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Application of Residue Inversion Formula for Laplace Transform to Initial Value Problem of Linear Ode's

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Abstract: *The application of Laplace transformation in solving Initial value Problems (IVP's) of Ordinary Differential Equations (ODE's) of order $n, n \in \mathbb{Z}^+$ is well known to scholars. The inversion of Laplace transformation in solving initial value problems of ODE's by the traditional algebraic method (i.e. through resolving into partial fraction and the use of Laplace Transforms table) can be very tedious and time consuming, especially when the Laplace transforms table is not readily available, thus renders the researcher handicapped. In this paper, we have reviewed the traditional algebraic method and now show how the Residue Theorem of complex analysis can best be applied directly to obtain the inverse Laplace transform which circumvents the rigor of resolving into partial fraction and the use of Laplace transforms table which normally resolve into resultant time wastage as always the case with the traditional method. Results obtained by applying the Residue approach in solving initial value problems of linear ODE's are experimented and proven to be elegant, efficient and valid.*
***Key Words**: Laplace transforms, Residue, Partial fractions, Poles, etc.*

I. Introduction

Laplace transforms helps in solving differential equations with initial values without finding the general solution and values of the arbitrary constants.

Definition 1: let $f(t)$ be a function defined for all positive values of t, then

$$F(s) = \int_0^\infty e^{-st} f(t)dt \tag{1.1}$$

Provided the integral exist, is called the Laplace transform of $f(t)$ [2]

(1.1) is denoted as $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$

The Laplace transform of simple functions especially those of derivatives are well known to scholars and are normally tabulated.

Consider

$$L\left\{\frac{d^n y}{dt^n}\right\} = S^n L\{y\} - S^{n-1}y(0) - S^{n-2}y'(0) - S^{n-3}y''(0) - \cdots - y^{n-1}(0)$$

where $L\{y''\} = S^2 L\{y\} - Sy(0) - y'(0)$.

The Laplace inverse transform of $F(S)$ written as

$$L^{-1}\{F(S)\} = f(t)$$

is a reverse process of finding $f(t)$ when $F(S)$is known. The traditional method of finding the inverse Laplace transform of say $F(S) = \frac{Q(s)}{P(s)}$ where $P(s) \neq 0$, is to resolve $F(S)$ into partial fractions and use tables of Laplace transform to establish the inverse [1].

If $\frac{Q(s)}{P(s)} \equiv \frac{Q(s)}{(s-a)(s-b)}$ is a proper rational function

Then,

$$\frac{Q(s)}{(s-a)(s-b)} = \frac{A}{(s-a)} + \frac{B}{(s-b)},$$ where a, b are constants and A, B are to be determined.

Hence,

$$L^{-1}\{F(S)\} = L^{-1}\left\{\frac{A}{(s-a)}\right\} + L^{-1}\left\{\frac{B}{(s-b)}\right\}$$
$$= Ae^{at} + Be^{bt}$$

The extension of Cauchy's Integral Formula of complex analysis to cases where the integrating function is not analytic at some singularities within the domain of integration, leads to the famous Cauchy Residue theorem which makes the integration of such functions possible by circumventing those isolated singularities [4]. Here, each isolated singularity contributes a term proportional to what is called the *Residue* of the singularity [3]. Here, the residue theorem provides a straight forward method of computing these integrals.

Definition 2: If $f(z) has\ a\ simple\ pole\ at\ z = a\ \ then$

$$Res(z = a) = \lim_{z \to a}(z - a)f(z)$$

II. Methodology

Consider the function $(S) = \frac{s+1}{s^2+2s}$, which is the Laplace transform of a certain function $f(t)$.

But, $f(t) = L^{-1}\{F(S)\}$

$$= L^{-1}\left\{\frac{s+1}{s^2+2s}\right\} = L^{-1}\left\{\frac{s+1}{s(s+2)}\right\} \quad (2.1)$$

Resolving $F(S)$into partial fractions;

$$\frac{s+1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

$$\frac{s+1}{s(s+2)} = \frac{\left(\frac{1}{2}\right)}{s} + \frac{\left(\frac{1}{2}\right)}{s+2}$$

$$\therefore F(S) = \frac{\left(\frac{1}{2}\right)}{s} + \frac{\left(\frac{1}{2}\right)}{s+2}$$

$$L^{-1}\left\{\frac{s+1}{s^2+2s}\right\} = L^{-1}\left\{\frac{\left(\frac{1}{2}\right)}{s}\right\} + L^{-1}\left\{\frac{\left(\frac{1}{2}\right)}{s+2}\right\} \quad \text{(from table of Laplace transforms)}$$

$$L^{-1}\left\{\frac{s+1}{s^2+2s}\right\} = \frac{1}{2} + \frac{1}{2}e^{-2t} \quad (2.2)$$

2.1 Residue Inversion Approach For Finding Inverse Laplace Transform

The same result in (2.2) above can be obtained by the use of residue Inversion formula for Laplace transform:

THOEREM 1. If the Laplace Transform of $f(t)$ *is given by*

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Then

$$L^{-1}\{F(S)\} = f(t)$$

Where $f(t)$ is the sum of Residues of $e^{st}F(S)$ at the poles of $F(S)$ [3].

Now, consider (2.1) above, where$F(S) = \frac{s+1}{s(s+2)}$, has simple poles at $s = 0$ *and* $s = 2$

Then,

$$Res(s=0) = \lim_{s\to 0}(s-0)e^{st}\frac{s+1}{s(s+2)}$$

$$= \lim_{s\to 0} e^{st}\frac{s+1}{s(s+2)} = \frac{1}{2}e^0 = \frac{1}{2}$$

$$Res(s=-2) = \lim_{s\to -2}(s+2)e^{st}\frac{s+1}{s(s+2)}$$

$$= \lim_{s\to 0} e^{st}\frac{s+1}{s} = \frac{1}{2}e^{-2t}$$

Hence, from Theorem 1

$$L^{-1}\left\{\frac{s+1}{s^2+2s}\right\} = \text{is the sum of Residues of } e^{st}F(S) \text{at the poles of } F(S)$$

$$= \frac{1}{2} + \frac{1}{2}e^{-2t}$$

This is the exact result obtained using partial fractions method and Laplace transforms table.

III. Numerical Experiment

Consider the initial value problem of Linear Ordinary Differential equation with constant coefficients

$$y'' - y' - 2y = 4\ ,\ y(0) = 2 \text{ and } y'(0) = 3 \quad (3.1)$$

By solving (3.1) using Laplace transform method

$$\Rightarrow \text{Given} \quad y'' - y' - 2y = 4 \; with \; y(0) = 2 \; and \; y'(0) = 3$$

Take the Laplace transform of both sides

$$\Rightarrow L\{y''\} + L\{y'\} - 2L\{y\} = L\{4\}$$

$$\Rightarrow S^2L\{y\} - Sy(0) - y'(0) + SL\{y\} - y(0) - 2L\{y\} = L\{4\}$$

$Set \; L\{y\} = \mathfrak{I}$

$$\Rightarrow S^2\mathfrak{I} - Sy(0) - y'(0) + S\mathfrak{I} - y(0) - 2\mathfrak{I} = \frac{4}{S}$$

Applying the initial conditions

$$S^2\mathfrak{I} - 2S - 3 + S\mathfrak{I} - 2 - 2\mathfrak{I} = \frac{4}{S}$$

$$\Rightarrow \mathfrak{I}(S^2 + S - 2) = \frac{4}{S} + 2S + 5$$

$$\Rightarrow \mathfrak{I} = \frac{2S^2+5S+4}{S(S+2)(S-1)}$$

Take the inverse Laplace transform of both sides

$$\Rightarrow L^{-1}\{\mathfrak{I}\} = L^{-1}\left\{\frac{2S^2+5S+4}{S(S+2)(S-1)}\right\} = y \qquad (3.2)$$

3.1 Solve (3.2) Using The Traditional Algebraic Method

Express $\frac{2S^2+5S+4}{S(S+2)(S-1)}$ into partial fractions

$$\Rightarrow \frac{2S^2+5S+4}{S(S+2)(S-1)} = \frac{A}{S} + \frac{B}{S-1} + \frac{C}{S+2}$$

$$\Rightarrow \frac{2S^2+5S+4}{S(S+2)(S-1)} = \frac{-2}{S} + \frac{\frac{11}{3}}{S-1} + \frac{\frac{1}{3}}{S+2}$$

Now,

$$\Rightarrow L^{-1}\left\{\frac{2S^2+5S+4}{S(S+2)(S-1)}\right\} = y = L^{-1}\left\{\frac{-2}{S}\right\} + L^{-1}\left\{\frac{\frac{1}{3}}{S+2}\right\} + L^{-1}\left\{\frac{\frac{11}{3}}{S-1}\right\}$$

From Laplace transforms table

$$\Rightarrow y = -2 + \frac{1}{3}e^{-2t} + \frac{11}{3}e^{t} \qquad (3.3)$$

3.4. SOLVE (3.2) USING RESIDUE INVERSION APPROACH

Consider the working of problem (3.1) up to (3.2) above. The same result in (3.3) above can be obtained from (3.2) above by the application of residue Inversion approach for inverse Laplace transform as establish above; Recalled from Theorem 1

$$L^{-1}\{F(S)\} = \text{ the sum of Residues of } e^{st}F(S) \text{ at the poles of } F(S)$$

From (3.2), $F(S) = \frac{2S^2+5S+4}{S(S+2)(S-1)}$ have simple poles at $S = 0, \; S = 1 \; and \; S = -2$

But,

$$Res(S=0)=\lim_{s\to 0}(s-0)e^{st}\left[\frac{2S^2+5S+4}{S(S+2)(S-1)}\right]=\lim_{s\to 0}e^{st}\left[\frac{2S^2+5S+4}{(S+2)(S-1)}\right]=-2$$

Similarly,

$$Res(S=1)=\frac{11}{3}e^{t}$$

And

$$Res(S=-2)=\frac{1}{3}e^{-2t}$$

Hence,

$$L^{-1}\left\{\frac{2S^2+5S+4}{S(S+2)(S-1)}\right\}=-2+\frac{1}{3}e^{-2t}+\frac{11}{3}e^{t}=y$$

IV. Conclusion

In comparing the methods of finding the inverse Laplace transform from the Residue inversion approach and the traditional method of resolving into partial fraction with the use of tables, both results are exact and valid. However, the method by Residue inversion is more direct, precise, efficient, time saving and has no need of resolving into partial fraction nor referring to Laplace transform tables at anytime for the complete solution of the ODE. Also, the residue inversion approached has to its advantage a fast and direct way of obtaining the solution of the ODE without the rigor of solving for the Homogeneous solution first, then finding the particular solution before applying the initial conditions in other to get the final solution of the ODE as it is the case with other algebraic methods of solving linear ODE's.

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