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On Polyhedrons

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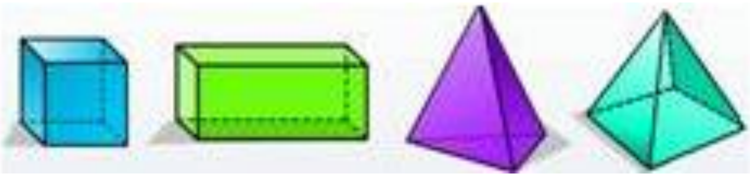
Abstract*. In this article we discuss some Geometric and Topological properties of the polyhedrons and reformulate Polyhedral Gauss Bonnet Theorem.*

I. Introduction

Polyhedrons have been studied extensively by mathematicians, both modern as well as ancient.These are building blocks of compact 3-manifolds and help in explaining many difficult concepts of Topology. Polyhedrons have had great attraction for mathematicians also because it is easy to play and experiment with them and their nature renders them computer friendly.We will discuss belowsome of their interesting topological and geometric properties.

Polyhedron. A convex polyhedron Pis a 3-dimensional solid with polygonal faces such that
(i) Intersection of any two of these polygonal facesis either a commonedge or a vertex or an empty set.
(ii) Each edge is shared by exactly two polygonal facesof P.
Some examples of the polyhedrons are given below.



{ Figure 1 }

Euler number. Let P be a polyhedron with V, E and F as its number of vertices, edges and faces respectively. The alternating sum V – E + F is called Euler number of P and is usually denoted by χ(P).

In 1750 Euler proved the following theorem regarding χ(P) of the *convexpolyhedrons*.

Theorem 1. The Euler number χ(P) = V – E + F is same for allconvex polyhedronsirrespective of their number of vertices, edges and faces. And the common value is 2.

Proof. Let P be a convex polyhedron such that its i^{th}polygonal face has n_i edges (say $e_{i1}, e_{i2}, \ldots e_{in_i}$where$n_i \geq 3$) and hence has n_i angles (say $\alpha_{i1}, \alpha_{i2}, \ldots \alpha_{in_i}$), then from Euclid's geometry we have

Sum of interior angles of the i^{th} polygonal face = $\sum_{j=1}^{n_i} \alpha_{ij} = (n_i - 2)\,\pi$

If we take sum over *i* = 1, 2, 3, …, F (i.e. over all faces of the Polyhedron), the above equation gives

$$\sum_{i=1}^{F}\sum_{j=1}^{n_i} \alpha_{ij} = \sum_{i=1}^{F}(n_i - 2)\pi$$

$\Rightarrow$ **Sum of all polygonal angles in P** = $\sum_{i=1}^{F}(n_i\pi) - 2F\pi$

$$= \mathbf{2E\pi - 2F\pi} \_\_\_\_\_\_\_\_\_\_ (*)$$

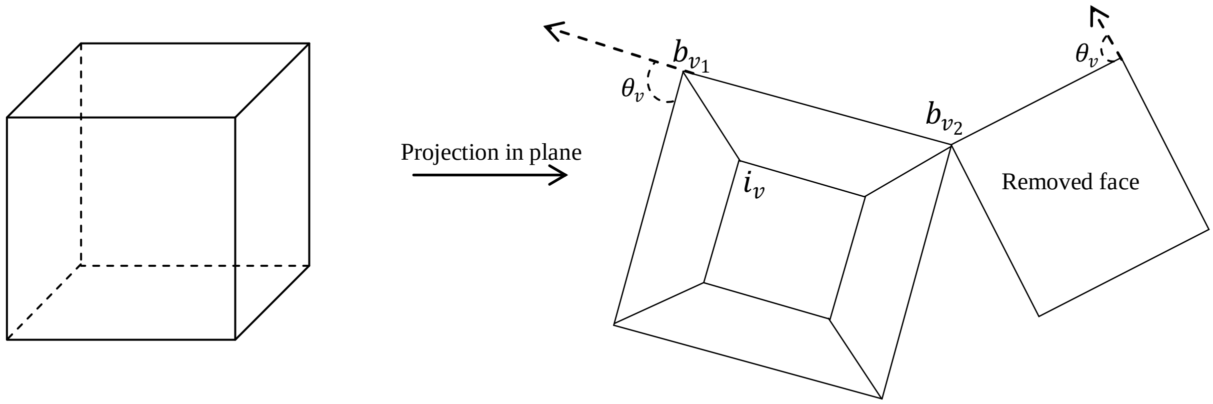
{$\because n_i$ is the number of edges on the i^{th} face, so if we count edges on all faces of P then each edge will get counted twice hence $\sum_{i=1}^{F} n_i = 2E$ }.

Now we shall find sum of all polygonal angles of P in a little different way as is explained below.

Remove one face of the polyhedron and embed rest of it in the plane to get a planar graph with straight edges, polygonal regions and two kinds of vertices viz. interior vertices i_v and boundary vertices b_v.

Notice that each interior vertex i_v is surrounded by polygons and the angle sum around each such vertex is 2π. At each boundary vertex b_v the angle of the polygonal face (or some of the angles of the polygonal faces at vertex b_v) is $(\pi - \theta_v)$, where θ_v denotes the exterior angle of the polygon at vertex b_v (see Figure 2).

If we take sum of all angles of all the polygonal regions corresponding to all faces of the polyhedron then for each $b_v \in$ P the expression $(\pi - \theta_v)$ appears twice in the sum because each boundary vertex b_v occurs one time in the removed polygonal face of the polyhedron and one time in the embedded polyhedron in the plane. So we have the following equation.



{ Figure 2 }

- **Sum of all angles** = $2\pi \times i_v + \sum_v 2(\pi - \theta_v)$
  $= 2\pi \times i_v + 2(\pi \times b_v - \sum_v \theta_v)$
  $= 2\pi (i_v + b_v) - 2 \times$sum of exterior angles of a polygon
  $= 2\pi V - 4\pi$ ___________(**)

  Combining (*) and (**) we get **V – E + F = 2**

Remarks.

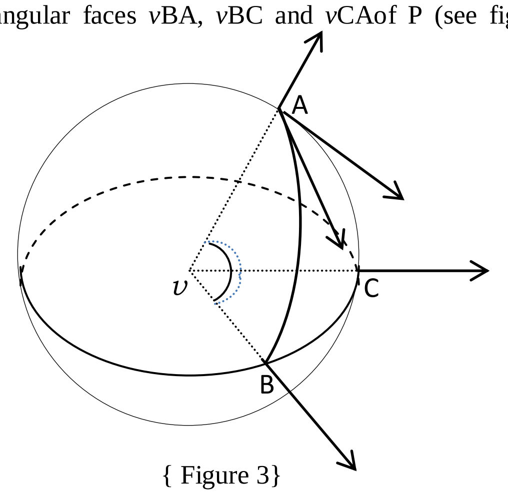
1. One may ask, how can we equate (*) and (**) ?Since in equation (*) we have taken sum of the angles of the polygonal faces of the original polyhedron whereas in equation (**) we have taken sum of the angles of the embedded polyhedron in the plane. These two kinds of angles are different and hence these sums may be quite different but surprisingly these are the same. This follows from equations (*) and (**) that the sum depends only on the number of vertices, edges and faces in P and not on their measurements.
2. In 1813 A.J. L′huilier (1750 -1840), who used to spent most of his time on problems relating to Euler's formula, noticed that the Euler's formula was wrong for polyhedrons having holes and he gave a correct formula for a polyhedron with g holes, which is: V – E + F = 2 – 2g.

II. Solid Angles in Polyhedrons

In a polyhedron there are four kinds of angles viz. (i) interior solid angles (ii) exterior solid angles (iii) dihedral angles and (iv) polygonalface angles. We shall discuss relation among all these angles.

(i)Interior solid angle in a polyhedron: We discuss an *interior solid angle*, at a vertex *v*,in case of tetrahedron onlyand same definition is applicable to any otherconvex polyhedron P.

Consider a spherewith radius *r* whose center coincides with avertex *v* of P.Part of this sphere, which is enclosedbya cone consisting oftriangular faces *v*BA, *v*BC and *v*CAof P (see figure 3 below) is a spherical triangle ABC.



{ Figure 3}

Angle subtended by this triangle at the center *v*, of the sphere, is defined as interior solid angle and is measured as follows.

$$\textbf{Interior solid angle at } v = \frac{Area of triangle ABC}{r^2}$$

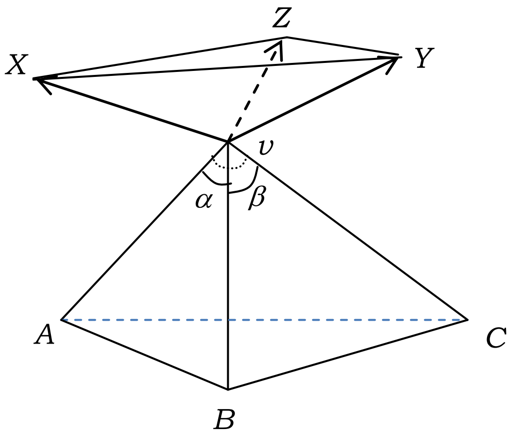
But area of *spherical triangle* ABCis $(\angle A + \angle B + \angle C - \pi)r^2$.So we have

Interior solid angle at *v* = $\frac{(\angle A + \angle B + \angle C - \pi)r^2}{r^2}$

$= \angle A + \angle B + \angle C - \pi$ steradians

This definition is exactly same as the definition of a plane angle θsubtended by an arc*l* of a circle of radius *r* at its center, which is given by θ = *l*/*r*.

(ii) **Exterior solid angle in a polyhedron**: In any polyhedron P (say tetrahedron *v*ABC for the sake of simplicity, shown in figure 4) exterior solid angle, at a vertex *v*, is defined as the interior solid angle enclosed in the cone generated by the normals to the faces of P meeting at *v*.



{ Figure 4}

(iii) **Dihedral angles**. Suppose two faces X_1 and $X_2$of P intersects in an edge *e*, then angle between these two plane faces is known as a *dihedral angle* and is equal to the angle between two lines, whichare normal to *e*and arecontained in the planesX_1 and X_2respectively.

(iv) **Polygonal face angles.** These are plane angles, of the faces of P, formed at itsvarious vertices.

Remarks.

1.Solid angle subtended by a unit sphere at its center is 4π.

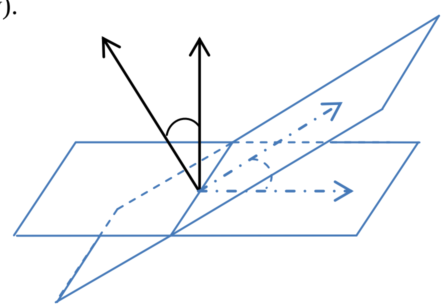
2.In figure3;∠A, ∠B and ∠C are the angles between the tangents -- to the pairs of arcs (AB, AC), (BA, BC) and (CA, CB) -- drawn at A, B and C respectively. Since these tangents are normal to the respective radii so ∠A of the spherical triangle ABC is same as the angle between two facesA*v*C and A*v*B of the polyhedron and hence is a dihedral angleat the edge A*v*. Similarly ∠B and∠C aredihedral anglesat the edges B*v* and C*v* respectively. This implies the following result for tetrahedrons.

Interior solid angle at v is equal to sum of the dihedral angles at theedges emerging from v minus π.

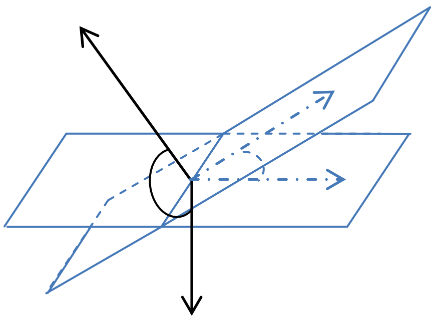
And in an arbitrary convex polyhedron if e_v edges are emerging at a vertex *v* then we have the following result.

$$\angle i = \sum_{e_v}(\textbf{dihedralangleatanedge}\, e_v\, \textbf{emergingfrom}\, v) - (e_v - 2)\pi.$$

3. Angle between two planes is same as the angle between their normals if they are drawn in the *same direction* and πminus the angle between their normals if they are drawn in *opposite direction* (see figure 5 below).



Normals in the same direction



Normals in the opposite direction

{ Figure 5 }

Theorem 2. Prove thatexterior angle of aPolyhedron at any vertexis equal to*2π minus sum of the polygonal angles at v.*

Proof.Consider the Figure 4 wherein *v*X,*v*Yand*v*Z are normals to the triangles AB*v*,BC*v* and CA*v* respectively. Notice that the total solid angle around *v* (*i.e.* 4π) has been sub-divided into various other solid angles, which we shall explain now.

A solid angleenclosed by the triangles X*v*A, X*v*B and A*v*B is equal to α (since it is equal to sum of dihedral angles minus π which is $\frac{\pi}{2} + \frac{\pi}{2} + \alpha - \pi$ also see remark 4 at the end). Similarly a solid angle enclosed by Y*v*B, Y*v*C and B*v*C is equal to β and a solid angle enclosed by Z*v*C, Z*v*A and C*v*A is equal to γ.

Since *v*X and *v*Y are normal to the triangles *v*AB and*v*BC drawn in opposite direction, so

$\angle XvY = \pi -$ dihedral angle at the edge *v*B

$= \pi - \alpha\beta$(if αβ denotes dihedral angle at the edge *v*B).

If $\angle i$ and $\angle e$ respectively denote solid interior and exterior angles of P at the vertex *v* then the angle subtended at *v* by the unit sphere centered at *v* is given by

$$4\pi = \angle i + \angle e + \angle\alpha + \angle\beta + \angle\gamma + \angle(\pi - \alpha\beta) + \angle(\pi - \beta\gamma) + \angle(\pi - \gamma\alpha)$$

We have already proved that $\angle i$ = sum of dihedral angles at *v* minus π i.e. $\angle\alpha\beta + \angle\beta\gamma + \angle\gamma\alpha - \pi$. Using this and the last equation we get the following result.

$$\angle e = 2\pi - (\angle\alpha + \angle\beta + \angle\gamma) \quad \text{-----------------------} \quad (***)$$

Remarks.

1. Right Hand Side of the last equation (*i.e.* (***) equation)is known as angle defect at the vertex *v*.
2. From equation (***) we get the following result

Sum of all exterior solid angles $= \Sigma(\angle e)$

$= \Sigma(2\pi - (\angle\alpha + \angle\beta + \angle\gamma))$

$= 2\pi V -$ sum of all polygonal angles

$= 2\pi V - 2E\pi + 2F\pi$ (using eq. (*))

$= 2\pi (V - E + F)$

$= 2\pi\chi(P)$

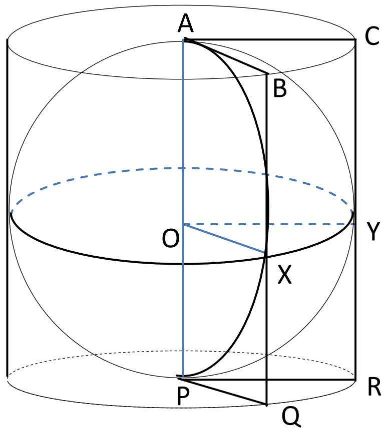
$$\Rightarrow \frac{\text{Sum of all exterior solid angles}}{2\pi} = \chi(P)$$

Notice that left hand side of the above equation (known as a Gauss number) is a geometric object while right hand side (known as the Euler number) is a combinatorial object, so this equation gives a connection among Geometry, Topologyand Combinatorics.

3. In the light of Theorem 2 and the previous remark, we can restate the classical Gauss Bonet Theorem for polyhedrons as follows.

"*Sum of all exterior solid angles of a convexpolyhedron P = 2π × χ(P).*"

4. In the following figure a sphere of unit radius, circumscribed by a cylinder of unit base radius and height 2units, has been shown. Let the angle XOY be α then curved area AXY is calculated as follows.



{ Figure 6 }

According to Archimedes Sphere-Cylinder Theorem area of the crescent AXPYA is same as the rectangular area BXQRYC on the cylinder. But area of the rectangle is length × breadth = 2 × α. So the area of the half crescent i.e. Area (AXY) = α.

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