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Universal Portfolios Generated by Reciprocal Functions of Price Relatives

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Abstract : *In this paper, a study of the empirical performance of the universal portfolios generated by certain reciprocal functions of the price relatives is presented. The portfolios are obtained from the zero-gradient sets of specific logarithmic objective functions containing the estimated daily growth rate of the investment wealth. No solution of the zero-gradient equations is available and hence the pseudo Lagrange multiplier is used to generate the portfolios.*
Keywords: *Empirical performance, investment wealth, pseudo Lagrange multiplier, reciprocal functions of price relatives, universal portfolio*

I. Introduction

Investment in a market where no assumption is made on the stochastic model of the stock prices has been of interest to many researchers (see [1]). A general method of generating a universal portfolio using the moments of probability distributions has been proposed by Cover and Ordentlich in [2]. However, the study in [2] is confined to the Dirichlet distribution with two special parametric vectors. Another method of generating a universal portfolio using a probability mass function is proposed by Ordentlich and Cover in [4]. By considering investment in a game-theoretic setting, a mini-max bound is obtained for the ratio of the universal wealth to the best constant-rebalanced-portfolio wealth in [4]. The Cover-Ordentlich universal portfolio consumes a substantial amount of computer memory and implementation time. To overcome this problem, Tan [5] introduces the memory-cum-time saving finite-order universal portfolios generated by probability distributions. Empirically, it has been shown that the finite-order portfolios can perform as well the Cover-Ordentlich portfolios and can even outperform them.

A method of generating a universal portfolio using the zero-gradient set of an objective function containing the Kullback-Leibler divergence of two portfolio vectors is introduced by Helmbold et al. [3]. The performance of the Helmbold universal portfolios is studied in [3,6]. An extension of this method to the zero-gradient set of an objective function containing the Mahalanobis squared-divergence of two portfolio vectors is proposed by Tan and Lim in [7]. In this paper a different logarithmic objective function is used to generate a universal portfolio using a pseudo Lagrange multiplier.

II. Theoretical Results

Definition 2.1: Investment in a stock market with m stocks is considered. The price relative of a given stock on a given day is the ratio of its closing price to its opening price. Let $\boldsymbol{x}_n = (x_{ni})$ denote the price-relative vector on the n^{th} trading day, where x_{ni} is the price relative of stock i on day n, for $i = 1, 2, \ldots, m;\ n = 1, 2, \ldots$. The investment portfolio or strategy on day n is denoted by $\boldsymbol{b}_n = (b_{ni})$ where b_{ni} is the proportion of the current wealth on day n invested on stock i, for $i = 1, 2, \ldots, m$, where $0 \le b_{ni} \le 1$ and $\sum_{i=1}^{m} b_{ni} = 1$. The initial wealth is assumed to be 1 unit and the wealth $S_n(\boldsymbol{x}_n)$ at the end of day n is calculated recursively as:

$$S_n(\boldsymbol{x}_n) = (\boldsymbol{b}_n^t \boldsymbol{x}_n) S_{n-1}(\boldsymbol{x}_{n-1}), \tag{1}$$

where $\boldsymbol{b}_n^t \boldsymbol{x}_n = \sum_{j=1}^{m} b_{nj} x_{nj}$. Alternatively, $S_n(\boldsymbol{x}_n)$ can be written as:

$$S_n(\boldsymbol{x}_n) = \prod_{k=1}^{m} \boldsymbol{b}_k^t \boldsymbol{x}_k, \tag{2}$$

where $\boldsymbol{b}_k^t \boldsymbol{x}_k = \sum_{j=1}^{m} b_{kj} x_{kj}$ for $k = 1, 2, \ldots, m$.

Propositions 2.2: The universal portfolios to be introduced in this paper are additive-update portfolios where the updates depend on reciprocal functions of price relatives. Hence they will be called RPR (Reciprocal-Price-Relative) universal portfolios. The *gradient vector* of an objective function $\hat{F}(\boldsymbol{b}_{n+1})$ is defined as: $\nabla \hat{F} = \left(\frac{\partial \hat{F}}{\partial b_{n+1,i}}\right)$. It is more convenient to consider the portfolio components $b_{n+1,1}, \ldots, b_{n+1,m}$ as free variables subject to the constraint $\sum_{i=1}^{m} b_{n+1,i} = 1$. Hence, we include the Lagrange multiplier λ and regard $\hat{F}(\boldsymbol{b}_{n+1}, \lambda; \boldsymbol{b}_n, \boldsymbol{x}_n)$ as a function of $\boldsymbol{b}_{n+1}$ and λ given $\boldsymbol{b}_n$ and $\boldsymbol{x}_n$. The zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ is the set

$$\{\boldsymbol{b}_{n+1}: \nabla\hat{F}(\boldsymbol{b}_{n+1}, \lambda) = \boldsymbol{0}\} \tag{3}$$

The *pseudo Lagrange multiplier* λ is a function of the variable $\boldsymbol{b}_{n+1}$ obtained by some mathematical operation on the zero-gradient equations of the objective function $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$. Since it is a variable, it is not a valid solution of the zero-gradient equations. The pseudo λ is said to be a *pseudo solution* of the zero-gradient equations.

The quantity $\log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_{n+1})$ is the rate of growth of wealth on day $n+1$ which can be estimated by $\log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)$ since $\boldsymbol{x}_{n+1}$ is unknown on day n. If the first-order Taylor series $\left[\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right) - 1\right]$ is used to approximate $\log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)$, the resulting portfolio is known as the Type 1 RPR universal portfolio.

Proposition 2.2.1: *Let $C = (c_{ij})$ be a non-negative matrix satisfying $\boldsymbol{1}^t C = \boldsymbol{1}$ where $\boldsymbol{1} = (1, 1, \ldots, 1)$ and $\alpha \geq 0$ are given. Given $\xi > 0$ and $\boldsymbol{b}_1$, consider the objective function*

$$\hat{F}(\boldsymbol{b}_{n+1}, \lambda) = \xi\left[log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right) - 1\right] - log\left\{\prod_{i=1}^{m}\left[\eta_i + \left(b_{n+1,i} - b_{ni}\right)\right]\right\} + \lambda\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right) \tag{4}$$

where

$$v_i = [\alpha(\boldsymbol{b}_n^t \boldsymbol{x}_n) + x_{ni}]^{-1} \tag{5}$$

for $i = 1, 2, \ldots, m$ and

$$\eta_i = \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)\sum_{j=1}^{m} c_{ij} v_j \tag{6}$$

for $i = 1, 2, \ldots, m$.

The pseudo Type 1 RPR (C, α) universal portfolio generated by the zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ is given by

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)[\boldsymbol{v} - C\boldsymbol{v}], \tag{7}$$

for $n = 1, 2, \ldots$, where ξ is any positive scalar satisfying $\boldsymbol{b}_{n+1} \geq \boldsymbol{0}$.

***Proof**:* Differentiate (4) with respect to $b_{n+1,i}$ to obtain

$$\frac{\partial\hat{F}}{\partial b_{n+1,i}} = \xi\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \frac{1}{\left[\eta_i + \left(b_{n+1,i} - b_{ni}\right)\right]} + \lambda = 0 \tag{8}$$

for $i = 1, 2, \ldots, m$. From (8),

$$\left[\xi\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \lambda\right]\left[\eta_i + \left(b_{n+1,i} - b_{ni}\right)\right] = 1 \tag{9}$$

for $i = 1, 2, \ldots, m$. Multiply (8) by b_{ni} and sum over i to get

$$\lambda = \sum_{i=1}^{m}\frac{b_{ni}}{\left[\eta_i + \left(b_{n+1,i} - b_{ni}\right)\right]} - \xi. \tag{10}$$

The variable λ in (10) is known as a pseudo Lagrange multiplier.
Substitute the value of λ in (10) into (9) to get

$$\left[\xi\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \sum_{j=1}^{m}\frac{b_{nj}}{\left[\eta_j + \left(b_{n+1,j} - b_{nj}\right)\right]} - \xi\right]\left[\eta_i + \left(b_{n+1,i} - b_{ni}\right)\right] = 1 \tag{11}$$

for $i = 1, 2, \ldots, m$. Let

$$z_i = \left[\eta_i + \left(b_{n+1,i} - b_{ni}\right)\right]^{-1} \tag{12}$$

for $i = 1, 2, \ldots, m$ and hence from (11),

$$\sum_{j=1}^{m} b_{nj} z_j - z_i = \xi\left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right]. \tag{13}$$

The system of equations (13) in matrix form is

$$\boldsymbol{A}\boldsymbol{z} = \boldsymbol{y} \tag{14}$$

where

$$y_i = \xi \left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] \tag{15}$$

for $i = 1, 2, \ldots, m$ and $A = (a_{ij})$ is defined by

$$\begin{aligned} a_{ii} &= b_{ni} - 1 \text{ for } i = 1, 2, \ldots, m \\ a_{ij} &= b_{nj} \text{ for } i \neq j. \end{aligned} \tag{16}$$

The solution to $\boldsymbol{A}\boldsymbol{z} = \boldsymbol{y}$ is $\boldsymbol{z} = \zeta\boldsymbol{1} - \boldsymbol{y}$ where ζ is any real scalar (see the next lemma). Thus, $z_i = \frac{(\zeta-\xi)(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \xi x_{ni}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}$ for $i = 1, 2, \ldots, m$. By reparametrizing $\zeta = (\alpha + 1)\xi$,

$$z_i^{-1} = \frac{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}{\xi\left[\alpha\left((\boldsymbol{b}_n^t \boldsymbol{x}_n) + x_{ni}\right)\right]} \tag{17}$$

for $i = 1, 2, \ldots, m$ and from (5), (6), (12), we obtain

$$b_{n+1,i} = b_{ni} + z_i^{-1} - \eta_i = b_{ni} + \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)\left[v_i - \sum_{j=1}^{m} c_{ij} v_j\right],$$

where (7) obtains.

Lemma 2.3: *Let A be the $m \times m$ matrix defined by (16).*

(i) The solution to $\boldsymbol{A}\boldsymbol{z} = \boldsymbol{y}$ where $\boldsymbol{y}$ is defined by (15) is $\boldsymbol{z} = \zeta\boldsymbol{1} - \boldsymbol{y}$ for any real scalar ζ.

(ii) The solution to $\boldsymbol{A}\boldsymbol{s} = \boldsymbol{q}$ where $\boldsymbol{q}$ is defined by

$$q_i = \phi(\boldsymbol{b}_n, \boldsymbol{x}_n)\left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] \tag{18}$$

for $i = 1, 2, \ldots, m$ and

$$\phi(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi\left[1 + \left(\frac{\boldsymbol{\sigma}^t \boldsymbol{x}_n - (\boldsymbol{s}^{-1})^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)\right] \tag{19}$$

is $\boldsymbol{s} = \gamma\boldsymbol{1} - \boldsymbol{q}$ for any real scalar γ where $\boldsymbol{s}^{-1} = (s_i^{-1})$.

Proof.

(i) Any $\boldsymbol{z}$ of the form $\boldsymbol{z} = \zeta\boldsymbol{1} - \boldsymbol{y}$ satisfies $A\boldsymbol{z} = \zeta(A\boldsymbol{1}) - A\boldsymbol{y} = \zeta(A\boldsymbol{1}) - (-\boldsymbol{y}) = \boldsymbol{y}$. Conversely, if $\boldsymbol{z}$ is a solution to $A\boldsymbol{z} = \boldsymbol{y}$, then $A\mathbf{z} = (\mathbf{b}_n^t \mathbf{z})\mathbf{1} - \mathbf{z} = \mathbf{y}$, implying that $\boldsymbol{z} = \zeta\boldsymbol{1} - \boldsymbol{y}$ where $\zeta = \boldsymbol{b}_n^t \boldsymbol{z}_n$

(ii) First note that $A\boldsymbol{q} = (\boldsymbol{b}_n^t \boldsymbol{q})\boldsymbol{1} - \boldsymbol{q} = -\boldsymbol{q}$ since $\boldsymbol{b}_n^t q = \phi(\boldsymbol{b}_n, \boldsymbol{x}_n)\sum_{i=1}^{m}\left[b_{ni} - \frac{b_{ni}x_{ni}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}\right] = 0$. Any $\boldsymbol{s}$ of the form $\boldsymbol{s} = \gamma\boldsymbol{1} - \boldsymbol{q}$ will satisfy $A\boldsymbol{s} = \gamma(A\boldsymbol{1}) - A\boldsymbol{q} = \boldsymbol{q}$. Conversely, any solution $\boldsymbol{s}$ to $A\boldsymbol{s} = \boldsymbol{q}$ must satisfy $A\boldsymbol{s} = (\boldsymbol{b}_n^t \boldsymbol{s})\boldsymbol{1} - \boldsymbol{s} = \boldsymbol{q}$, implying that $\boldsymbol{s} = \gamma\boldsymbol{1} - \boldsymbol{q}$ where $\gamma = \boldsymbol{b}_n^t \boldsymbol{s}$.

The portfolio obtained from the zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ where the second-order Taylor series is used to approximate $log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)$ is known as the Type 2 RPR universal portfolio.

Proposition 2.2.2:*Let $C = (c_{ij})$ be a non-negative matrix satisfying $\boldsymbol{1}^t C = \boldsymbol{1}$ and α, a real scalar be given. Given $\xi > 0$ and $\mathbf{b}_1$, consider the objective function*

$$\begin{aligned} \hat{F}(\boldsymbol{b}_{n+1}, \lambda) = \xi &\left[log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right) - \frac{1}{2}\left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1\right)^2\right] \\ &- log\left\{\prod_{j=1}^{m}\left[\sigma_j + \left(b_{n+1,j} - b_{nj}\right)\right]\right\} + \lambda\left(\sum_{j=1}^{m} b_{n+1,j} - 1\right) \end{aligned} \tag{20}$$

where

$$\sigma_i = \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \sum_{j=1}^{m} c_{ij}\eta_j \tag{21}$$

for $i = 1, 2, \ldots, m$,

$$r_i = \left[\alpha(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2 + \left(x_{ni} - \beta(\boldsymbol{b}_n, \boldsymbol{x}_n)\right)(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \beta(\boldsymbol{b}_n, \boldsymbol{x}_n)x_{ni}\right]^{-1} > 0 \quad (22)$$

for $i = 1, 2, \dots, m$ *and*

$$\beta(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi^{-1}(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2\boldsymbol{x}_n^t[C\boldsymbol{r} - \boldsymbol{r}]. \quad (23)$$

The pseudo Type 2 RPR (C, α) *universal portfolio generated by the zero-gradient set of* $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ *is given by*

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \xi^{-1}(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2[\boldsymbol{r} - C\boldsymbol{r}] \quad (24)$$

for $n = 1, 2, \dots$, *where* ξ *is any positive scalar satisfying* $\boldsymbol{b}_{n+1} \geq 0$, *provided* $\boldsymbol{r}$ *is a consistent solution of (22).*

Proof. Differentiate $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ in (20) to get

$$\begin{aligned}\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi\left[\frac{2x_{ni}}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)} - \frac{(\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n)x_{ni}}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2}\right] \\ -\left[\sigma_i\left(b_{n+1,i} - b_{ni}\right)\right]^{-1} + \lambda \\ = 0\end{aligned} \quad (25)$$

for $i = 1, 2, \dots, m$.

Multiply (25) by b_{ni} and sum over i to get the pseudo Lagrange multiplier

$$\begin{aligned}\lambda = -\xi\left[2 - \frac{(\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n)}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)}\right] \\ + \left[\sum_{j=1}^m b_{nj}\left(\sigma_j + \left(b_{n+1,j} - b_{nj}\right)\right)^{-1}\right]\end{aligned} \quad (26)$$

Substitute the value of λ in (26) into (25) to obtain

$$\xi\left[\frac{2x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n} - \frac{(\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n)x_{ni}}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2} - 2 + \frac{(\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n)}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)}\right] - \left[\sigma_i + \left(b_{n+1,i} - b_{ni}\right)\right]^{-1} + \sum_{j=1}^m b_{nj}\left[\sigma_j + \left(b_{n+1,j} - b_{nj}\right)^{-1}\right] = 0 \quad (27)$$

for $i = 1, 2, \dots, m$. Let

$$s_i^{-1} = b_{n+1,i} - b_i + \sigma_i \quad (28)$$

for $i = 1, 2, \dots, m$ and from (27),

$$\sum_{j=1}^m b_{nj}s_j - s_i = \phi(\boldsymbol{b}_n, \boldsymbol{x}_n)\left[1 - \frac{x_{ni}}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)}\right] \quad (29)$$

for $i = 1, 2, \dots, m$ where

$$\phi(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi\left[2 - \left(\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right)\right]. \quad (30)$$

We use the vector notation $\boldsymbol{s}^{-1} = (s_i^{-1})$. From (28), $(\boldsymbol{s}^{-1})^t\boldsymbol{x}_n = (\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n) - (\boldsymbol{b}_n^t\boldsymbol{x}_n) + (\boldsymbol{\sigma}^t\boldsymbol{x}_n)$, implying that

$$\frac{(\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n)}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)} = 1 + \frac{(\boldsymbol{s}^{-1})^t\boldsymbol{x}_n - (\boldsymbol{\sigma}^t\boldsymbol{x}_n)}{\boldsymbol{b}_n^t\boldsymbol{x}_n}. \quad (31)$$

Substituting (31) into (30), the equivalent definition of $\phi(\boldsymbol{b}_n, \boldsymbol{x}_n)$ in (19) is obtained. The matrix form of the set of equations in (29) is $A\boldsymbol{s} = \boldsymbol{q}$ where A and $\boldsymbol{q}$ are defined by (16) and (18) respectively. From Lemma 1, the solution to $A\boldsymbol{s} = \boldsymbol{q}$ is $\boldsymbol{s} = \gamma\boldsymbol{1} - \boldsymbol{q}$ for any real scalar γ. An equivalent definition of $\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)$ in (23) is

$$\beta(\boldsymbol{b}_n, \boldsymbol{x}_n) = (\boldsymbol{x}_n^t\boldsymbol{\sigma} - \boldsymbol{x}_n^t\boldsymbol{s}^{-1}). \quad (32)$$

Hence, from (18), (19) and (32), $s_i = \gamma - q_i = \gamma - \xi\left[1 + \frac{\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)}{(\boldsymbol{b}_n^t\boldsymbol{x}_n)}\right]$ for $i = 1, 2, \dots, m$.

Reparametrizing γ as $(\alpha + 1)\xi$,

$s_i = (\boldsymbol{b}_n^t\boldsymbol{x}_n)^{-2}\xi\left[\alpha(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2 + \left(x_{ni} - \beta(\boldsymbol{b}_n, \boldsymbol{x}_n)\right)(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \beta(\boldsymbol{b}_n, \boldsymbol{x}_n)x_{ni}\right] = (\boldsymbol{b}_n^t\boldsymbol{x}_n)^{-2}\xi r_i^{-1}$

for $i = 1, 2, \dots, m$ where r_i is defined by (22). From (21) and (28), $b_{n+1,i} = b_{ni} + s_i^{-1} - \sigma_i = b_{ni} + \xi^{-1}(\boldsymbol{b}_n^t\boldsymbol{x}_n)^2\left[r_i - \sum_{j=1}^m c_{ij}r_j\right]$ for $i = 1, 2, \dots, m$ and (24) is proved.

Remarks.

(i) From (23),

$$\beta(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \sum_{j=1}^{m} y_{nj} r_j \tag{33}$$

where

$$y_{nj} = \sum_{k=1}^{m} c_{kj} x_{nk} - x_{nj} \tag{34}$$

for $j = 1, 2, \ldots, m$.

The system of equations (22) can be expressed as:

$$r_i = \left[\alpha(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 + x_{ni}(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \sum_{j=1}^{m} y_{nj}\,[x_{ni} - (\boldsymbol{b}_n^t \boldsymbol{x}_n)] r_j\right]^{-1} \tag{35}$$

for $i = 1, 2, \ldots, m$. In general, a consistent solution for $\boldsymbol{r}$ may be impossible.

(ii) Some comments on the value of $\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)$ is in order. From (21) and (28),

$$\sum_{i=1}^{m} \sigma_i = \sum_{i=1}^{m} s_i^{-1} = \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \sum_{i=1}^{m} r_i \tag{36}$$

since $\mathbf{1}^t C = \mathbf{1}$ and furthermore $\sigma_i > 0$ and $s_i^{-1} > 0$ for all $i = 1, 2, \ldots, m$. Now,

$$\left(\sum_{j=1}^{m} \sigma_j\right) \min_i\{x_{ni}\} \leq \sum_{i=1}^{m} \sigma_i x_{ni} \leq \left(\sum_{j=1}^{m} \sigma_j\right) \max_i\{x_{ni}\}$$

and

$$\left(\sum_{j=1}^{m} s_j^{-1}\right) \min_i\{x_{ni}\} \leq \sum_{i=1}^{m} s_i^{-1} x_{ni} \leq \left(\sum_{j=1}^{m} s_j^{-1}\right) \max_i\{x_{ni}\}.$$

Using the definition of $\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)$ in (32) and (36),

$$0 \leq |\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)| \leq \left[\xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \sum_{i=1}^{m} r_i\right] \left\{\max_i\{x_{ni}\} - \min_i\{x_{ni}\}\right\} \tag{37}$$

(iii) The pseudo type 2 RPR universal portfolio may be relaxed by assuming that $\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)$ is a constant, not depending on the $\boldsymbol{b}_n$ and $\boldsymbol{x}_n$. The pseudo relaxed type 2 portfolio has parametric set (C, α, β) where the choice of β may be $-1 \leq \beta \leq 1$. The scalar α is chosen so that $r_i > 0$ for all $i = 1, 2, \ldots, m$ in (22). This is always possible for a large enough α.

The portfolio obtained from the zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ where the second-order Taylor series is used to approximate $log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)$ is known as the Type 2 RPR universal portfolio.

III. Empirical Performance

The Malaysian companies selected for the empirical study are listed in Table 1. There are five data sets D, E, F, G and H. The stocks of the Malaysian companies are traded from 1st March 2006 until 2nd August 2012, consisting of a total of 1500 trading days.

The pseudo Type 1 RPR (C, α) universal portfolio is run on the five data sets D, E, F, G and H where the selected matrix C is

$$C = \begin{pmatrix} 0.24 & 0.20 & 0.23 & 0.31 & 0.21 \\ 0.17 & 0.20 & 0.33 & 0.22 & 0.11 \\ 0.30 & 0.27 & 0.30 & 0.03 & 0.24 \\ 0.00 & 0.04 & 0.02 & 0.15 & 0.31 \\ 0.29 & 0.29 & 0.13 & 0.29 & 0.13 \end{pmatrix} \tag{38}$$

with the elements being randomly generated and having the property that each column sum is 1 or close to 1. It is more convenient to use the parameter $\zeta = \xi^{-1}$. The initial wealth of the investor is assumed to be 1 unit and the initial portfolio $\boldsymbol{b}_0 = (0.2, 0.2, 0.2, 0.2, 0.2)$. Eleven integer values of the parameter α from 0 until 10 are used in the study. The ζ intervals searched for each α are listed in Table 2, together with the best wealth

S_{1500} obtained after 1500 trading days corresponding to the best ζ in the interval. The next portfolios b_{1501} after 1500 days is also listed in the table. It is observed from Table 2 that the best wealth of 3.1154 units is obtained for data set E corresponding to $\alpha = 6$ and $\zeta = 0.0051$. The lowest wealth of 1.26784 units is obtained for data set F corresponding to $\alpha = 0, 1, \dots, 10$ and $\zeta = 0$. Average wealths of 2.385, 4.443 and 5.200 units are obtained for data sets D, G and H respectively. It is also observed from Table 2 that for sets D, E and H, a proportion 0.7 of the current wealth after 1500 trading days tends to be invested in the fourth company of the portfolio, whereas the proportion invested in the first company tends to zero. This indicates that the fourth and the first stocks are the best and worst stocks respectively. For sets F and G, the portfolios become constant after a long run.

Similarly, the pseudo relaxed Type 2 RPR portfolios with parametric set (C, α, β) are run on the data sets D, E, F, G and H. The matrix C selected is an equal-entry matrix with each entry 0.2 and eleven integer values of α from 0 until 10 and $\beta = 0.6$ are investigated. The best wealths S_{1500} obtained after 1500 trading days corresponding to a $\zeta = \xi^{-1}$ in the zeta interval searched are listed in Table 3. The next portfolios b_{1501} after 1500 trading days are also listed in the table. It is observed that average wealths of 2.57, 9.27, 1.46, 4.44 and 4.57 units are obtained for data sets D, E, F, G and H respectively. The best portfolio corresponds to E with G and H exhibiting good performance. In set E, current-wealth proportions of around 0.37 and 0.26 tend to be invested in the first and fifth stocks respectively, whereas in set D, proportions of 0.25 and 0.23 tend to be invested in the second and third stocks. For sets G and H, the portfolios tend to be constant after a long run.

The performance of the pseudo Type 1 and Type 2 RPR portfolios seem to be comparable, with neither exhibiting a superior performance over the other.

Table 2

Table 1

LIST OF MALAYSIAN COMPANIES IN THE DATA SETS D, E, F, G AND H

Data Set	Five Malaysian Companies in Each Set
D	IOI Corporation, Carlsberg Brewery Malaysia, British American Tobacco, Nestle, Digi
E	Public Bank, Kulim, KLCC Property Holdings, AEON Corporation, Kuala Lumpur Kepong
F	AMMB Holdings, Berjaya Sports TOTO, Air Asia, Gamuda, Genting
G	AEON Corporation, British American Tobacco, Kulim, Nestle, Digi
H	Digi, Public Bank, KLCC Property Holdings, Carlsberg Brewery Malaysia, Kuala Lumpur Kepong

The best wealths obtained after 1500 trading days by running the pseudo Type 1 RPR (C, α) universal portfolios over data sets D, E, F, G, H where C is given by (37), eleven integer values of α, the ζ intervals searched and the final portfolios after 1500 trading days are listed

Set D

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.00072]	0.00072	2.38548	0.00246	0.16739	0.05745	0.71236	0.06035
1	[0,0.00145]	0.00145	2.38526	0.00098	0.16712	0.05647	0.71589	0.05953
2	[0,0.00218]	0.00218	2.38518	0.00051	0.16703	0.05613	0.71707	0.05926
3	[0,0.00291]	0.00291	2.38514	0.00027	0.16698	0.05596	0.71766	0.05912
4	[0,0.00364]	0.00364	2.38512	0.00013	0.16695	0.05586	0.71801	0.05904
5	[0,0.00437]	0.00437	2.38510	0.00004	0.16694	0.05579	0.71822	0.05899
6	[0,0.00509]	0.00509	2.38504	0.00036	0.16699	0.05603	0.71740	0.05923
7	[0,0.00582]	0.00582	2.38504	0.00027	0.16697	0.05596	0.71765	0.05916
8	[0,0.00655]	0.00655	2.38504	0.00020	0.16696	0.05590	0.71785	0.05911
9	[0,0.00728]	0.00728	2.38503	0.00013	0.16695	0.05585	0.71800	0.05907
10	[0,0.00801]	0.00801	2.38503	0.00008	0.16693	0.05582	0.71813	0.05904

Set E

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.00072]	0.00072	10.2858	0.00258	0.16725	0.05175	0.71226	0.06076
1	[0,0.00145]	0.00145	10.3008	0.00111	0.16702	0.05631	0.71585	0.05972
2	[0,0.00218]	0.00218	10.3058	0.00060	0.16695	0.05602	0.71704	0.05939
3	[0,0.00291]	0.00291	10.3083	0.00035	0.16692	0.05588	0.71764	0.05922
4	[0,0.00364]	0.00364	10.3098	0.00020	0.16690	0.05579	0.71799	0.05912
5	[0,0.00437]	0.00437	10.3108	0.00009	0.16689	0.05574	0.71823	0.05905
6	[0,0.00509]	0.00509	10.3115	0.00002	0.16688	0.05569	0.71840	0.05900
7	[0,0.00582]	0.00582	10.3079	0.00031	0.16693	0.05591	0.71764	0.05921
8	[0,0.00655]	0.00655	10.3088	0.00023	0.16692	0.05586	0.71784	0.05915
9	[0,0.00728]	0.00728	10.3095	0.00016	0.16692	0.05582	0.71799	0.05911
10	[0,0.00801]	0.00801	10.3101	0.00011	0.16691	0.05579	0.71812	0.05908

Set F

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.00072]	0	1.26784	0.2	0.2	0.2	0.2	0.2
1	[0,0.00145]	0	1.26784	0.2	0.2	0.2	0.2	0.2
2	[0,0.00218]	0	1.26784	0.2	0.2	0.2	0.2	0.2
3	[0,0.00291]	0	1.26784	0.2	0.2	0.2	0.2	0.2
4	[0,0.00364]	0	1.26784	0.2	0.2	0.2	0.2	0.2
5	[0,0.00437]	0	1.26784	0.2	0.2	0.2	0.2	0.2
6	[0,0.00509]	0	1.26784	0.2	0.2	0.2	0.2	0.2
7	[0,0.00582]	0	1.26784	0.2	0.2	0.2	0.2	0.2
8	[0,0.00655]	0	1.26784	0.2	0.2	0.2	0.2	0.2
9	[0,0.00728]	0	1.26784	0.2	0.2	0.2	0.2	0.2
10	[0,0.00801]	0	1.26784	0.2	0.2	0.2	0.2	0.2

Set G

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.00072]	0	4.44326	0.2	0.2	0.2	0.2	0.2
1	[0,0.00145]	0	4.44326	0.2	0.2	0.2	0.2	0.2
2	[0,0.00218]	0	4.44326	0.2	0.2	0.2	0.2	0.2
3	[0,0.00291]	0	4.44326	0.2	0.2	0.2	0.2	0.2
4	[0,0.00364]	0	4.44326	0.2	0.2	0.2	0.2	0.2
5	[0,0.00437]	0	4.44326	0.2	0.2	0.2	0.2	0.2
6	[0,0.00509]	0	4.44326	0.2	0.2	0.2	0.2	0.2
7	[0,0.00582]	0	4.44326	0.2	0.2	0.2	0.2	0.2
8	[0,0.00655]	0	4.44326	0.2	0.2	0.2	0.2	0.2
9	[0,0.00728]	0	4.44326	0.2	0.2	0.2	0.2	0.2
10	[0,0.00801]	0	4.44326	0.2	0.2	0.2	0.2	0.2

Set H

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.00072]	0.00072	5.19812	0.00246	0.16726	0.05670	0.71299	0.06029
1	[0,0.00145]	0.00145	5.20243	0.00101	0.16707	0.05622	0.71622	0.05948
2	[0,0.00218]	0.00218	5.20387	0.00533	0.16700	0.05596	0.71729	0.05922
3	[0,0.00291]	0.00291	5.20459	0.00294	0.16696	0.05584	0.71782	0.05909
4	[0,0.00364]	0.00364	5.20502	0.00151	0.16693	0.05576	0.71814	0.05902
5	[0,0.00437]	0.00437	5.20531	0.00006	0.16692	0.05571	0.71836	0.05897
6	[0,0.00509]	0.00509	5.20432	0.00038	0.16697	0.05595	0.71749	0.05921
7	[0,0.00582]	0.00582	5.20462	0.00028	0.16696	0.05589	0.71773	0.05915
8	[0,0.00655]	0.00655	5.20486	0.00020	0.16694	0.05584	0.71792	0.05910
9	[0,0.00728]	0.00728	5.20505	0.00014	0.16693	0.05580	0.71807	0.05906
10	[0,0.00801]	0.00801	5.20520	0.00009	0.16692	0.05577	0.71819	0.05903

Table 3

The best wealths obtained after 1500 trading days by running the pseudo, relaxed Type 2 RPR (C, α, β) universal portfolios over data sets D, E, F, G, H where each entry of C is 0.2, β is 0.6, the ζ intervals searched and the final portfolios after 1500 trading days are listed

Set D

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.1]	0.1	2.46438	0.24380	0.21627	0.21218	0.17654	0.15121
1	[0,0.5]	0.5	2.52712	0.23456	0.23665	0.22449	0.17672	0.12758
2	[0,1.3]	1.3	2.57281	0.23182	0.25012	0.23215	0.17530	0.11061
3	[0,2.2]	2.2	2.56920	0.22625	0.25142	0.23250	0.17754	0.11229
4	[0,3.5]	3.5	2.57869	0.22423	0.25482	0.23428	0.17777	0.10890
5	[0,5.1]	5.1	2.58523	0.22281	0.25718	0.23550	0.17794	0.10657
6	[0,6.9]	6.9	2.58608	0.22144	0.25797	0.23587	0.17839	0.10634
7	[0,9]	9	2.58756	0.22048	0.25876	0.23626	0.17865	0.10585
8	[0,11.4]	11.4	2.58923	0.21978	0.25950	0.23663	0.17881	0.10528
9	[0,14]	14	2.58892	0.21909	0.25969	0.23671	0.17906	0.10545
10	[0,16.9]	16.9	2.58920	0.21857	0.25998	0.23684	0.17923	0.10538

Set E

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.2]	0.2	9.17412	0.30314	0.19905	0.11999	0.11239	0.26543
1	[0,1]	1	9.46185	0.36914	0.15425	0.09283	0.10818	0.27560
2	[0,2.1]	2.1	9.36491	0.37176	0.14014	0.09788	0.12168	0.26854
3	[0,3.6]	3.6	9.31575	0.37245	0.13381	0.10079	0.12810	0.26485
4	[0,5.5]	5.5	9.28603	0.37265	0.13027	0.10264	0.13184	0.26260
5	[0,7.9]	7.9	9.28320	0.37495	0.12697	0.10265	0.13353	0.26190
6	[0,10.7]	10.7	9.27700	0.37604	0.12487	0.10298	0.13492	0.26119
7	[0,13.9]	13.9	9.27005	0.37654	0.12345	0.10341	0.13606	0.26054
8	[0,17.5]	17.5	9.26328	0.37674	0.12244	0.10385	0.13700	0.25997
9	[0,21.6]	21.6	9.26318	0.37761	0.12128	0.10381	0.13753	0.25977
10	[0,26]	26	9.25639	0.37741	0.12079	0.10427	0.13824	0.25929

Set F

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.1]	0.1	1.37957	0.13895	0.13657	0.29070	0.20205	0.23173
1	[0,0.6]	0.6	1.43715	0.11958	0.12320	0.33978	0.16434	0.25310
2	[0,1.5]	1.5	1.45959	0.11436	0.12027	0.36283	0.14144	0.26110
3	[0,2.7]	2.7	1.46457	0.11518	0.12207	0.36952	0.13025	0.26298
4	[0,4.2]	4.2	1.46533	0.11672	0.12416	0.37151	0.12423	0.26338
5	[0,6]	6	1.46486	0.11817	0.12595	0.37187	0.12068	0.26333
6	[0,8.1]	8.1	1.46401	0.11941	0.12743	0.37161	0.11842	0.26313
7	[0,10.5]	10.5	1.46310	0.12045	0.12864	0.37112	0.11689	0.26290
8	[0,13.2]	13.2	1.46221	0.12132	0.12965	0.37055	0.11582	0.26266
9	[0,16.2]	16.2	1.46140	0.12206	0.13049	0.36998	0.11503	0.26244
10	[0,19.6]	19.6	1.46167	0.12228	0.13081	0.37047	0.11388	0.26256

Set G

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.1]	0.1	4.44326	0.2	0.2	0.2	0.2	0.2
1	[0,0.5]	0.5	4.44326	0.2	0.2	0.2	0.2	0.2
2	[0,1.1]	1.1	4.44326	0.2	0.2	0.2	0.2	0.2
3	[0,1.9]	1.9	4.44326	0.2	0.2	0.2	0.2	0.2
4	[0,2.9]	2.9	4.44326	0.2	0.2	0.2	0.2	0.2
5	[0,4.1]	4.1	4.44326	0.2	0.2	0.2	0.2	0.2
6	[0,5.6]	5.6	4.44326	0.2	0.2	0.2	0.2	0.2
7	[0,7.3]	7.3	4.44326	0.2	0.2	0.2	0.2	0.2
8	[0,9.2]	9.2	4.44326	0.2	0.2	0.2	0.2	0.2
9	[0,11.3]	11.3	4.44326	0.2	0.2	0.2	0.2	0.2
10	[0,13.7]	13.7	4.44326	0.2	0.2	0.2	0.2	0.2

Set H

α	ζ	Best ζ	S_{1500}	b_1	b_2	b_3	b_4	b_5
0	[0,0.1]	0.1	4.57123	0.2	0.2	0.2	0.2	0.2
1	[0,0.5]	0.5	4.57123	0.2	0.2	0.2	0.2	0.2
2	[0,1.1]	1.1	4.57123	0.2	0.2	0.2	0.2	0.2
3	[0,1.9]	1.9	4.57123	0.2	0.2	0.2	0.2	0.2
4	[0,3]	3	4.57123	0.2	0.2	0.2	0.2	0.2
5	[0,4.4]	4.4	4.57123	0.2	0.2	0.2	0.2	0.2
6	[0,5.9]	5.9	4.57123	0.2	0.2	0.2	0.2	0.2
7	[0,7.8]	7.8	4.57123	0.2	0.2	0.2	0.2	0.2
8	[0,9.8]	9.8	4.57123	0.2	0.2	0.2	0.2	0.2
9	[0,12.1]	12.1	4.57123	0.2	0.2	0.2	0.2	0.2
10	[0,14.7]	14.7	4.57123	0.2	0.2	0.2	0.2	0.2

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