{k(n)}\right)$ is a nonthin subsequence of x if either $\delta(K)$ is a positive number or K fails to have natural density.

Definition 3.6 [Fridy, (1993)]: The number λ is a statistical limit point of the number sequence x provided that there is a nonthin subsequence of x that converges to λ.

Definition 3.7 [Fridy, (1993)]: The number γ is a statistical cluster point or accumulation point of the number sequence x provided that for each $\varepsilon > 0$, the set $\{k \in \mathbb{N}: |x_k - \gamma| < \varepsilon\}$ does not have natural density zero.

If $\Lambda_x, \Gamma_x\ and\ L_x$ denote the set of all statistical limit points, set of all statistical cluster points and set of (ordinary) limit points, respectively, then it has been proved that

$\Lambda_x \subseteq \Gamma_x \subseteq L_x$.

Example 3.1: let $x_k = 1$, if $k = n^2$ and $x_k = 0$ otherwise $L_x = \{0,1\}$ and $\Lambda_x = \{0\}$.

It is clear that

$\Lambda_x \subseteq L_x$ for any sequence x. To show that $\Lambda_x\ and\ L_x$ can be different, consider a sequence x for which $\Lambda_x = \phi$ while $L_x = \mathbb{R}$, the set of real numbers.

Example 3.2: let us consider the sequence $l = \{x_k: k = 1,2,3, \dots\}$ whose terms are

$$x_k = \begin{cases} k, & when\ k\ is\ a\ square \\ \\ 1/k, & otherwise \end{cases}$$

Then, it is easy to see that l is divergent in the ordinary sense, while 0 is the statistical limit of l since $\delta(K) = 0$, where $K = \{n^2\ for\ n = 1,2,3, \dots\}$.

Not all properties of convergent sequences are true for statistical convergence. For example, it is known that a subsequence of convergent sequence is convergent. However, for statistical convergence, this is not so. Indeed, the sequence $y = \{k: k = 1,2,3, \dots\}$ is a subsequence of the statistical convergent sequence l from example 3.2. At the same time, y is statistically divergent.

Definition 3.8 [Fridy and Orhan, (1993)]: A sequence $\theta = \{k_r\}$ satisfying:

i. $k_0 = 0$,

ii. $h_r = k_r - k_{r-1} \to \infty,\ r \to \infty$

Is called a Lacunary sequence.

For each Lacunary sequence θ, we define the interval $I_r := (k_{r-1}, k_r]$ and the fraction $q_r = \frac{k_r}{k_{r-1}}$. Lacunary statistical convergence has been introduced by Fridy and Orhan in the following way:

Example 3.3: The sequence $\theta = \{2^r\}$ is a Lacunary sequence with $I_r := (2^{r-1}, 2^r]$ and $q_r := 2$.

Definition 3.9 [Fridy and Orhan, 1993)]: A sequence x is called Lacunary statistical convergent if for each $\varepsilon > 0$, we have

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - l| \geq \varepsilon\}| = 0$$

And denoted by $x_k \to l(\theta - S)$

Lemma 1.3: For a lacunary sequence $\theta = \{k_r\}, x_k \to l$ implies $x_k \to l\ (\theta - S)$ if and only if $\liminf_r q_r > 1$.

Lemma 1.4: For a lacunary sequence $\theta = \{k_r\}, x_k \to l$ implies $x_k \to l\ (\theta - S)$ if and only if $\limsup_r q_r < \infty$.

As a consequence of the above lammas, we have:

Theorem 3.4: Let $\theta = \{k_r\}$ be a lacunary sequence. Then statistically convergence and $\theta -$statistical convergence of $x_k \to l$ if and only if

$$1 < \liminf_r q_r < \limsup_r q_r < \infty$$

Obviously, example 3.3 above satisfies the conditions of the above theorem for $r > 0$.

Remark 3.1: To defined $\lambda -$Statistical Convergence, first we need a sequence $\{\lambda_r\}$ of positive non decreasing numbers such that $\lambda_r \to \infty, as\ r \to \infty, \lambda_1 = 1\ and\ \lambda_{r+1} \leq \lambda_r + 1$. Assume that ω is the space of all sequences satisfying these conditions. Then for each $\{\lambda_r\} \in \omega$, and for each r, we defined intervals

$$M_r = [r - \lambda_r + 1, r].$$

Definition 3.10 [Mursaleen, (2000)]: A sequence x is said to be $\lambda -$statistical covergent to l, if for each $\varepsilon > 0$,

$$\lim_r \frac{1}{\lambda_r} \{|k \in M_r : |x_k - l| \geq \varepsilon|\} = 0.$$

The $\lambda -$statistical convergence of x to l is represented by the notation $x_k \to l(\lambda - S)$.

Remark 3.2: For $\lambda_r = r, \lambda -$statistical convergence coincides with the statistical convergence.

Further remarks:

The idea of statistical convergence goes back to the first edition (published in Warsaw in 1935) of the monograph of Zygmund (1979). Formally, the concept of statistical convergence was introduced by Steinhaus (1951) and Fast (1951) and later reintroduced by Schoenberg (1959). Different mathematicians studied properties of statistical convergence and applied this concept in various fields such as Measure theory Miller, (1995), Trigonometric series Zygmund, (1979), Approximation theory Duman *et al*., (2003), Locally convex spaces Maddox, (1988), Finitely additive set functions Connor and Kline, (1996), in the study of subsets of the Stone-Chech compactification of the set of natural numbers Connor and Swardson, (1993) and Banach spaces Connor *et al*., (2000).

However, in general case, neither limits nor statistical limit can be calculated or measured with absolute precision. To reflect this imprecision, and to model it by mathematical structures, several approaches in mathematics have been developed: fuzzy set theory, fuzzy logic, interval analysis, set valued analysis, etc. One of these approaches is the neoclassical analysis (Burgin, 1995, 2000). In it ordinary structure of analysis, that is, functions, sequences, series and operators are studied by means of fuzzy concepts: fuzzy limit, fuzzy continuity and fuzzy derivatives. For example continuous functions which are studied in the classical analysis become a part of the set of the fuzzy continuous functions in neoclassical analysis. Neoclassical analysis extends methods of calculus to reflect uncertainties that arise in computations and measurements.

Belen and Yildirim (2015) defined the concepts of A-statistical convergence and A^j-statistical convergence in a 2-normed space and present an example to show the importance of generalized form of convergence through an ideal. And they also introduced some new sequence spaces in a 2-Banach space and examine some inclusion relations between these spaces. Kukul (2014) studied αβ-statistical convergence and started with the discussion of statistical convergence. The concept of αβ-statistical convergence which is the main interest of his study has been considered in his work. He also showed that αβ-statistical convergence is a non-trivial extension of statistical, λ-statistical and lacunary statistical convergences. Finally, he also introduced boundedness of a sequence in the sense of αβ-statistical convergence.

Nuray (1999) derived algebraic and order properties, preservation under uniform convergence, Cauchy properties, and properties of cluster points. And also showed that for certain types of statistical convergence stronger convergence properties also hold and also he further discussed the relationship between generalized statistical convergence and subsets of the Stone-Cech compactification of the integers.

Bala (2011) introduced the concept of weak statistical convergence of sequence of functionals in a normed space. He also showed that in a reflexive space, weak statistically convergent sequences of functional are the same as weakly statistically convergent sequences of functionals. Gumus (2015) in his study, he provided a new approach to *I* - statistical convergence. And he also introduced a new concept with *I* - statistical convergence and weak convergence together and call it weak *I* - statistical convergence or WS(I) - convergence. Then he introduced this concept for lacunary sequences and obtained lacunary weak *I*- statistical convergence i.e. WSq (I) - convergence.

Hazarika and Sava (2013) in their study introduced the concept of λ-statistical convergence in n-normed spaces. Some inclusion relations between the sets of statistically convergent and λ-statistically convergent sequences are established. They also find its relations to statistical convergence, (C,1)-summability and strong (V; λ)-summability in n-normed spaces. Kaya, Kucukaslan and Wagner (2013) investigated the properties of statistically convergent sequences. Also they introduced definition of statistical monotonicity and upper (or lower) peak points of real valued sequences. They also studied the interplay between the statistical convergence and the above concepts. Finally, the statistically monotonicity is generalized by using a matrix transformation.

Borgohain (2011). Studied the concept of density for sets of natural numbers in some lacunary A-convergent sequence spaces. He also investigated some relation between the ordinary convergence and module statistical convergence for every unbounded modulus function. Moreover he also studied some results on the newly defined lacunary f -statistically A-convergent sequence spaces with respect to some Musielak-Orlicz function.

Dutta, (2013) introduces a new concept of strong summability and statistical convergence of sequence of fuzzy numbers and established relations between them. Erkus and Duman (2003) use the concept of A-statistical convergence which is a regular (non-matrix) summability method, and obtained a general Korovkin type approximation theorem which concerns the problem of approximating a function f by means of a sequence $\{L_n f\}$ of positive linear operators.

Vinod et al (2012) conducted a study to see how weak ideal convergence 'looks like' in lp spaces and extend the recently introduced concept of weak$*$ statistical convergence to have a new concept of weak$*$ ideal convergence. And they gave the necessary and sufficient conditions for a sequence of bounded linear functional on a Banach space to be weak$*$ ideal convergent. Esi (summitted) defined the concept of lacunary double Δ^m $-$statistical convergent sequences in probabilistic normed space and gave some results. The main purpose of his study was to generalize the results for double sequences on statistical convergence in probabilistic normed space given by Esi and Özdemir (2003) earlier and obtained some results on lacunary statistical convergence for double generalized difference sequences on probabilistic normed spaces. The results they obtained are more general than the results of Esi (2003).

Miller and Orhan (2004) Studied the basis for comparing rates of convergence of two null sequences which shows that "$x = \{x_n\}$ converges (stat T) faster than $z = \{z_n\}$ provided that($(x_n = z_n)$) is T-statistically convergent to zero" where $T = (t_{mn})$ is a mean. They also extended the previously known results either on the ordinary convergence or statistical rates of convergence of two null sequences. They also considered lacunary statistical rates of convergence.

Based on the concept of new type of statistical convergence defined by Aktuglu, Srivastava (2015) has introduced the weighted $\alpha\beta$ $-$statistical convergence of order θ in case of fuzzy functions and classified it into pointwise, uniform and equi-statistical convergence. He has checked some basic properties and investigated convergences in terms of their α $-$cuts. The interrelations among them are also established. He has also proved that continuity, boundedness etc are preserved in the equi-statistical sense under some suitable conditions, but not in pointwise sense.

Bardaro *et al* (2015) dealt with a new type of statistical convergence for double sequences, called Ψ-A-statistical convergence, and proved a Korovkin-type approximation theorem with respect to this type of convergence in modular spaces. Finally, they gave some applications to moment-type operators in Orlicz spaces.

Kostyrko et al (2000) introduced the concept of I-convergence of sequences of real numbers based on the notion of the ideal of subsets of N. The I-convergence gives a unifying look on several types of convergence related to the statistical convergence. In a sense it is equivalent to the concept of μ-statistical convergence introduced by J. Connor (μ being a two valued measure defined on a subfield of 2^N).

Savas and Esi (2012) defined statistical analogues of convergence and Cauchy for triple sequences on probabilistic normed space. Ulusu and Nuray (2012) defined lacunary statistical convergence for sequences of sets and study in detail the relationship between other convergence concepts.

Note: It should be noted that, Statistical convergence is a natural generalization of the usual (classical) convergence of sequences. The idea which is used to define new type of convergence was the following; a sequence may have infinitely many terms which are not including in ε $-$neighbourhoods of the limit point for ε small enough but the set of indices of such terms have density zero. As it is well known this is not possible in ordinary sense. Therefore, new type of convergences defined in this way give us a new type of convergence which is different from the ordinary convergence. In many years, researchers focused on convergences which are obtained from different density functions. But a careful observation shows that all density functions are based on different class of intervals. For example, statistical convergence and lacunary convergence are based on intervals [1, n] and $(k_{n-1}, k_n]$. it should be noted that, any convergence sequence is statistically convergent but the converse is not true.

Recall that all finite subsets of natural numbers have density zero. If we combine this fact by that of ordinary convergence of a sequence to a real number l, implies that

$$\{k: |x_k - l| \geq \varepsilon\}$$

is a finite set which shows that ordinary convergence implies statistical convergence. Also the boundedness property does not hold by statistical convergence. Recall that in the sense of ordinary convergence, convergent sequences are all bounded. So this also shows that statistical convergence is different from ordinary convergence. Every convergence sequence is statistically convergent with same limit. Sequences which are statistically convergent may neither be convergent nor bounded.

Example 3.4: Consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} k, & if\ k\ is\ a\ square \\ 0, & otherwise \end{cases}$$

It can be seen that this sequence is statistically convergent to zero but it neither convergent nor bounded ordinarily.

Example 3.5: Consider a sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & if\ k = 2n \\ 2, & if\ k = 2n-1 \end{cases}, \text{for } n = 1,2,3, \ldots$$

This sequence is not statistically convergent

IV. Statistics in Statistical Convergence

Statistics is a branch of mathematics concerned with the collection, organising, analysing, summarising and interpretation of numerical facts about a given situation in order to draw valid conclusion. There are two branches of statistics: inferential and descriptive. Inferential statistics is usually used for two tasks: to estimate the properties of a population given sample characteristics and to predict property of a system given its past and current properties. To do this, specific statistical constructions were invented. The most popular and useful of them are the average or mean (or more precisely arithmetic mean) μ and standard deviation σ (variance σ^2).

To make prediction for future, statistics accumulates data for some period of time. To know about the whole population, samples are used. Normally, such inferences (for future or for population) are based on some assumptions on limit process and their convergence. Iterative processes are used widely in statistics. For example, the empirical approach to probability based on the law (or better to say supposition) of large numbers, states that a procedure repeated again and again, the relative frequency probability approaches the actual probability. The foundation for estimating population parameters and hypothesis testing is formed by the central limit theorem which tell us how sample mean change when the sample size grows. In experiments, scientists measure how statistical characteristics (e.g., means or standard deviations converge Harris and Chiang (1999).

Boshernitzan et al (2000) consider generalizations of the pointwise and mean ergodic theorems to ergodic theorems averaging along different subsequences of the integers or real numbers. The Birkhoff and Von Neumann ergodic theorems give conclusions about convergence of average measurements of systems when the measurements are made at integer times. They also considered the case when the measurements are made at times a(n) or ([a(n)]) where the function a(x) is taken from a class of functions called a Hardy field, and also assume that $|a(x)|$ goes to infinity slower than some positive power of x. A special, well-known Hardy field is Hardy's class of logarithmico-exponential functions. Their main interest is to point out that for a function a(x) as described above, a complete characterization of the ergodic averaging behaviour of the sequence ([a(n)]) is possible in terms of the distance of a(x) from (certain) polynomials.

Chu (2008) studied weighted polynomial multiple ergodic averages. A sequence of weights is called universally good if any polynomial multiple ergodic average with this sequence of weights converges in L^2. They also found a necessary condition and show that for any bounded measurable function $\emptyset$ on an ergodic system, the sequence $\emptyset(T^n x)$ is universally good for almost every x.

Host and Kra (2003) studied the L^2-convergence of two types of ergodic averages. The first is the average of a product of functions evaluated at return times along arithmetic progressions, such as the expressions appearing in Furstenberg's proof of Szemerėdi's Theorem. The second average is taken along cubes whose sizes tend to +∞. For each average and showed that it is sufficient to prove the convergence for special systems, the characteristic factors. They built these factors in a general way, independent of the type of the average. To each of these factors they associated a natural group of transformations and give them the structure of a nilmanifold. From the second convergence result they derived a combinatorial interpretation for the arithmetic structure inside a set of integers of positive upper density.

Convergence of means/averages and standard deviations have been studied by many authors and applied to different problems (Akcoglu and Sucheston, 1975: Akcoglu and Del Junco, 1975: Assani, 2003, 2005: Dunford and Schwartz, 1955: Frantzikinakis and Kra, 2005: Host and Kra, 2005: Jones and Rosenblatt, 1992: Leibman, 2002, 2005, Vapnik and Chervonenkis, 1981). Convergence of statistical characteristics such as mean and standard deviations are related to statistical convergence.

Note: Finally it is clear that, statistical convergence is not a natural generalization of convergence in statistics. But rather a conditional equivalent to convergence in statistics as follows:
A sequence l is statistically convergent if and only if its sequence of partial averages $\mu(l)$ converges and its sequence of partial standard deviations $\sigma(l)$ converges to zero.
To each sequence $l = \{x_k ; k = 1,2,3,\dots\}$ of real numbers, it is possible to correspond a new sequence $\mu(l) = \{\mu_n = (1/n)\sum_{k=1}^{n} x_k;\ n = 1,2,3,\dots\}$ of its partial averages (means). Here a partial average of l is equal to $\mu_n = (1/n)\sum_{k=1}^{n} x_k$.
Sequences of partial averages/means play an important role in the theory of ergodic systems Billingsley (1965). Indeed, the definition of an ergodic system is based on the concept of the "time average" of the values of some appropriate function g arguments for which are dynamic transformations T of a point x from the manifold of the dynamical system. This average is given by the formula
$g(x) = \lim (1/n)\sum_{k=1}^{n-1} g(T^k x)$.
In other words, the dynamic average is the limit of the partial averages/means of the sequence $\{T^k x\ ;\ k = 1,2,3,\dots\}$.
Let $l = \{x_k : k = 1,2,3,\dots\}$ be a bounded sequence, i.e., there is a number m such that $|x_k| < m$ for all $k \in \mathbb{N}$. This condition is usually true for all sequences generated by measurements or computations, i.e., for all sequences of data that come from real life.

Theorem 4.1 [Burgin and Duman, (2006)]: If $x = S$-lim l, then $x = \lim \mu(l)$.
Proof: Since $x = S - lim\ l, for\ every\ \varepsilon > 0$, we have

$$\lim_{n\to\infty}(1/n)\,|\{k \le n, k \in \mathbb{N}: |x_k - x| \ge \varepsilon\}| = 0$$

As $|x_k| < m$ for all $k \in \mathbb{N}$, there is a number p such that $|x_k - x| < p$ for all $k \in \mathbb{N}$. Namely, $|x_k - x| \le |x_k| + |x| = p$.
Taking the set $L_{n,\varepsilon}(x) = \{k \le n, k \in \mathbb{N}: |x_k - x| \ge \varepsilon\}$, denoting
$|L_{n,\varepsilon}(x)|$ by u_n, and using the hypothesis $|x_k| < m$ for all $k \in \mathbb{N}$, we have the following system of inequalities:

$$\begin{aligned} |\mu_n - x| &= \left|(1/n)\sum_{k=1}^{n} x_k - x\right| \\ &\le (1/n)\sum_{k=1}^{n}|x_k - x| \\ &\le \frac{1}{n}\{ku_n + (n - u_n)\varepsilon\} \\ &\le \frac{1}{n}\{ku_n + n\varepsilon\} \\ &= \varepsilon + k\left(\frac{u_n}{n}\right) \end{aligned}$$

From equation (1) above, we get, for sufficiently large n, the inequality $|\mu_n - x| < \varepsilon + k\varepsilon$ Thus, $x = \lim \mu(l)$. Hence, the theorem is proved.
Remark 4.1. However, convergence of the partial averages/means of a sequence does not imply statistical convergence of this sequence as the following example demonstrates.
Example 4.1: Let us consider the sequence $l = \{x_k : k = 1,2,3,\dots\}$ whose terms are $x_k = (-1)^k\sqrt{k}$. This sequence is statistically divergent although $\lim \mu(l) = 0$.
Taking a sequence $l = \{x_k : k = 1,2,3,\dots\}$ of real numbers, it is possible to construct not only the sequence $\mu(l) = \{\mu_n = \left(\frac{1}{n}\right)\sum_{k=1}^{n} x_k ; n = 1,2,3,\dots\}$ of its partial averages (means) but also the sequences $\sigma(l) = \{\sigma_n = (\frac{1}{n}\sum_{k=1}^{n}(x_k - \mu_n)^2)^{1/2},\ n = 1,2,3,\dots\}$ of its partial standard deviations σ_n and $\sigma_n^2(l) = \{\sigma_n^2 = \frac{1}{n}\sum_{k=1}^{n}(x_k - \mu n)2,\ n = 1,2,3,\dots\}$ of its partial variances $\sigma n2$.
Theorem 4.2 [Burgin and Duman, (2006)]: If $x = S - \lim l$ and $|x_k| < m$ for all $k \in \mathbb{N}$, then, $\lim \sigma(l) = 0$.
Proof: We will show that $\lim \sigma^2(l) = 0$. By the definition $\sigma_n^2 = \frac{1}{n}\sum_{k=1}^{n}(x_k - \mu_n)^2 = \left(\frac{1}{n}\right)\sum_{k=1}^{n}(x_k)^2 - \mu_n^2$. Thus $\sigma^2(l) = \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{k=1}^{n}(x_k)^2 - \lim_{n\to\infty}\mu_n^2$.
If $|x_k| < m$ for all $k \in \mathbb{N}$, then there exists a number p such that $|x_k^2 - x^2| < p$ for all $k \in \mathbb{N}$.
Namely, $|x_k^2 - x^2| \le |x_k|^2 + |x|^2 < m^2 + |x|^2 < m^2 + |x|^2 + m + |x| = p$. Let us consider the absolute value of the difference $\mu_n^2 - \left(\frac{1}{n}\right)\sum_{k=1}^{n}(x_k)^2 = \sigma_n^2$. Taking the set $L_{n,\varepsilon}(x) = \{k \le n, k \in \mathbb{N}: |x_k - x| \ge \varepsilon\}$, denoting $|L_{n,\varepsilon}(x)|$ by u_n, and using the hypothesis $|x_k| < m$ for all $k \in \mathbb{N}$, we have the following inequalities:

$$|\sigma_n^2| = \left|\left(\frac{1}{n}\right)\sum_{k=1}^{n}(x_k)^2 - \mu_n^2\right|$$
$$= \left|\left(\frac{1}{n}\right)\sum_{k=1}^{n}(x_k^2 - x^2) - (\mu_n^2 - x^2)\right|$$
$$\leq \left(\frac{1}{n}\right)\sum_{k=1}^{n}|x_k^2 - x^2| + |\mu_n^2 - x^2|$$
$$< \left(\frac{p}{n}\right)\sum_{k=1}^{n}|x_k - x| + |\mu_n^2 - x^2|$$
$$< \left(\frac{p}{n}\right)(u_n + (n - u_n)\varepsilon) + |\mu_n^2 - x^2|$$
$$< \left(\frac{p}{n}\right)(u_n + n\varepsilon) + |\mu_n^2 - x^2|$$
$$= p\left(\frac{u_n}{n}\right) + \varepsilon p + |\mu_n^2 - x^2|$$

As $|x_k^2 - x^2| = |x_k - x|\ |x_k + x| = p.|x_k - x|$. By theorem 4.1, we have $x = \lim \mu(l)$, which guarantees that $\lim \mu_n^2 = x^2$. Also by theorem 4.1, $\lim \left(\frac{\mu_n}{n}\right) = 0$.

Since $\varepsilon > 0$, was arbitrary, the right hand side of the above inequality tends to zero as $n \to \infty$. Therefore, we have $\lim \sigma(l) = 0$. Hence, the theorem is proved.

Corollary 4.1: If $x = S - lim\ l$ and $|\ x_k\ | < m$ for all $k \in \mathbb{N}$, then $\lim \sigma^2(l) = 0$.

Theorem 4.3 [Burgin and Duman, (2006)]: A sequence l is statistically convergent if its sequence of partial averages $\mu(l)$ converges and $x_k \leq \lim \mu(l)$ $(or \quad x_k \geq \lim \mu(l)$ for all $k = 1, 2, 3, \ldots)$

Proof: Let us assume that $x = lim\ \mu(l)$, $x_k \leq \lim \mu(l)$ and take some $\varepsilon > 0$, the set $L_{n,\varepsilon}(x) = \{k \leq n, k \in \mathbb{N}: |x_k - x| \geq \varepsilon\}$, and denote $|L_{n,\varepsilon}(x)|$ by u_n. then we have

$$|x - u_n| = \left|x - \left(\frac{1}{n}\right)\sum_{k=1}^{n} x_k\right|$$
$$= \left|\left(\frac{1}{n}\right)\sum_{k=1}^{n}(x - x_k)\right|$$
$$= \left(\frac{1}{n}\right)\sum_{k=1}^{n}(x - x_k)$$
$$\geq \left(\frac{1}{n}\right)\sum_{k=1}^{n}|x - x_k|$$
$$\geq \varepsilon(x - x_k)$$
$$\geq \left(\frac{u_n}{n}\right)\varepsilon$$

Consequently, $\lim_{n\to\infty}|x - \mu_n| \geq \lim_{n\to\infty}\left(\frac{u_n}{n}\right)\varepsilon$. as $\lim_{n\to\infty}|x - \mu_n| = 0$, and ε is a fixed number, we have $\lim_{n\to\infty}\left(\frac{1}{n}\right)|\{k \leq n, k \in \mathbb{N}: |x_k - x| \geq \varepsilon\}| = 0$, i.e, $x = S - \lim l$.

The case when $x_k \geq \lim \mu(l)$ for all $k = 1, 2, 3, \ldots$ is considered in a similar way. Hence, the theorem is proved as ε is an arbitrary positive number.

Theorem 4.4 [Burgin and Duman, (2006)]: A sequence l is statistically convergent if and only if it sequence of partial averages $\mu(l)$ converges and its sequences of partial standard deviation $\sigma(l)$ converges to zero.

Proof: Necessary condition follows from theorem 4.1 and 4.2.

Sufficient condition: let us suppose that $x = \lim \mu(l)$, $\lim \sigma(l) = 0$, and take some $\varepsilon > 0$. This implies that for any $\lambda > 0$, there exists a number n such that $\lambda > |x - \mu_n|$. then taking a number n such that it implies the inequality $\varepsilon > \lambda$, we have

$$\sigma_n^2 = \left(\frac{1}{n}\right)\sum_{k=1}^{n}(x_k - \mu_n)^2$$
$$\geq \left(\frac{1}{n}\right)\sum_{k=1}^{n}\{(x_k - \mu_n)^2;\ |x_k - \mu_n| \geq \varepsilon\}$$
$$\left(\frac{1}{n}\right)\sum_{k=1}^{n}\left\{\left((x_k - x) + (x - \mu_n)\right)^2;\ |x_k - x| \geq \varepsilon\right\}$$
$$\left(\frac{1}{n}\right)\sum_{k=1}^{n}\left\{\left((x_k - x) \pm \lambda\right)^2;\ |x_k - x| \geq \varepsilon\right\} \qquad (4.1)$$

$$\left(\frac{1}{n}\right)\sum_{k=1}^{n}\{(x_k - x)^2 \pm 2\lambda(x_k - x) + \lambda^2);\ |x_k - x| \geq \varepsilon\}$$

$$\left(\frac{1}{n}\right)\sum_{k=1}^{n}\{(x_k - \mu_n)^2;\ |x_k - x| \geq \varepsilon\} \pm 2\lambda\left(\frac{1}{n}\right)\sum_{k=1}^{n}\{(x_k - x);\ |x_k - x| \geq \varepsilon\} + \lambda^2 \qquad (4.2)$$

As $(x_k - \mu_n) = (x_k - x) + (x - \mu_n)$ and we take λ or $-\lambda$ in the expression (4.1) according to the following rules:

1) If $(x_k - x) \geq 0$ and $(x - \mu_n) \geq 0$, then $(x_k - x) + (x - \mu_n) \geq (x_k - x) > (x_k - x) - \lambda$, we take $-\lambda$;
2) If $(x_k - x) \geq 0$ and $(x - \mu_n) \leq 0$, $(x_k - x) + (x - \mu_n) \geq (x_k - x) - |x - \mu_n| > (x_k - x) - \lambda$, and we take $-\lambda$;
3) If $(x_k - x) \leq 0$ and $(x - \mu_n) \geq 0$, then $|(x_k - x) + (x - \mu_n)| = |(x - x_k) - (x - x_k)| > |(x - x_k) - \lambda = xk - x + \lambda$, and we take $+\lambda$;
4) If $(x_k - x) \leq 0$ and $(x - \mu_n) \leq 0$, then $|(x_k - x) + (x - \mu_n)| \geq |x - x_k| > |(x_k - x) + \lambda|$ as $x_k - x < -\varepsilon$, and we take $+\lambda$.

In the expression (4.2), it is possible to take a sequence $\{\lambda_p; p = 1,2,3, \dots\}$ such that the sequence λ_k^2 converges to 0 because the sequence $\{\mu_n; n = 1,2,3, \dots\}$ converges to x when n tends to ∞. The sum $2\lambda_k \left(\frac{1}{n}\right)\sum_{k=1}^{n}\{(x_k - x; xk - x \geq \varepsilon$ also converges to 0 when p tends to ∞ because λp converges to 0 and

$$\left(\frac{1}{n}\right)\sum(x_k - x) < \left(\frac{1}{n}\right)\sum(|x_k| + |x|) \leq m + |x|.$$

At the same time, the sequence $\{\sigma_n: n = 1,2,3, \dots\}$ also converges to 0. Thus,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum\{(x_k - \mu_n)^2; |x_k - x| \geq \varepsilon\} = 0$$

This implies that

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum\{|x_k - \mu_n|^2; |x_k - x| \geq \varepsilon\} = 0$$

At the same time,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum\{|x_k - \mu_n|^2; |x_k - x| \geq \varepsilon\} \geq \varepsilon \cdot \lim_{n\to\infty}\left(\frac{1}{n}\right)|\{k \leq n, k \in N; |x_k - x| \geq \varepsilon\}| = 0$$

For any $\varepsilon > 0$ as ε is an arbitrary positive number, that is,

$$x = S - \lim$$

Hence, the theorem is proved.

V. Conclusion

We have reviewed the classical theory of ordinary convergence of sequences which successfully shows the link between ordinary convergence and statistical convergence of sequences. Finally, we exposed profoundly the link between statistical convergence and convergence in statistics such as mean (average) and standard deviation.

It should noted here that, all results on statistical convergence of single sequence have similar versions for statistical convergence of double sequences. For further reading on results for double sequences [see Mursaleen and Edely (2003)], [Mursaleen (2004)], Cakan, C. et al (2006), Asi, A. (2013), Miller, H. I. And Miller, V. (2008), Sarabadan, S. And Talabi, S. (2012), Alotaibi, A. M. (2010), Vinod, C. T. And Sarma, B. (2006), Patterson, R. F. (1999, 2002) and many others.

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