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Numerical treatments to nonlocal Fredholm –Volterra integral equation with continuous kernel

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Abstract: *In this paper, we consider the nonlocal Fredholm- Volterra integral equation of the second kind, with continuous kernels. We consider three different numerical methods,the Trapezoidal rule, Simpson rule and Collocation method to reduce the nonlocal* ***F-VIE*** *to a nonlocal algebraic system of equations. The algebraic system is computed numerically, when the historical memory of the problem (nonlocal function) takes three cases: when there is no memory, when the memory is linear and when the memory is nonlinear. Moreover, the estimate error, in each method and each case, is computed. Here, we deduce that, the error in the absence of memory is larger than in the linear memory. Moreover, the error of the linear memory is larger than the nonlinear memory.*

Keyword: *nonlocal Fredholm-Volterra integral equation (nonlocal F-VIE), numerical methods, algebraic system (AS), the error estimate.*
MSC (2010): 45B05, 45G10, 60R.

I. Introduction:

Many problems in mathematical physics, contact problems in the theory of elasticity and mixed boundary value problems in mathematical physics are transformed into integral equations of linear and nonlinear cases. The books edited by Green [1], Hochstadt [2], Kanwal [3] and Schiavone et al.

[4]contained many different methods to solve the linear integral equation analytically. At the same time the sense of numerical methods takes an important place in solving the linear integral equations. More information for the numerical methods can be found in Linz[5], Golberg [6], Delves and Mohamed[7], Atkinson[8].The **F-VIE**of the first kind in one, two and three dimensions is considered in [9]. In [10-13] the authors consider many numerical methods to solve the integral equations. In all previous work, the nonlocal term (historicalmemory of the problem) is considered equal zero. Now, and in the following series of work, we will consider the memory historical term and its effect in computing the error.

Consider, in the space $C\,[0,T\,]$, the nonlocal **F-VIE** of the second:

$$\mu\phi(t)=f\,(t)-H\,(t,\phi(t))+\lambda\int_0^1 k\,(t,s)\phi(s)ds+\lambda\int_0^t v\,(t,s)\phi(s)ds. \tag{1.1}$$

Where, the free term $f\,(t)$ and the historical memory of the integral equation $H\,(t,\phi(t))$ are known functions. The two functions $k\,(t,s)$ and $v\,(t,s)$ are continuous kernels of **FI** and **VI** term respectively. While, $\phi(t)$ is unknown function represents the solution of (1.1). In addition, μ is a constant defined the kind of the integral equation; while λ has a physical meaning.

In order to guarantee the existence of a unique solution of (1.1), we assume the following

(i) For a constant $\ell>\{\ell_1,\ell_2\}$, we have

$$(a).\left|H\,(t,\phi(t))\right|\le\ell_1\left|\phi(t)\right|; \qquad (b).\left|H\,(t,\phi(t))-H\,(t,\psi(t))\right|\le\ell_2\left|\phi(t)-\psi(t)\right|.$$

(ii) The continuous kernels $k\,(t,s)$ and $v\,(t,s)$, for all $t,s\in[0,T\,]$ satisfies,

$$\left|k\,(t,s)\right|\le M\,,\ \left|v\,(t,s)\right|\le S\,, \qquad (M\,,S \ \text{are constants}).$$

(iii) The continuous function $f\,(t)$ satisfies $\left\|f\,(t)\right\|_{C\,[0,T\,]}=\max_{0\le t\le T}\left|f\,(t)\right|=F$, (F is a constant).

Theorem 1(without proof): the **nonlocal F-VIE** (1. 1) has a unique solution in the space $C\,[0,T\,]$ under the condition $\left(\ell+|\lambda|M+T\,|\lambda|S\right)<|\mu|;\ max\ t=T$.

The aim of this paper is using three different numerical methods, the Trapezoidal rule, Simpson rule and Collocation method to reduce the nonlocal **F-VIE** to a nonlocal algebraic system of equations. Finally, numerical results are calculated and the error estimate, in each method, is computed.

II. Numerical methods:

In this section, we discuss the solution of the nonlocal **F-VIE** (1.1) numerically using three different methods Trapezoidal rule, Simpson rule and Collocation method, and determine the error in each method.

2.1. Trapezoidal rule:

For solving equation (1. 1) numerically**,** we divide the interval $[0,1]$ into N subintervals with length $h=1/N$; N can be even or odd, where $t=t_i, s=t_j,\ \ 0<i,j<N$.

Then the nonlocal **F-VIE** (1.1) reduce to the following nonlocal AS

$$\mu\phi(t_i)=f(t_i)-H(t_i,\phi(t_i))+\lambda\sum_{j=0}^{N}u_j k(t_i,t_j)\phi(t_j)+\lambda\sum_{j=0}^{N}w_j v(t_i,t_j)\phi(t_j)+R_N. \tag{2.1}$$

Where R_N is the error of the method and u_j, w_j are the weights defined by

$$u_j=\begin{cases}h/2 & j=0,N\\ h & 0<j<N.\end{cases}\qquad w_j=\begin{cases}h/2 & j=0,i\\ h & 0<j<i\\ 0 & j>i.\end{cases} \tag{2.2}$$

After neglecting the error, and then, using the following notations $\phi_i=\phi(t_i),\ f_i=f(t_i), H_i(\phi_i)=H(t_i,\phi(t_i)),\ \ k_{i,j}=k(t_i,t_j), v_{i,j}=v(t_i,t_j)$; the formula (2.1) can be re-written in the following form:

$$\mu\phi_i=f_i-H_i(\phi_i)+\lambda\sum_{j=0}^{N}u_j k_{i,j}\phi_j+\lambda\sum_{j=0}^{N}w_j v_{i,j}\phi_j,\qquad 0\le i\le N. \tag{2.3}$$

The formula (2.3) represents system of $(N+1)$ equations and $(N+1)$ unknowns coefficients. By solving them, we can obtain the approximation solution of (1.1).

Definition 1: The estimate local error R_N of Trapezoidal rule is determined by

$$\begin{aligned}R_N&=\left|\lambda\int_0^1 k(t,s)\phi(s)ds+\lambda\int_0^t v(t,s)\phi(s)ds-\lambda\sum_{j=0}^{N}u_j k_{i,j}\phi_j-\lambda\sum_{j=0}^{N}w_j v_{i,j}\phi_j\right|,\ \ i=0,1,2,...,N.\\ &=-\frac{\lambda}{12}h^2\frac{d^2}{d\xi^2}\left|\left(k(t_N,\xi)\phi(\xi)\right)+\left(v(t_N,\xi)\phi(\xi)\right)\right|,\quad \xi\in[0,1]\end{aligned} \tag{2.4}$$

In order to guarantee the existence of a unique solution of (2.3), we assume the following:

(i') For a constant $\ell'>\{\ell_1',\ell_2'\}$, we have

$(a')\ \ |H_i(\phi_i)|\le\ell_1'|\phi_i|$; $\ \ (b')\ |H_i(\phi_i)-H_i(\psi_i)|\le\ell_2'|\phi_i-\psi_i|$;

$(ii')\ \sup_j\sum_{j=0}^{N}|u_j k_{i,j}|\le M',\qquad \sup_j\sum_{j=0}^{N}|w_j v_{i,j}|\le S'$, ($M', S'$ are constants).

$(iii')\ \|f\|_{\ell_\infty}=\sup_i|f_i|=F'$, ($F'$ is constant).

Theorem 2(without proof): the nonlocal **AS** (2.3) has a unique solution in the space ℓ_∞ under the condition $(\ell'+|\lambda|M'+|\lambda|S')<|\mu|$. •

If $N\to\infty$, then $\{\lambda\sum_{j=0}^{N}u_j k_{i,j}\phi_j+\lambda\sum_{j=0}^{N}w_j v_{i,j}\phi_j\}\to\{\lambda\int_0^1 k(t,s)\phi(s)ds+\lambda\int_0^t v(t,s)\phi(s)ds\}$.

Thus, the solution of the nonlocal **AS** (2.3) becomes the solution of the nonlocal **F-VIE** (1.1).

Corollary 1: If the condition of theorem 2 is satisfied, then $\lim_{N\to\infty}R_N=0$.

2.2. Simpson rule:

For using Simpson rule to solve the nonlocal **F-VIE** (1.1) numerically, we divide the interval $[0,1]$ into N sub-intervals with length $h=1/N$, N is even, $0<i,j<N$. Then, after approximating the integrals term and neglecting the error $\tilde{R}_N$, we have

$$\mu\phi_i=f_i-H_i(\phi_i)+\lambda\sum_{j=0}^{N}\rho_j k_{i,j}\phi_j+\lambda\sum_{j=0}^{N}\vartheta_j v_{i,j}\phi_j,\qquad 0\le i\le N. \tag{2.5}$$

Where the weight ρ_j is defined as

$(\rho_j=h/3, j=0,N);(\rho_j=4h/3,\ 0<j<N,\ j\text{ odd})$ and $(\rho_j=2h/3, 0<j<N,\ \ j\text{ even})$. While, the weight ϑ_j takes two forms depending on the value of i odd or even

1. If i is odd we use Trapezoidal rul**e** and then $\vartheta_j=\tilde{\omega}_j$; $(\tilde{\omega}_j=h/2, j=0,i)$; $(\tilde{\omega}_j=h,\ 0<j<i)$ and $(\tilde{\omega}_j=0, j>i)$.
2. If i is even we use Simpson rul**e** and then $\vartheta_j=\omega_j$, $(\omega_j=h/3,\ j=0,i)$; $(\omega_j=(4h/3),$ $0<j<i,\ \ j\text{ odd})$; $(\omega_j=(2h/3), 0<j<i,\ \ j\text{ even})$ and $(\omega_j=0, j>i)$.

Definition 2: The estimate local error $\tilde{R}_N$ of **Simpson rule** is determined by

$$\tilde{R}_N=\left|\lambda\int_0^1 k(t,s)\gamma(s,\phi(s))ds+\lambda\int_0^t v(t,s)g(s,\phi(s))ds-\lambda\sum_{j=0}^{N}\rho_j k_{ij}\gamma_j(\phi_j)-\lambda\sum_{j=0}^{N}\vartheta_j v_{ij}g_j(\phi_j)\right|,\ \ i=0,1,2,...,N. \tag{2.6}$$

$$=-\frac{1}{180}h^4\frac{d^4}{d\xi^4}\Big|\big(k(t_N,\xi)\gamma(\xi,\phi(\xi))\big)+\big(v(t_N,\xi)g(\xi,\phi(\xi))\big)\Big|,\quad \xi\in[0,1].$$

The nonlocal **AS has a unique solution, under the conditions** (i'); (iii') and replacing (ii') by the following condition

(ii^*) $\sup_j\sum_{j=0}^{N}\left|\rho_j k_{i,j}\right|\le M^*,\quad \sup_j\sum_{j=0}^{N}\left|\vartheta_j v_{i,j}\right|\le S^*$, ($M^*,S^*$ are constants).

Theorem 3(without proof): the nonlocal **AS** (2.5) has a unique solution in the space ℓ_∞ under the condition $(\ell+|\lambda|M^*+|\lambda|S^*)<|\mu|$.•

2.3. Collocation method:

We present the collocation method to obtain the numerical solution of (1.1). The solution is based on approximating $\phi(t)$ in Eq. (1.1) by $Q_N(t)=\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t)$ of $(N+1)$ linearly independent functions

$\psi_0(t),\psi_1(t),...,\psi_N(t)$ on the interval $[0,1]$. Using the principal basic of the collocation method, see [7, 8], we can obtain

$$\mu\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t_i)=f_i-H_i\left(\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t_i)\right)+\lambda\sum_{j=0}^{N}u_j k_{i,j}\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t_j)+\lambda\sum_{j=0}^{N}w_j v_{i,j}\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t_j),\ \ 0\le i\le N. \tag{2.7}$$

The formula (2.7) represents system of $(N+1)$ nonlinear equations for $(N+1)$ unknowns $c_0,c_1,...,c_N$. By solving them we can obtain $c_0,c_1,...,c_N$ and then we get the approximate solution $Q(t)$

Definition 3: The estimate error R'_N of the collocation method is given by

$$R'_N=\left|\lambda\int_0^1 k(t,s)\phi(s)ds+\lambda\int_0^t v(t,s)\phi(s)ds-\lambda\sum_{j=0}^{N}u_j k_{i,j}\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t_j)-\lambda\sum_{j=0}^{N}w_j v_{i,j}\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(t_j)\right|,\ \ i=0,1,2,...,N.$$

$$=-\frac{1}{12}h^2\frac{d^2}{d\xi^2}\left|\left(k(t_N,\xi)\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(\xi)\right)+\left(v(t_N,\xi)\sum_{\varsigma=0}^{N}c_\varsigma\psi_\varsigma(\xi)\right)\right|,\quad \xi\in[0,1].$$

The existence of a unique solution of the nonlocal **AS** (2.7) in the space ℓ_∞ can be proved directly after replacing the condition $(i\,')$ in theorem 2 by the following condition

(i*) For the function $h_i\,(Q_{i,N})$, we assume

$$(a)\left|H_i\left(Q_{i,N}\right)\right| \le \ell^*_{1,i}\left|Q_{i,N}\right|; \quad (b).\left|H_i\left(Q_{i,N}\right)-H_i\left(Q'_{i,N}\right)\right| \le \ell^*_{2,i}\left|Q_{i,N}-Q'_{i,N}\right|;$$

Theorem 4.(without proof): the nonlocal **AS** (2.7) has a unique solution in the Banach space ℓ_∞ under the condition $\left(\ell^*_i+|\lambda|M'+|\lambda|S'\right)<|\mu|,\ \ell^*_i=\max.\{\ell^*_{1,i},\ell^*_{2,i}\}$. ●

III. Numerical Examples

Consider the nonlocal **F-VIE:**

$$\mu\phi(t)=f(t)-H(t,\phi(t))+\lambda\int_0^1 t\,s^2\phi(s)ds+\lambda\int_0^t t\,s\,\phi(s)ds,\ \ (\mu=0.001,\ \lambda=0.01,\ 0\le t\le T\le 1). \qquad (3.1)$$

We use the Trapezoidal method, Simpson method and collocation method to obtain the numerical solution of (3.1) for different value of $\mu=0.1, 0.5 \text{ and } 1$ when $H(t,\phi(t))=0$, and for different value of $h=0.25,\ 0.125$ and 0.0625. When $H(t,\phi(t))$ takes two values $t\phi(t)$, and $\phi^2(t)$, where $\lambda=0.01$, (exact solution is $\phi(t)=t^2$) as following:

(I) When there is no memory term ($H(t,\phi(t))=0$). Here we solve, numerically (3.1) for different value of $\mu=(0.1,\ 0.5, 1)$, $\lambda=0.01$, and $h=0.625$.

case I (**F-VIE**) : Trapezoidal method when $H(t,\phi(t))=0, \lambda=0.01, h=0.625$.							
t	ϕ	$\mu=0.1,\ h=0.625,\ N=16$		$\mu=0.5,\ h=0.625,\ N=16$		$\mu=1,\ h=0.625,\ N=16$	
		ϕ^{Tr}	E^{Tr}	ϕ^{Tr}	E^{Tr}	ϕ^{Tr}	E^{Tr}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25354E-02	3.54000E-05	6.25069E-02	6.90000E-06	6.25034E-02	3.40000E-06
0.5	2.50000E-01	2.50080E-01	8.00000E-05	2.50016E-01	1.60000E-05	2.50008E-01	8.00000E-06
0.75	5.62500E-01	5.62644E-01	1.44000E-04	5.62528E-01	2.80000E-05	5.62514E-01	1.40000E-05
1	1.00000E+00	1.00024E+00	2.40000E-04	1.00005E+00	5.00000E-05	1.00002E+00	2.00000E-05

Table (1)

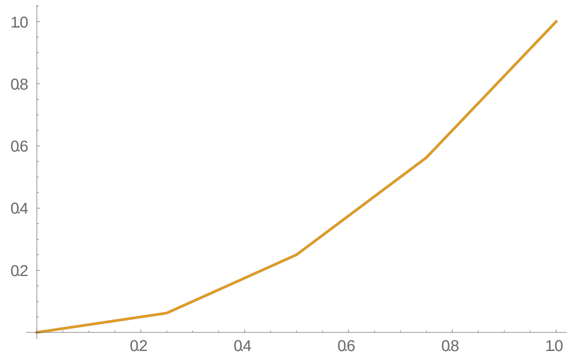

Fig. (1-i) $\mu=0.1, h=0.625$

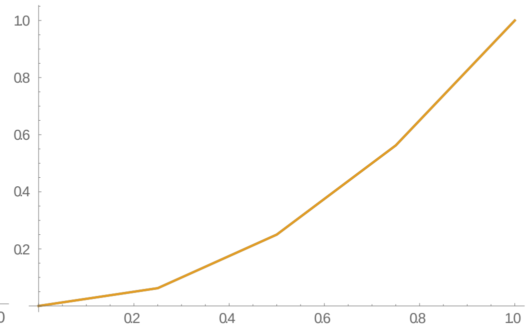

Fig. (1-ii) $\mu=0.5, h=0.625$

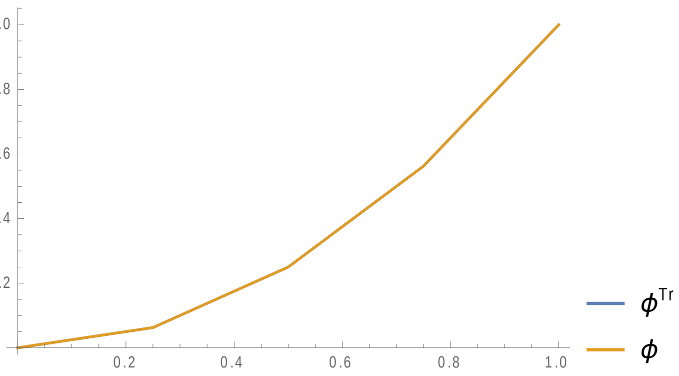

Fig.((1-iii) $\mu=1,\ h=0.625$

(I-1) Figs. (1) describe the relation between the exact solution and numerical solution, when $H\left(t,\phi(t)\right)==0$, using **Trapezoidal method**, with $\left(\lambda=0.01,\ h=0.652,\ \text{and}\ N=16\right)$ at $\mu=0.1$ in Fig. (1.i), $\mu=0.5$ in Fig (1.ii) and $\mu=1$ in Fig. (1.iii) .

case I (**F-VIE**) : Simpson method when $H\left(t,\phi(t)\right)=0, \lambda=0.01, h=0.625.$							
t	ϕ	$\mu=0.1,\ h=0.625,\ N=16$		$\mu=0.5,\ h=0.625,\ N=16$		$\mu=1,\ h=0.625,\ N=16$	
		ϕ^S	E^S	ϕ^S	E^S	ϕ^S	E^S
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25003E-02	3.28764E-07	6.25000E-02	2.09546E-08	6.25000E-02	7.77174E-09
0.5	2.50000E-01	2.50001E-01	6.78743E-07	2.50000E-01	4.26915E-08	2.50000E-01	1.57370E-08
0.75	5.62500E-01	5.62501E-01	1.22643E-06	5.62500E-01	7.20683E-08	5.62500E-01	2.56043E-08
1	1.00000E+00	1.00000E+00	2.65702E-06	1.00000E+00	1.35847E-07	1.00000E+00	4.40445E-08

Table (2)

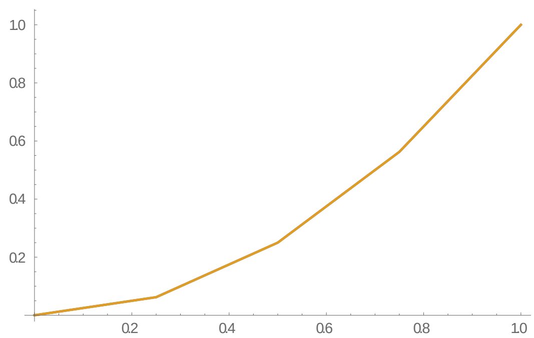

Fig. (2-i) $\mu=0.1,\ h=0.625$

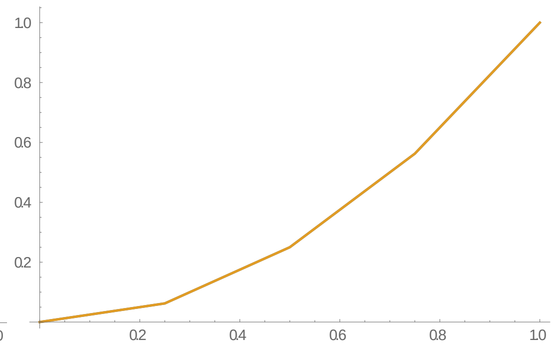

Fig. (2-ii) $\mu=0.5,\ h=0.625$

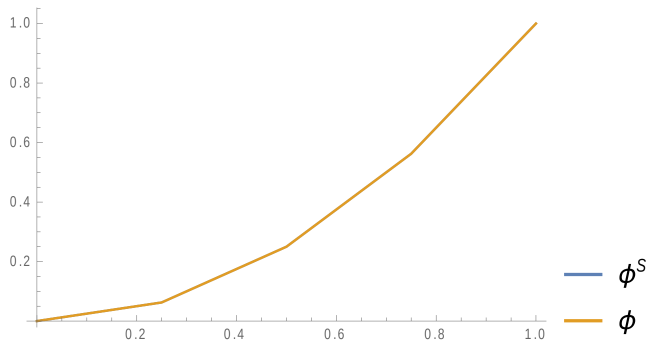

Fig. (2-iii) $\mu=1,\ h=0.625$

(I-2) Figs. (2) describe the relation between the exact solution and numerical solution, when $H(t,\phi(t)) == 0$, using **Simpson method**, with $(\lambda = 0.01,\ h = 0.652, \text{ and } N = 16)$ at $\mu = 0.1$ in Fig. (2.i), $\mu = 0.5$ in Fig (2.ii) and $\mu = 1$ in Fig. (2.iii).

case I (**F-VIE**): collocation method when $H(t,\phi(t)) = 0, \lambda = 0.01, h = 0.625.$							
t	ϕ	$\mu = 0.1,\ h = 0.625,\ N = 16$		$\mu = 0.5,\ h = 0.625,\ N = 16$		$\mu = 1,\ h = 0.625,\ N = 16$	
		ϕ^{Co}	E^{Co}	ϕ^{Co}	E^{Co}	ϕ^{Co}	E^{Co}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25354E-02	3.54000E-05	6.25069E-02	6.90000E-06	6.25034E-02	3.40000E-06
0.5	2.50000E-01	2.50080E-01	8.00000E-05	2.50016E-01	1.60000E-05	2.50008E-01	8.00000E-06
0.75	5.62500E-01	5.62644E-01	1.44000E-04	5.62528E-01	2.80000E-05	5.62514E-01	1.40000E-05
1	1.00000E+00	1.00024E+00	2.40000E-04	1.00005E+00	5.00000E-05	1.00002E+00	2.00000E-05

Table (3)

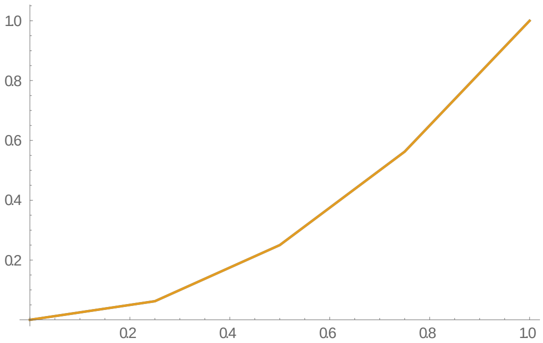

Fig. (3-i) $\mu = 0.1,\ h = 0.625$

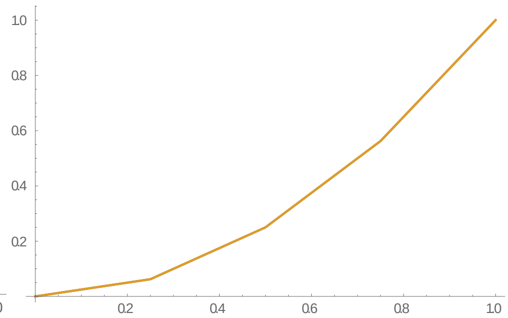

Fig. (3-ii) $\mu = 0.5, h = 0.625$

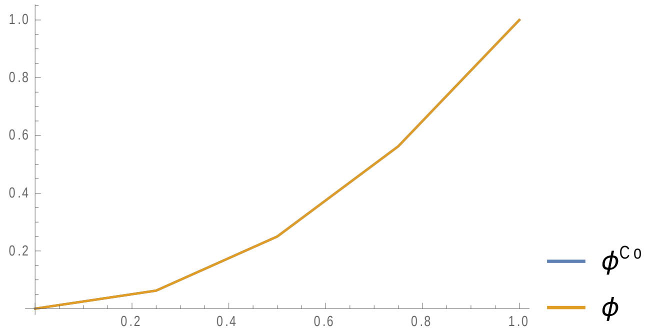

Fig. (3-iii) $\mu = 1,\ h = 0.625$

(I-3) Figs. (3) describe the relation between the exact solution and numerical solution, when $H(t,\phi(t)) == 0$, using **Collocation method**, with $(\lambda = 0.01,\ h = 0.652, \text{ and } N = 16)$ at $\mu = 0.1$ in Fig. (3.i), $\mu = 0.5$ in Fig (3.ii) and $\mu = 1$ in Fig. (3.iii).

(II) When the memory in a linear form ($H(t,\phi(t)) = t\phi(t)$).Here we solve, numerically (3.1) for different value of $h = (0.25,\ 0.125, 0.625)$, $\lambda = 0.01$, and $\mu = 0.001$.

case II (**F-VIE**): Trapezoidal method when $H(t,\phi(t)) = t\phi(t), \lambda = 0.01, \mu = 0.001$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.0625, N = 16$	
		ϕ^{Tr}	E^{Tr}	ϕ^{Tr}	E^{Tr}	ϕ^{Tr}	E^{Tr}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.27171E-02	2.17058E-04	6.25545E-02	5.44981E-05	6.25136E-02	1.36391E-05

0.5	2.50000E-01	2.50247E-01	2.46952E-04	2.50062E-01	6.19713E-05	2.50016E-01	1.55074E-05
0.75	5.62500E-01	5.62796E-01	2.96311E-04	5.62574E-01	7.43101E-05	5.62519E-01	1.85921E-05
1	1.00000E+00	1.00037E+00	3.65434E-04	1.00009E+00	9.15896E-05	1.00002E+00	2.29118E-05

Table (4)

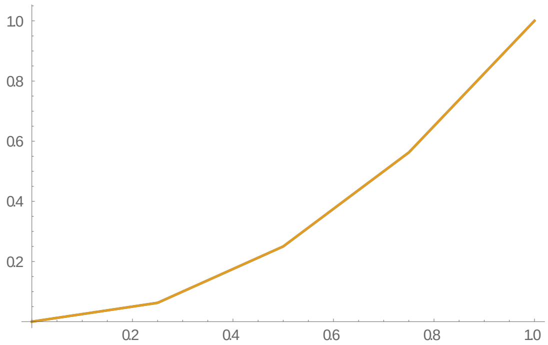

Fig. (4-i) $h=0.25, N=4$

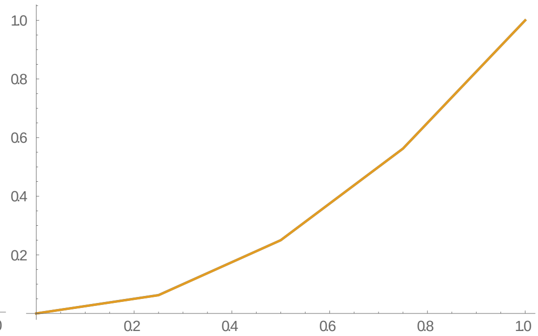

Fig. (4-ii) $h=0.125, N=8$

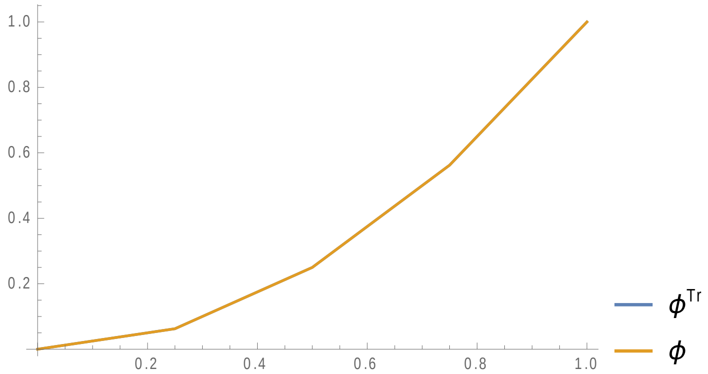

Fig. (4-iii) $h=0.0625, N=16$

(II-1) Figs. (4) describe the relation between the exact solution and numerical solution, when $H(t,\phi(t)) == t\phi(t)$, using **Trapezoidal method**, with $\lambda = 0.01, \mu = 0.001$ at $h = 0.25(N = 4)$; $h = 0.25(N = 8); h = 0.625(N = 16)$ in Fig. (4.i), Fig (4.ii) and Fig.(4.iii), respectively.

case II (**F-VIE**) : Simpson method when $H(t,\phi(t)) = t\phi(t), \mu = 0.001, \lambda = 0.01.$							
t	ϕ	$h=0.25, N=4$		$h=0.125, N=8$		$h=0.0625, N=16$	
		ϕ^S	E^S	ϕ^S	E^S	ϕ^S	E^S
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25151E-02	1.51032E-05	6.25004E-02	3.74939E-07	6.25000E-02	3.32442E-08
0.5	2.50000E-01	2.50005E-01	5.39734E-06	2.50000E-01	3.79457E-07	2.50000E-01	3.42676E-08
0.75	5.62500E-01	5.62593E-01	9.32632E-05	5.62500E-01	3.96161E-07	5.62500E-01	3.84272E-08
1	1.00000E+00	1.00001E+00	5.64260E-06	1.00000E+00	4.40716E-07	1.00000E+00	4.95787E-08

Table (5)

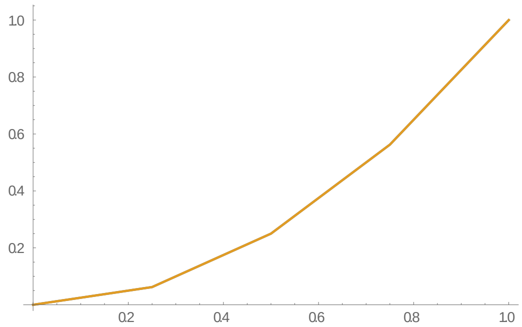

Fig. (5-i) $h = 0.25, N = 4$

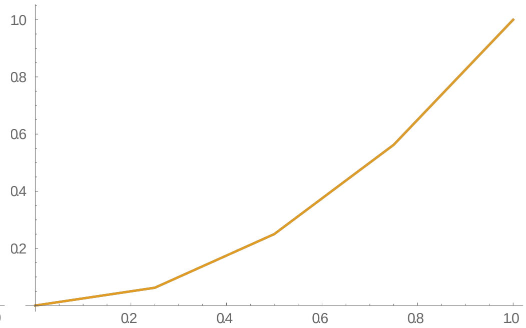

Fig. (5-ii) $h = 0.125, N = 8$

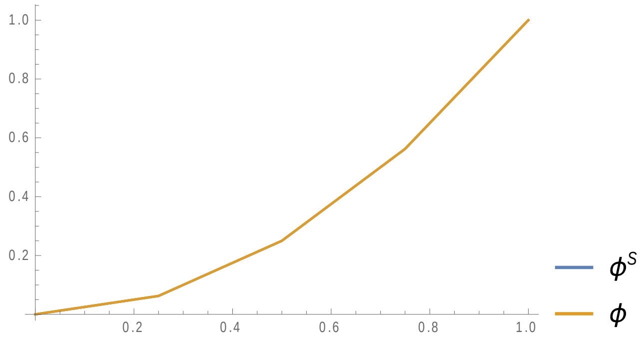

Fig. (5-iii) $h = 0.0625, N = 16$

(II-2) Figs. (5) describe the relation between the exact solution and numerical solution, when $H(t,\phi(t)) == t\phi(t)$, using **Simpson method**, with $\lambda = 0.01, \mu = 0.001$ at $h = 0.25(N = 4);$ $h = 0.25(N = 8); h = 0.625(N = 16)$ in Fig. (5.i), Fig (5.ii) and Fig.(5.iii), respectively.

case II (**F-VIE**) : collocation method when $H(t,\phi(t)) = t\phi(t), \mu = 0.001, \lambda = 0.01.$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.0625, N = 16$	
		ϕ^{Co}	E^{Co}	ϕ^{Co}	E^{Co}	ϕ^{Co}	E^{Co}
0	0.00000E+00	1.41790E-30	1.41790E-30	1.00536E-27	1.00536E-27	1.24261E-18	1.24261E-18
0.25	6.25000E-02	6.27171E-02	2.17100E-04	6.25545E-02	5.45000E-05	6.25136E-02	1.36000E-05
0.5	2.50000E-01	2.50247E-01	2.47000E-04	2.50062E-01	6.20000E-05	2.50016E-01	1.60000E-05
0.75	5.62500E-01	5.62796E-01	2.96000E-04	5.62574E-01	7.40000E-05	5.62519E-01	1.90000E-05
1	1.00000E+00	1.00037E+00	3.70000E-04	1.00009E+00	9.00000E-05	1.00002E+00	2.00000E-05

Table (6)

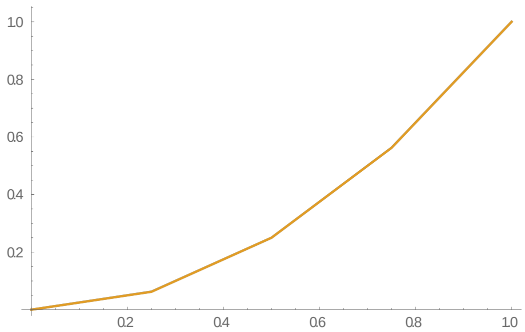

Fig. (6-i) $h = 0.25, N = 4$

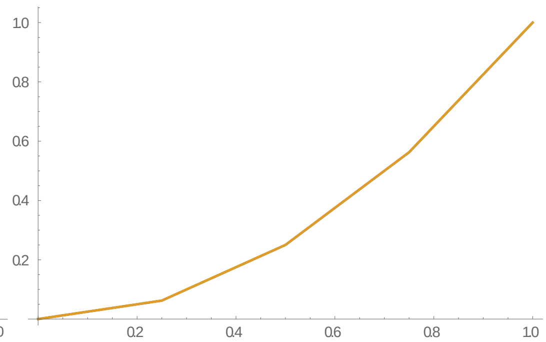

Fig. (6-ii) $h = 0.125, N = 8$

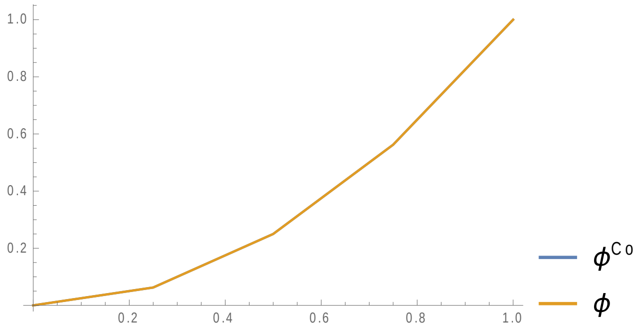

Fig. (6-iii) $h = 0.0625, N = 16$

(II-2) Figs. (6) describe the relation between the exact solution and numerical solution, when $H(t,\phi(t)) == t\phi(t)$, using **Collocation method**, with $\lambda = 0.01, \mu = 0.001$ at $h = 0.25(N = 4)$; $h = 0.25(N = 8); h = 0.625(N = 16)$ in Fig. (6.i), Fig (6.ii) and Fig.(6.iii), respectively.

(III) When the memory in a nonlinear form ($H(t,\phi(t)) = \phi^2(t)$).Here we solve, numerically (3.1) for different value of $h = (0.25, \ 0.125, 0.625)$, $\lambda = 0.01$, and $\mu = 0.001$

case III (**F-VIE**) : Trapezoidal method when $H(t,\phi(t)) = \phi^2(t), \mu = 0.001, \ \lambda = 0.1$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.0625, N = 16$	
		ϕ^{Tr}	E^{Tr}	ϕ^{Tr}	E^{Tr}	ϕ^{Tr}	E^{Tr}
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.29304E-02	4.30400E-04	6.26084E-02	1.08400E-04	6.25271E-02	2.71000E-05
0.5	2.50000E-01	2.50247E-01	2.47000E-04	2.50062E-01	6.20000E-05	2.50016E-01	1.60000E-05
0.75	5.62500E-01	5.62697E-01	1.97000E-04	5.62550E-01	5.00000E-05	5.62512E-01	1.20000E-05
1	1.00000E+00	1.00018E+00	1.80000E-04	1.00005E+00	5.00000E-05	1.00001E+00	1.00000E-05

Table (7)

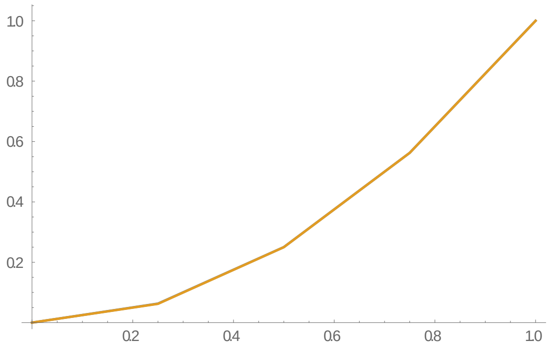

Fig. (7-i) $h = 0.25, N = 4$

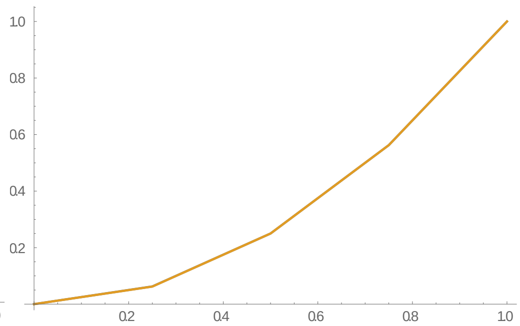

Fig. (7-ii) $h = 0.125, N = 8$

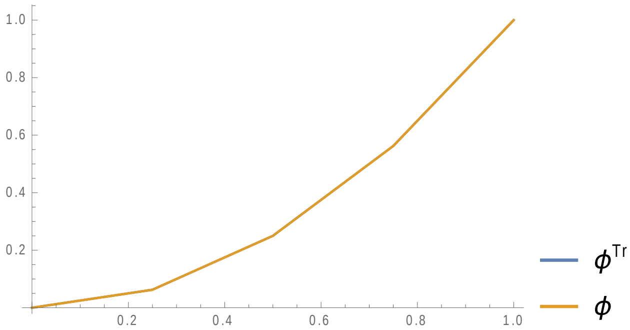

Fig. (7-iii) $h = 0.0625, N = 16$

(III-1) Figs. (7) describe the relation between the exact solution and numerical solution, when $H\left(t,\phi(t)\right) == \phi^2(t)$, using **Trapezoidal method**, with $\lambda = 0.01, \mu = 0.001$ at $h = 0.25(N = 4)$; $h = 0.25(N = 8); h = 0.625(N = 16)$ in Fig. (7.i), Fig (7.ii) and Fig.(7.iii), respectively.

case III (**F-VIE**) : Simpson method when $H\left(t,\phi(t)\right) = \phi^2(t), \mu = 0.001, \lambda = 0.1$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.0625, N = 16$	
		ϕ^S	E^S	ϕ^S	E^S	ϕ^S	E^S
0	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.25	6.25000E-02	6.25300E-02	2.99743E-05	6.25007E-02	7.11923E-07	6.25001E-02	5.69003E-08
0.5	2.50000E-01	2.50005E-01	5.35208E-06	2.50000E-01	3.63095E-07	2.50000E-01	2.98265E-08
0.75	5.62500E-01	5.62562E-01	6.21480E-05	5.62500E-01	2.51097E-07	5.62500E-01	2.20679E-08
1	1.00000E+00	1.00000E+00	2.75999E-06	1.00000E+00	2.01083E-07	1.00000E+00	1.97067E-08

Table (8)

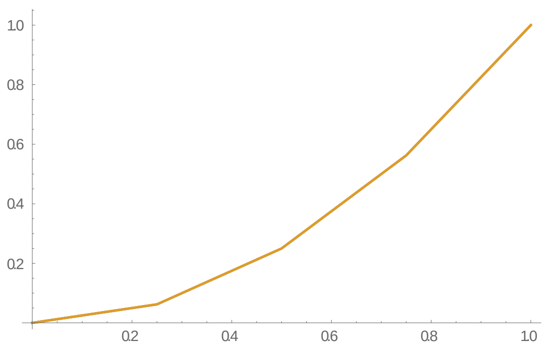

Fig. (8-i) $h = 0.25, N = 4$

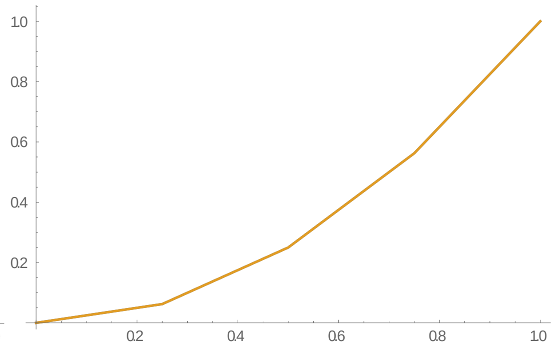

Fig. (8-ii) $h = 0.125, N = 8$

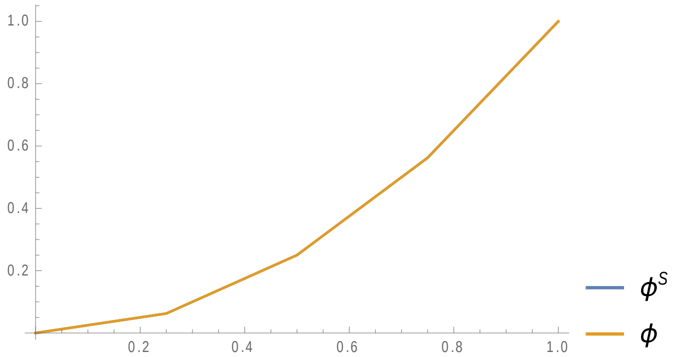

Fig. (8-iii) $h = 0.0625, N = 16$.

(III-2) Figs. (8) describe the relation between the exact solution and numerical solution, when $H\left(t,\phi(t)\right) == \phi^2(t)$, using **Simpson method**, with $\lambda = 0.01, \mu = 0.001$ at $h = 0.25(N = 4)$; $h = 0.25(N = 8); h = 0.625(N = 16)$ in Fig. (8.i), Fig (8.ii) and Fig.(8.iii), respectively.

case III (**F-VIE**) : collocation method when $H\left(t,\phi(t)\right) = \phi^2(t), \mu = 0.001, \lambda = 0.1$							
t	ϕ	$h = 0.25, N = 4$		$h = 0.125, N = 8$		$h = 0.0625, N = 16$	
		ϕ^{Co}	E^{Co}	ϕ^{Co}	E^{Co}	ϕ^{Co}	E^{Co}
0	0.00000E+00	-3.83633E-31	3.83633E-31	2.17138E-27	2.17138E-27	4.72070E-17	4.72070E-17
0.25	6.25000E-02	6.29304E-02	4.30392E-04	6.26084E-02	1.08383E-04	6.25271E-02	2.71466E-05
0.5	2.50000E-01	2.50247E-01	2.46629E-04	2.50062E-01	6.19371E-05	2.50016E-01	1.55025E-05
0.75	5.62500E-01	5.62697E-01	1.97397E-04	5.62550E-01	4.95274E-05	5.62512E-01	1.23935E-05
1	1.00000E+00	1.00018E+00	1.82484E-04	1.00005E+00	4.57528E-05	1.00001E+00	1.14468E-05

Table (9)

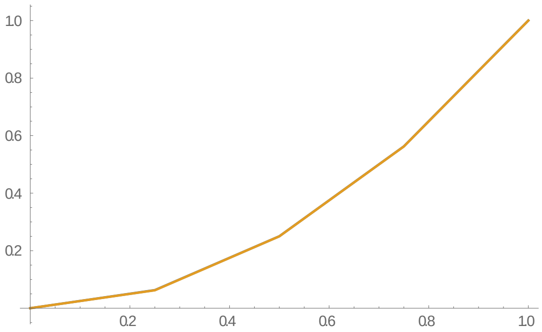

Fig. (9-i) $h = 0.25, N = 4$

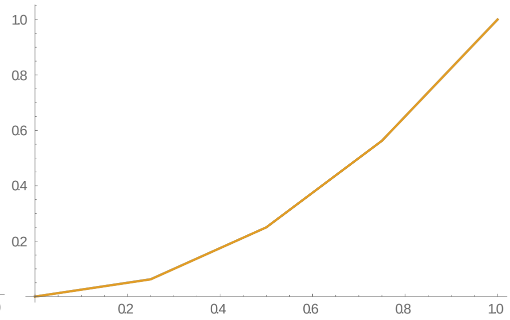

Fig. (9-ii) $h = 0.125, N = 8$

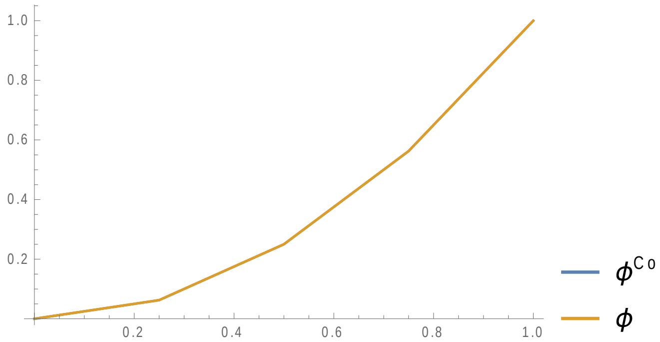

Fig. (9-iii) $h = 0.0625, N = 16$

(III-3) Figs. (9) describe the relation between the exact solution and numerical solution, when $H\left(t,\phi(t)\right) == \phi^2(t)$, using **Collocation method**, with $\lambda = 0.01, \mu = 0.001$ at $h = 0.25(N = 4)$; $h = 0.25(N = 8); h = 0.625(N = 16)$ in Fig. (9.i), Fig (9.ii) and Fig.(9.iii), respectively.

In all figures the y-axis represents the exact and numerical solution with respect to each method and x-axis represents the time.

IV. Conclusions

From the above results and others results we obtained, we can see that the proposed methods are efficient and accurate, also we notes the following

1- The value of absolute error is decreasing when the value of h decreases in the three methods.

2-The smallest error is obtained, with respect to the three methods, when the nonlocal function in the nonlinear form when $\mu \le 0.001$.

3-The error of the Simpson method is smaller than the corresponding error of the other two methods. So, the **Simpson method** is the best method in this studied

4-The error of the **Trapezoidal method** is close of the error of the **collocation method**.

5-The absolute value of the error when the memory term $H\left(t,\phi(t)\right)$ takes a nonlinear form is less than the corresponding error of the linear form in the three method.

6- When the memory term $H\left(t,\phi(t)\right) = 0,$ the absolute value of the error is large when $\mu \le 0.001\left(\mu << 1\right)$.

7- The value of absolute error is decreasing when the value of μ increases when the memory term $H\left(t,\phi(t)\right) = 0,$ in the three methods.

7- In the nonlocal integral equations μ is called the phase-lag of the integral equations.

8. The Max. E. and Min. E. in all cases in the three methods are given as follow

(I). First: when the memory term vanishes

1- For the **Trapezoidal method** without the non- local term $H\left(x,t,\phi(x,t)\right)$ we have Max. E. and Min. E. as follow:

●In Table (1) when $(h = 0.625)$, $\mu = 0.1$: (at t=1) 2.40000E-04and (at t=0) 0.00000E+00, respectively. when $\mu = 0.5$: (at t=1) 5.00000E-05 and (at t=0) 0.00000E+00, respectively. when $\mu = 1$:(at t=1) 2.00000E-05 and (at t=0) 0.00000E+00, respectively.

2- For the Simpson **method** without the non- local term $H\left(x,t,\phi(x,t)\right)$ we have

Max. E. and Min. E. as follow:

●In Table (2) when $(h = 0.625)$, $\mu = 0.1$: (at t=1) 2.65702E-06 and (at t=0) 0.00000E+00, respectively .when $\mu = 0.5$: (at t=1) 1.35847E-07and (at t=0) 0.00000E+00, respectively. when $\mu = 1$: (at t=1) 4.40445E-08 and (at t=0) 0.00000E+00, respectively.

3- For the **Collocation method** without the non- local term $H\left(x,t,\phi(x,t)\right)$ we have Max. E. and Min. E. as follow:

●In Table (3) when $(h = 0.625)$, $\mu = 0.1$: (at t=1) 2.40000E-04and (at t=0) 0.00000E+00, respectively .when $\mu = 0.5$: (at t=1) 5.00000E-05 and (at t=0) 0.00000E+00, respectively. when $\mu = 1$: (at t=1) 2.00000E-05 and (at t=0) 0.00000E+00, respectively.

(II).Second: when the memory term is linear

1- For the **Trapezoidal method** and the linear non- local term $H\left(x,t,\phi(x,t)\right)$ we haveMax. E. and Min. E. as follow:

●In Table (4) when $h = 0.25$: (at t=1) 3.65434E-04 and (at t=0) 0.00000E+00, respectively. when $h = 0.125$: (at t=1) 9.15896E-05 and (at t=0) 0.00000E+00, respectively. when $h = 0.625$: (at t=1) 2.29118E-05and (at t=0) 0.00000E+00, respectively.

2- For the Simpson **method** and the linear non- local term $H\left(x,t,\phi(x,t)\right)$ we have

Max. E. and Min. E. as follow:

●In Table (5) when $h = 0.25$: (at t=0.75) 9.32632E-05 and (at t=0) 0.00000E+00, respectively .when $h = 0.125$: (at t=1) 4.40716E-07 and (at t=0) 0.00000E+00, respectively. when $h = 0.625$: (at t=1) 4.95787E-08 and (at t=0) 0.00000E+00, respectively.

3- For the **Collocation method** and the linear non- local term $H\left(x,t,\phi(x,t)\right)$ we have Max. E. and Min. E. as follow:

●In Table (6) when $h = 0.25$: (at t=1) 3.70000E-04and (at t=0) 1.41790E-30, respectively.when $h = 0.125$: (at t=1) 9.00000E-05and (at t=0) 1.00536E-27, respectively. when $h = 0.625$: (at t=1) 2.00000E-05and (at t=0) 1.24261E-18, respectively.

(III).Third: when the memory term is nonlinear.

1- For the **Trapezoidal method** and the nonlinear non- local term $H\left(x,t,\phi(x,t)\right)$ we have Max. E. and Min. E. as follow:

●In Table (7) when $h = 0.25$: (at t=0.25) 4.30400E-04and (at t=0) 0.00000E+00, respectively.
when $h = 0.125$: (at t=0.25) 1.08400E-04 and (at t=0) 0.00000E+00, respectively. when $h = 0.625$: (at t=0.25) 2.71000E-05 and (at t=0) 0.00000E+00, respectively.

2- For the Simpson **method** and the nonlinear non- local term $H\left(x,t,\phi(x,t)\right)$ we have

Max. E. and Min. E. as follow:

●In Table (8) when $h = 0.25$: (at t=0.75) 6.21480E-05 and (at t=0) 0.00000E+00, respectively .when $h = 0.125$: (at t=0.25) 7.11923E-07 and (at t=0) 0.00000E+00, respectively. when $h = 0.625$: (at t=0.25) 5.69003E-08 and (at t=0) 0.00000E+00, respectively.

3- For the **Collocation method** and the nonlinear non- local term $H\left(x,t,\phi(x,t)\right)$ we have Max. E. and Min. E. as follow:

●In Table (9) when $h = 0.25$: (at t=0.25) 4.30392E-04 and (at t=0) 3.83633E-31, respectively .when $h = 0.125$: (at t=0.25) 1.08383E-04and (at t=0) 2.17138E-27, respectively. when $h = 0.625$: (at t=0.25) 2.71466E-05 and (at t=0) 4.72070E-17, respectively.

Future work

In the next paper, we consider the integral terms in the nonlinear cases. The historical memory and the nonlinear integral terms will be considered.

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