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The Discretized Backstepping Method for Stabilizing and Solving Nonlinear Parabolic Partial Differential Equations

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Abstract: *The discretized approach for the space variable will be used in this paper to transform nonlinear parabolic partial differential equation into system of ordinary differential equations, then using the backstepping transformation approach to stabilize and solve the obtained system of nonlinear ordinary differential equations, based on evaluating the Lyapunov function of the system which stabilize the original nonlinear partial differential equation.*

Keyword: *Backstepping method, Parabolic partial differential equation, Stabilization of boundary control problems, Lyapunov function.*

I. Introduction:

In the last ten years of the last century, the feedback design for nonlinear systems has experienced a growing popularity and many issues of major interest, with the rapid development of electrical circuit, mechanical systems, control systems and other engineering and scientific disciplines, they need to absorb and digest a wide range of nonlinear analysis tools, [5].

These nonlinear phenomena make the study of stabilization need an active method that can pass and make it stable. In that time the backstepping method appeared and became as a robust version of feedback linearization for nonlinear systems with uncertainties. Backstepping was particularly inspired by situations in which a plant nonlinearity, and the control input that needs to compensate for the effects of the nonlinearity are in different equations, [8].

Because backstepping method has the ability to cope with not only control synthesis challenges of this type but also much broader classes of systems and problems, such as unmeasured states, unknown parameters, zero dynamics, stochastic disturbances, and systems that are neither feedback linearizable nor even completely controllable, [8].

Backstepping method is a particular approach to stabilize dynamic systems and is particularly successful in the area of nonlinear control problems, [7]. Backstepping is unlike any of the methods previously developed in literatures for controlling partial differential equations (PDEs). It differs from optimal control methods in that it sacrifices optimality, [10].

The idea of integrator backstepping seems to be appeared simultaneously and often implicitly, in the works of Koditschek in 1987 [6], Sonntag and Sussmann in 1988 [11], Tsinias in 1989 [12] and Byrnes and Isidori in 1989 [3].

We will introduce in this paper a new discretized backstepping approach for finding the boundary controller function which stabilizes the nonlinear parabolic PDEs by transformation into an equivalent stable closed loop. This approach has its basic idea on transforming the PDE into a system of ordinary differential equations (ODE) and using the backstepping method to solve the resulting system which make our system stable. This approach is more easy and powerful than other approaches.

II. Fundamental of Backstepping Method

The method that we will present here reveals a key issue for finding the backstepping controls for arbitrarily unstable systems of nonlinear parabolic PDEs. By coordinate transformation into a target system which is equivalent to the original one, [5].

The stabilization problems for nonlinear systems are today the most commonly solved problems using the methods of feedback linearization and backstepping. These methods apply diffeomorphic coordinate transformations that transforms the system equations in the form where the stabilization problem becomes easy (the control input has access to all the nonlinearities).

The difference between the feedback linearization and backstepping methods is that feedback linearization was invented for systems with perfect models, while backstepping, developed later, allows some flexibility to deal with systems that contain perturbations, disturbances, and unmodeled dynamics therefore mixing these two methods will give a very accurate results, [1].

The main idea of backstepping method for nonlinear equations is to find the coordinate transformation:

$$w = u - \alpha(u) \quad \ldots(1)$$

which transforms the unstable nonlinear parabolic PDEs:

$$u_t(t,x) = u_{xx}(t,x) + f(u_i) \quad \ldots(2)$$

with initial and boundary conditions:

$$u(0,x) = g(x),\ x \in [0,1] \quad \ldots(3)$$

$$u(t,0) = 0,\ u(t,1) = U(t),\ t \geq 0 \quad \ldots(4)$$

into the exponentially stable target system:

$$w_t(t,x) = w_{xx}(t,x) \quad \ldots(5)$$

with boundary conditions

$$w(t,0) = 0, w(t,1) = 0 \quad \ldots(6)$$

where$0 < x < 1$, $t \geq 0$ and f is a nonlinear functions of u and $U(t)\colon C[0,1] \to C[0,1]$ is the nonlinear feedback control function.

III. Solution of Nonlinear Parabolic Partial Differential Equations:

The nonlinear parabolic PDEs (2)-(4), will be discretized into an equivalent system of nonlinear ODEs and upon using the coordinate transformation (1) to transform this system of ODEs to an equivalent one related to the target system (5)-(6) which is exponentially stable.

This approach may be divided into three steps:

Step 1: Fix $n \in \mathbb{N}$ and $h = \frac{1}{n+1}$ is the step size of the discretization over [0, 1]. Also let $u_i(t) = u(t, ih)$ for all $i = 0,1,\ldots,n+1$, where it is assumed that $u_0(t)$ is the first boundary condition and $u_{n+1}(t)$ is the control function and using the central difference discretization for $u_{xx}(t,x)$, we have:

$$u_0 = 0 \quad \ldots(7)$$

$$\frac{du_i}{dt} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f(u_i) \quad \ldots(8)$$

$$u_{n+1} = U(t) \quad \ldots(9)$$

Similar discretization for the target system (5)-(6), will give:

$$w_0 = 0 \quad \ldots(10)$$

$$\frac{dw_i}{dt} = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} \quad \ldots(11)$$

$$w_{n+1} = 0 \quad \ldots(12)$$

Step 2: Using the discretized backstepping coordinate transformation:

$$w_i = u_i - \alpha_{i-1}(u_1, u_2, \ldots, u_{i-1}),\ i = 1,2,\ldots,n \quad \ldots(13)$$

To find the value of α_i, which make the nonlinear parabolic PDEs (2)-(4) stable, driving equation with respect to t and using the chain ruleone may get:

$$\frac{dw_i}{dt} = \frac{du_i}{dt} - \left[\frac{\partial \alpha_{i-1}}{\partial u_1}\frac{\partial u_1}{\partial t} + \frac{\partial \alpha_{i-1}}{\partial u_2}\frac{\partial u_2}{\partial t} + \cdots + \frac{\partial \alpha_{i-1}}{\partial u_{i-1}}\frac{\partial u_{i-1}}{\partial t}\right] \quad \ldots(14)$$

Then substituting equations (8) and (11) in (14) and multiply the resulting equation by h^2, to get:

$$w_{i+1} - 2w_i + w_{i-1} = u_{i+1} - 2u_i + u_{i-1} + h^2 f\big(u_i(t)\big) - h^2 \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial u_j}\frac{\partial u_j}{\partial t} \quad \ldots(15)$$

From equation (14) in connection with equation (15)

$$u_{i+1} - \alpha_i - 2u_i + 2\alpha_{i-1} + u_{i-1} - \alpha_{i-2} =$$
$$u_{i+1} - 2u_i + u_{i-1} + h^2 f\big(u_i(t)\big) - h^2 \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial u_j}\frac{\partial u_j}{\partial t} \quad \ldots(16)$$

Eliminate$u$ that have the same index from both sides of equation (16) to get:

$$\alpha_i = 2\alpha_{i-1} - \alpha_{i-2} - h^2 f\big(u_i(t)\big) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial u_j}\left[u_{j+1} - 2u_j + u_{j-1} + h^2 f\big(u_i(t)\big)\right], \quad \ldots(17)$$

for all $i = 1, 2, \ldots, n$.

It is necessary to note that from equations (7) and (11), we have $\alpha_0 = \alpha_{-1} = 0$.

At last from equations (9), (12) and (13) the controller boundary function is given by:

$$U(t) = \alpha_n(u_1, u_2, \ldots, u_n). \quad \ldots(18)$$

which is the nonlinear boundary condition that make equation (2) stable (see [4], [13]).

Step 3: Substitute $U(t)$given by equation (18) back into equation (8) for $i = n$, a system of nonlinear$n$ first order ODEs, is obtained.

The solution of obtained system of ODEs may be found by the linearization method or any other numerical method for solving systems of nonlinear ODEs (see [2]) and (see [9]). The more easy method is using any suitable computer programs like Maple computer software to solve the nonlinear system of ODEs by three steps only.

IV. Illustrative Example

Consider the heat equation:

$$u_t(t,x) = u_{xx}(t,x) + u^2(t,x) \qquad \dots(19)$$

with boundary conditions:

$$u(t,0) = 0,\ u(t,1) = U(t)$$

where$0 \le x \le 1,\ t \ge 0$ and U is the unknown control functions.

Hence using the same steps given above, we proceed as follows:

Step 1: Using the finite difference discretization for the space variable will give:

$$\frac{du_i}{dt} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + f(u_i),$$

where $h = \frac{1}{N+1}$, $u_i = 1,2,\dots,N$ and u_0, u_{N+1} are the boundary conditions.

For simplicity, let $N = 3$ and hence $h = \frac{1}{3+1} = 0.25$. Therefore, the nonlinear discretized system of ODEs is given by:

$$\dot{u}_1 = \frac{u_2 - 2u_1 + u_0}{(0.25)^2} + u_1^2 \Longrightarrow \dot{u}_1 = 16u_2 - 32u_1 + u_1^2$$

$$\dot{u}_2 = \frac{u_3 - 2u_2 + u_1}{(0.25)^2} + u_2^2 \Longrightarrow \dot{u}_2 = 16u_3 - 32u_2 + 16u_1 + u_2^2$$

$$\dot{u}_3 = \frac{u_4 - 2u_3 + u_2}{(0.25)^2} + u_3^2 \Longrightarrow \dot{u}_3 = 16U - 32u_3 + 16u_2 + u_3^2$$

Step 2: Now, the resulting system of differential equation will be solved by backstepping method, as follows:

For $n = 1$:

Let $w_1 = u_1$ and hence $\dot{w}_1 = \dot{u}_1 = 16u_2 - 32u_1 + u_1^2$.

Let $u_2 = \alpha_1(u_1)$, with error $w_2 = u_2 - \alpha_1(u_1)$.

Since $u_2 = w_2 + \alpha_1$, hence $\dot{w}_1 = 16(w_2 + \alpha_1) - 32w_1 + u_1^2$.

Now, consider the control Lyapunov function

$$V_1 = \frac{1}{2}w_1^2,$$

and drive V_1 with respect to time.

$$\begin{aligned}\dot{V}_1 &= \frac{dV_1}{dt}\\ &= \frac{dV_1}{dw_1}\frac{dw_1}{dt}\\ &= w_1(16w_2 + 16\alpha_1 - 32w_1 + u_1^2)\\ &= w_1(16\alpha_1 - 32w_1 + u_1^2) + 16w_1w_2\end{aligned}$$

Now, select:

$$16\alpha_1 = -k_1w_1 - 32w_1 + u_1^2$$

and since $w_1 = u_1$, hencedrive with respect to time, give:

$$\begin{aligned}16\dot{\alpha}_1 &= (-k_1 + 32 - 2u_1)\dot{u}_1\\ &= (-k_1 + 32 - 2u_1)(16u_2 - 32u_1 + u_1^2)\end{aligned}$$

Then:

$\dot{V}_1 = -k_1w_1^2 + 16w_1w_2$, where $k_1 > 0$.

Clearly if $w_2 = 0$, then $\dot{V}_1 = -k_1w_1^2$, and w_1 is guaranteed to converge to zero asymptotically.

For $n = 2$:

From the results when $n = 1$

$$w_2 = u_2 - \alpha_1$$

and hence

$$\begin{aligned}\dot{w}_2 &= \dot{u}_2 - \dot{\alpha}_1\\ &= 16u_3 - 32u_2 + 16u_1 + u_2^2 + \frac{1}{16}(-k_1 + 32 - 2u_1)(16u_2 - 32u_1 + u_1^2)\end{aligned} \qquad \dots(20)$$

In which u_3 is considered as a virtual control input.

Now, define a virtual control low α_2 (error) for w_3 by:

$$w_3 = u_3 - \alpha_2(u_1,u_2) \Longrightarrow u_3 = w_3 + \alpha_2(u_1,u_2) \qquad \dots(21)$$

then equation (21) will be reduced to:

$\dot{w}_2 = 16(w_3 + \alpha_2) - 32u_2 + 16u_1 + u_2^2 + \frac{1}{16}(-k_1 + 32 - 2u_1)(16u_2 - 32u_1 + u_1^2)$

the objective is to ensure $w_2 \to 0$, thus we consider the Lyapunov function:

$V_2 = V_1 + \frac{1}{2}w_2^2$

and therefore:

$$\dot{V}_2 = \dot{V}_1 + w_2\dot{w}_2$$
$$= -k_1w_1^2 + 16w_1w_2 + w_2\left(16(w_3 + \alpha_2) - 32u_2 + 16u_1 + u_2^2 + \frac{1}{16}(-k_1 + 32 - 2u_1)(16u_2 - 32u_1 + u_1^2)\right)$$
$$= -k_1w_1^2 - k_2w_2^2 + 16w_1w_2 + 16w_2w_3 + w_2\left(16\alpha_2 + 16u_1 + k_2w_2 - 32u_2 + u_2^2 + \frac{1}{16}(-k_1 + 32 - 2u_1)(16u_2 - 32u_1 + u_1^2)\right)$$

while w_3 cannot be removed, let:

$16\alpha_2 = -16u_1 - k_2w_2 + 32u_2 - u_2^2 - \frac{1}{16}(-k_1 + 32 - 2u_1)(16u_2 - 32u_1 + u_1^2)$

hence:

$\dot{V}_2 = -k_1w_1^2 - k_2w_2^2 + 16w_2w_3$

if $w_3 = 0$, then $\dot{V}_2 = -\sum_{i=1}^{2} k_iw_i^2$, and w_1, w_2 are converge to zero asymptotically.

For $n = 3$:

As in the case when $n = 1, 2$, and from equation (22)

$w_3 = u_3 - \alpha_2(u_1, u_2)$

Therefore

$\dot{w}_3 = \dot{u}_3 - \dot{\alpha}_2 = \dot{u}_3 - \frac{d\alpha_2}{du_1}\dot{u}_1 - \frac{d\alpha_2}{du_2}\dot{u}_2$

and the new Lyapunov function:

$V_3 = V_2 + \frac{1}{2}w_3^2$

with total derivative:

$$\dot{V}_3 = \dot{V}_2 + w_3\dot{w}_3$$
$$= -\sum_{i=1}^{2} k_iw_i^2 + 16w_2w_3 + w_3\left((16U - 32u_3 + 16u_2 + u_3^2) - \frac{d\alpha_2}{du_1}(16u_2 - 32u_1 + u_1^2) - \frac{d\alpha_2}{du_2}(16u_3 - 32u_2 + 16u_1 + u_2^2)\right)$$
$$= -\sum_{i=1}^{2} k_iw_i^2 - k_3w_3^2 + w_3\left((16U - 32u_3 + 16u_2 + u_3^2) + k_3w_3 + 16w_2 - \frac{d\alpha_2}{du_1}(16u_2 - 32u_1 + u_1^2) - \frac{d\alpha_2}{du_2}(16u_3 - 32u_2 + 16u_1 + u_2^2)\right)$$

Since $w_3 \neq 0$, then:

$$\left(16U - 32u_3 + 16u_2 + u_3^2 + k_3z_3 + 16z_2 - \frac{d\alpha_2}{du_1}(16u_2 - 32u_1 + u_1^2) - \frac{d\alpha_2}{du_2}(16u_3 - 32u_2 + 16u_1 + u_2^2)\right) = 0$$

which will make the system stable, i.e.,

$\dot{V}_3 = -\sum_{i=1}^{3} k_iz_i^2 \leq 0, k_i > 0, i = 1,2,3.$

Finally, the controller $U(t) = u(t, 1)$ is given by:

$$U(t) = \frac{1}{16}\left(32u_3 - 16u_2 - u_3^2 - k_3w_3 - 16w_2 + \frac{d\alpha_2}{du_1}(16u_2 - 32u_1 + u_1^2) + \frac{d\alpha_2}{du_2}(16u_3 - 32u_2 + 16u_1 + u_2^2)\right)$$

Step 3: Since $k_i > 0, i = 1,2,3$ and for computation and comparison purpose let $k_i = 32, i = 1,2,3$, then:

$$U(t) = -3u_2 + \frac{213}{8}u_1u_2 - \frac{66}{16}u_1u_3 - \frac{1}{8}u_2u_3 - \frac{395}{16}u_1^2 - 4u_2^2 - \frac{1}{16}u_3^2 + \frac{973}{64}u_1^3 - \frac{1}{128}u_2^3 - \frac{66}{256}u_1u_2^2 - \frac{1521}{128}u_2u_1^2 - \frac{93}{128}u_1^4$$

Therefore the resulting nonlinear system of ODEs is given by:

$$\left.\begin{array}{l}\dot{u}_1 = 16u_2 - 32u_1 + u_1^2 \\ \dot{u}_2 = 16u_3 - 32u_2 + 16u_1 + u_2^2 \\ \dot{u}_3 = -32u_3 - 32u_2 + 426u_1u_2 - 66u_1u_3 - 2u_2u_3 - \\ \qquad 395u_1^2 - 64u_2^2 + \frac{973}{4}u_1^3 - \frac{1}{8}u_2^3 - \frac{66}{16}u_1u_2^2 - \frac{1521}{8}u_2u_1^2 - \frac{93}{8}u_1^4\end{array}\right\} \quad \ldots(22)$$

Solve system (22) using linearization method with the corporation of the computer software Maple results the following solution:

$$u_1 = e^{-32t}\left(c_1 + c_2 - c_2cos(16t) + c_3sin(16t)\right)$$
$$u_2 = e^{-32t}\left(c_2sin(16t) + c_3cos(16t)\right)$$
$$u_3 = -e^{-32t}\left(c_1 + c_2 - 2c_2cos(16t) + 2c_3sin(16t)\right)$$

where c_1,c_2 and c_3 are any arbitrary constants depending on the initial condition of the PDE (19) or equivalently the system of ODEs (22).

Figure (1) illustrate the numerical solution of system (22) for different values of $t \in [0,1]$ with initial condition $u(0,x) = 1$, which is equivalent to the solution of the original PDE (19).

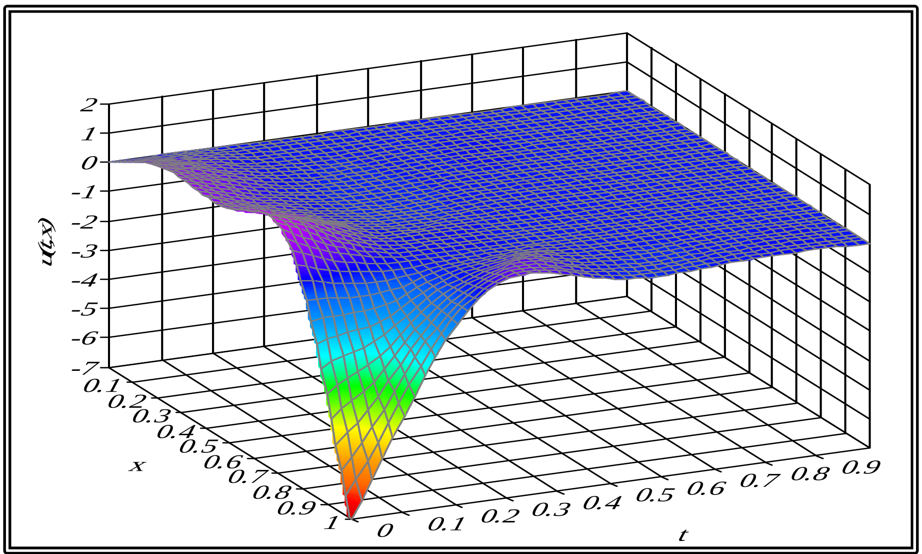

Fig.(1) Closed-loop response with controller for the heat equation (19).

while the controlled function $U(t)$ is presented in Fig.(2).

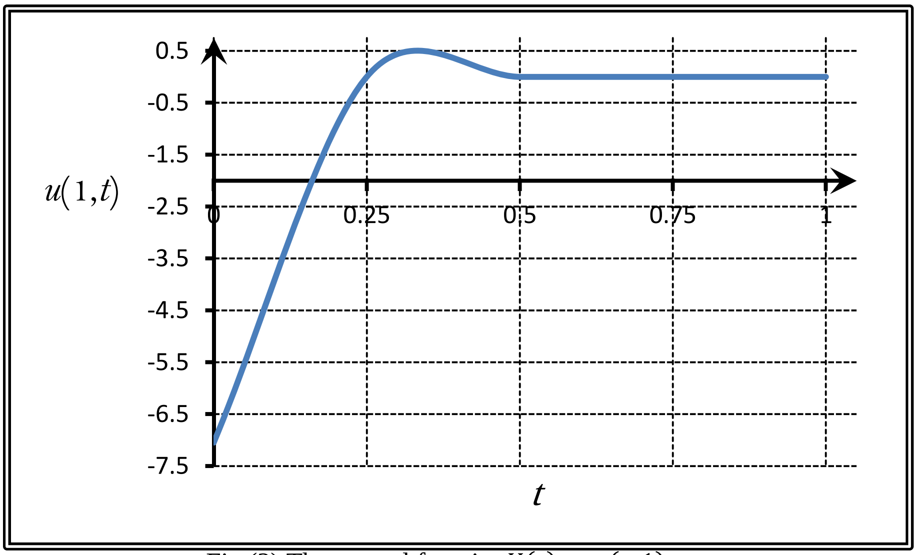

Fig.(2) The control function$U(t) = u(t,1)$.

V. Conclusions

A nonlinear controller based on Lyapunov function method and backstepping design achieves global asymptotic stabilization of unstable nonlinear heat equation has been derived. The result holds for any finite discretization in space of the original PDE model.

The followed approach in this work indicates that a control law designed using only three steps of backstepping can be successfully used to stabilize the nonlinear heat equation.

The followed approach of derivation is easy to apply for stabilizing and solving PDEs which depends on mixing the straightforward approach in the theory of discretization of PDEs, theory of system of ODEs and theory of stability using Lyapunov functions.

The obtained results of the undertaken illustrative example are very accurate in comparison with results obtained by other researchers.

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