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Pullbacks and Pushouts in the Category of Graphs

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Abstract: *In category theory the notion of a Pullback like that of an Equalizer is one that comes up very often in Mathematics and Logic. It is a generalization of both intersection and inverse image. The dual notion of Pullback is that of a pushout of two homomorphisms with a common domain. In this paper we prove that the Category G of Graphs has both Pullbacks and Pushouts by actually constructing them.*
Keywords: *homomorphism, pullbacks, pushouts, projection, surjective.*

I. Introduction

A graph G consists of a pair G = (V(G) , E(G)) (also written as G = (V , E) whenever the context is clear) where V(G) is a finite set whose elements are called vertices and E(G) is a set of unordered pairs of distinct elements in V(G) whose members are called edges. The graphs as we have defined above are called simple graphs. Throughout our discussions all graphs are considered to be simple graphs [1, 2].

Let G and G_1 be graphs. A homomorphisms f: G $\rightarrow$ G_1 is a pair $f = (f^*, \tilde{f})$ where f^*: V(G) $\rightarrow$ V(G_1) and $\tilde{f}$: E(G) $\rightarrow$ E(G_1) are functions such that $\tilde{f}$ ((u, v)) = (f^*(u) , f^*(v)) for all edges (u , v) $\in$ E (G) . For convenience if (u,v) $\in$ E (G) then $\tilde{f}$ ((u, v)) is simply denoted as $\tilde{f}$ (u, v) [3].

Then we have the category of graphs say $\mathscr{G}$, where objects are graphs and morphisms are as defined above, where equality, compositions and the identity morphisms are defined in the natural way. It is also proved that two homomorphisms $f = (f^*, \tilde{f})$ and $g = (g^*, \tilde{g})$ of graphs are equal if and only if $f^* = g^*$ (Lemma 1.6 in [3]).

II. Pullbacks

Definition 2.1: Given two graph homomorphisms f: X $\rightarrow$ Z and g : Y $\rightarrow$ Z a commutative diagram is called a pullback for f and g, if for every pair of morphisms $\beta_1 : Q \rightarrow X$ and $\beta_2 : Q \rightarrow Y$ such that $f\beta_1 = g\beta_2$, there exists a unique homomorphism $\gamma : Q \rightarrow P$ such that $\beta_1 = \alpha_1 \gamma$ and $\beta_2 = \alpha_2 \gamma$ [see figure 1].

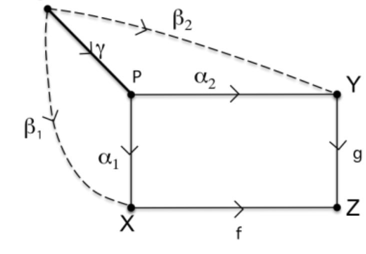

Figure 1

Proposition 2.2: The category of graphs $\mathscr{G}$ has pullbacks.

Proof: Consider any diagram where f and g are homomorphism of graphs [see figure 2, 3].

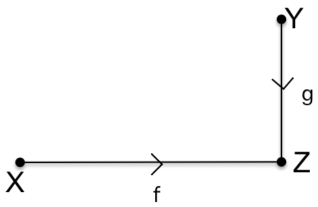

Figure 2

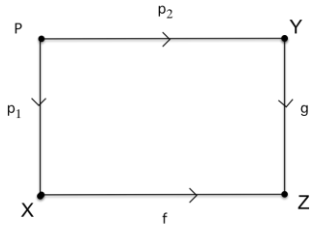

Figure 3

Let P be the graph defined as below. $V(P)=\{(x,y)\in V(X)\times V(Y) \text{ such that } f^*(x)=g^*(y)\}$;
Also $(x_1, y_1) \sim (x_2, y_2)$ in P if and only if $x_1 \sim x_2$ in X and $y_1 \sim y_2$ in Y.
Consider the projection maps $p_1 : P \to X$ and $p_2 : P \to Y$ as defined below: For $(x, y) \in V(P)$,

$$p_1^*: V(P) \to V(X) \quad \text{and} \quad p_2^*: V(P) \to V(Y)$$
$$(x,y) \mapsto x \text{ and } (x,y) \mapsto y$$

Then p_1^* and p_2^* are surjective maps. Moreover if $(x_1, y_1) \sim (x_2, y_2)$ in P, then by definition $x_1 \sim x_2$ and $y_1 \sim y_2$. This shows that $p_1^*(x_1, y_1) \sim p_1^*(x_2, y_2)$. Hence we have a well defined map

$$\widetilde{p_1} : E(P) \to E(X)$$
$$((x_1, y_1), (x_2, y_2)) \mapsto (p_1^*(x_1, y_1), p_1^*(x_2, y_2))$$

thus showing that $p_1 : P \to X$ is a homomorphism of graphs. Similarly $p_2 : P \to Y$ is also a homomorphism of graphs.
Moreover for all $(x,y) \in V(P)$

$$\begin{aligned}(f\, p_1)^*(x,y) &= f^*\, p_1^*(x,y) = f^*(x)\\ &= g^*(y) \quad (\text{by definition of } P)\\ &= g^*\, p_2^*(x,y)\\ &= (g\, p_2)^*(x,y)\end{aligned}$$

and hence $f\, p_1 = g\, p_2$ (by Lemma 1.6 in [3])

Suppose there exists homomorphism of graphs $\alpha_1 : Q \to X$ and $\alpha_2 : Q \to Y$ such that $f\alpha_1 = g\alpha_2$ [See figure 4].

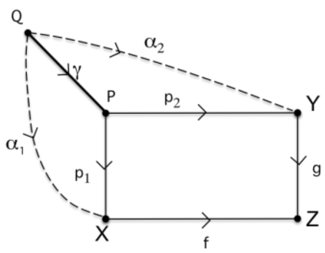

Figure 4

Then $(f\alpha_1)^* = (g\alpha_2)^*$. i.e. $f^*\alpha_1^* = g^*\alpha_2^*$. Now we define a homomorphism $\gamma : Q \to P$ as follows: If $u \in V(Q)$, then $f^*\alpha_1^*(u) = g^*\alpha_2^*(u)$ and so by definition of P, $(\alpha_1^*(u), \alpha_2^*(u)) \in V(P)$.
So define

$$\gamma^*(u) = (\alpha_1^*(u), \alpha_2^*(u)).$$

Then $p_1^* \gamma^* (u) = p_1^* \ (\alpha_1^*(u), \alpha_2^*(u))$

$= \alpha_1^*(u)$

so that $p_1 \gamma = \alpha_1$ (by Lemma 1.6 in [3]). Similarly $p_2 \gamma = \alpha_2$.

Suppose there exists $\delta : Q \rightarrow P$ such that $p_1 \delta = \alpha_1$ and $p_2 \delta = \alpha_2$.

For $u \in V(Q)$ let $\delta^* (u) = (x_1, y_1) \in P$

Then $p_1^* \delta^* (u) = x_1$

$= \alpha_1^* (u)$

and

$p_2^* \delta^* (u) = y_1$

$= \alpha_2^* (u)$

Therefore $\gamma^*(u) = (\alpha_1^*(u), \alpha_2^*(u))$

$= (x_1, y_1)$

$= \delta^* (u)$

and so (by Lemma 1.6 in [3]) $\gamma = \delta$, proving the uniqueness of γ .

Thus P is a pull back for *f* and *g* [4].

Example 2.3: Let $\overline{\mathbb{K}}$ denote the full subcategory of complete graphs. Then $\overline{\mathbb{K}}$ has pull backs.

Proof: Since any two pull backs are isomorphic we follow the construction as in the Proposition 2.2. Consider the diagram [see figure 5] where X , Y, Z are complete graphs.

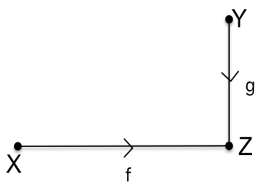

Figure 5

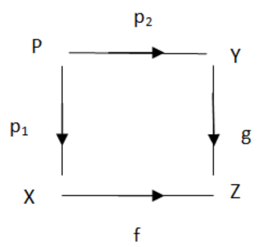

Figure 6

Let P be the graph with the obvious adjacency relation, where V(P) = { (x, y) ∈ V(X) × V(Y) / f*(x) = g*(y)}. Consider the diagram [see figure 6] where p_1 and p_2 are the restrictions of the canonical projections from the product.

Claim: P is a complete graph. Let (x_1 , y_1) and (x_2 , y_2) be any two distinct vertices in v(P). Then either $x_1 \neq x_2$ or $y_1 \neq y_2$ or both. Suppose $x_1 = x_2$. Then $y_1 \neq y_2$ and so f (x_1) = g (y_1) and f (x_1) = f (x_2) = g (y_2). Therefore g (y_1) = g (y_2). However $y_1 \neq y_2$ and Y is a complete graph implies that $y_1 \sim y_2$ and hence g (y_1) ~ g (y_2) which is a contradiction. Hence $x_1 \neq x_2$. Similarly $y_1 \neq y_2$. Thus (x_1 , y_1) ≠ (x_2 , y_2) in V(P) implies that $x_1 \neq x_2$ and $y_1 \neq y_2$ which in turn implies that $x_1 \sim x_2$ and $y_1 \sim y_2$ .Thus (x_1 , y_1) ~ (x_2 , y_2). Therefore any two distinct vertices in P are adjacent and so P is a complete graph. Therefore P ∈ $\overline{\mathbb{K}}$ i.e. $\overline{\mathbb{K}}$ has pullbacks.

Example 2.4: The full subcategory C of connected graphs does not have pullbacks.

Proof: Let X , Y and Z be connected graphs defined by the following diagrams [see figure 7].

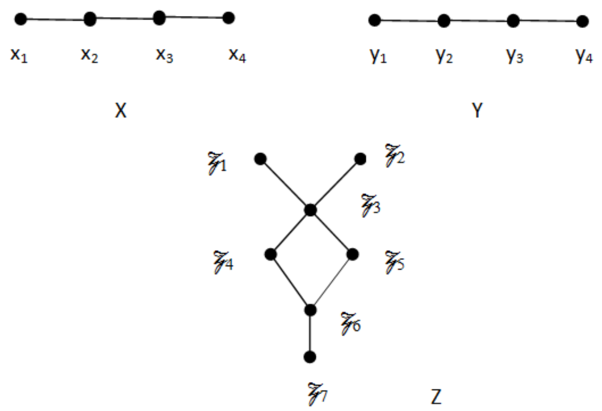

Figure 7

[X and Y are the same (isomorphic). However to avoid some confusions in constructing the pullbacks as in Proposition 2.2, we give different names to the vertices].

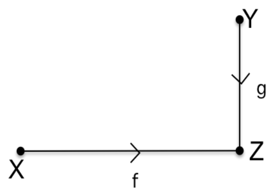

Figure 8

Consider the homomorphism of graphs $f : X \to Z$ and $g : Y \to Z$ defined as follows [see figure 8].

$f(x_1) = z_1$	$g(y_1) = z_2$
$f(x_2) = z_3$	$g(y_2) = z_3$
$f(x_3) = z_4$	$g(y_3) = z_5$
$f(x_4) = z_6$	$g(y_4) = z_6$

Then the pull back of f and g in G is given by the subgraph P of $X \times Y$ where
$V(P) = \{ (x, y) \in V(X) \times V(Y) \,/\, f(x) = g(y) \}$

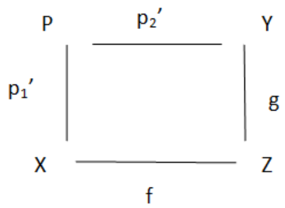

Figure 9

and p_1' and p_2' are the restrictions of the canonical projections p_1 and p_2,

$$X \times Y \xrightarrow{P_1} X \quad , \quad X \times Y \xrightarrow{p_2} Y$$ [see figure 9] .

In this example $V(P) = \{ (x_2, y_2), (x_4, y_4) \}$. Since $x_2 \nsim x_4$ (or $y_2 \nsim y_4$), $(x_2, y_2) \nsim (x_4, y_4)$.
Hence p is the empty graph on two vertices which is totally disconnected and so $p \notin C$.

Therefore C does not have pullbacks.

III. Pushouts

Definition 3.1: Given a diagram [figure 10] in the category of graphs, a commutative diagram [figure 11]

Figure 10

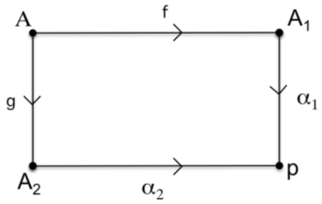

Figure 11

is called a pushout for f and g if for every pair of morphisms $\beta_1 : Q \rightarrow A_1$ and $\beta_2 : Q \rightarrow A$
such that β_1 f = β_2 g , there exists a unique morphism $\gamma : P \rightarrow Q$ such that $\gamma\alpha_1 = \beta_1$ and $\gamma\alpha_2 = \beta_2$
[see figure 12].

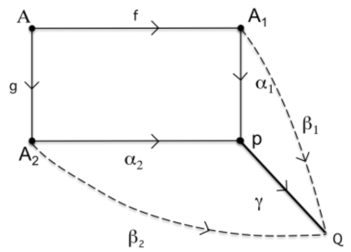

Figure 12

Proposition 3.2: The Category of graphs $\mathscr{G}$ has pushouts.

Proof: Consider any diagram in G as given below [see figure 13].

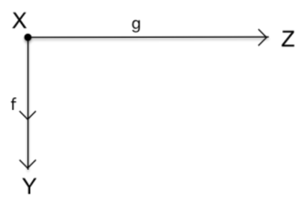

Figure 13

Step 1: We construct a graph T as follows:

$$V(T) = (V(Y) \times \{0\}) \cup (V(Z) \times \{1\})$$

(i.e. the disjoint union of sets V(Y) and V(Z)).

$$= \{ (y,0) / y \in V(Y) \} \cup \{(z, 1) / z \in Z \}.$$

The edges in T are defined as follows.

i) $((y_1,0),(y_2,0)) \in E(T)$ *if and only if* $(y_1, y_2) \in E(Y)$, and

ii) $((z_1,1),(z_2,1)) \in E(T)$ *if and only if* $(z_1, z_2) \in E(Z)$

In T define a relation R by declaring (y, 0) R (z, 1) if and only if there exists an
$x \in X$ such that y = f (x) and z = g (x) ……(1)

Let "~" be the smallest equivalence relation in T generated by R. Let A = T / ~ denote the quotient set. i.e. A = set of all equivalence classes of ~. Let any such equivalence class be denoted as [a] for a $\in$ T.

Then A = T / ~ = { [(y , 0)] , [(z, 1)] / y $\in$ V(Y) , $z \in$ V(Z) } where
[(y , 0)] = [(z, 1)] if and only if there exists x $\in$ X such that y = f(x) and z = g(x).
In particular [(f^*(x), 0)] = [(g^*(x) , 1)] by (1)………..(2)

Step: 2 Let us consider the graph Q where V(Q) = A ; The edges in Q are defined by

i) ([(y_1 , 0)] , [(y_2 , 0)]) $\in$ E(Q) if and only if (y_1, y_2) $\in$ E(Y)

ii) ([(z_1 , 1)] , [(z_2 , 1)]) $\in$ E(Q) if and only if (z_1 , z_2) $\in$ E(Z) ……(3)

Define $p_1 : Y \to Q$ and $p_2 : Z \to Q$ as follows [see figure 14].

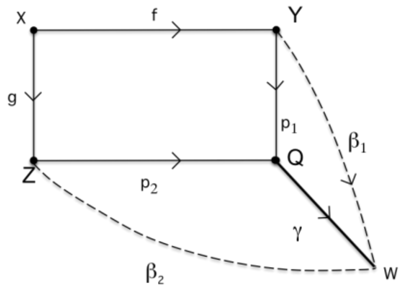

Figure 14

p_1^*: V (Y) $\to$ V(Q)
y $\to$ [(y , 0)]

and p_2^* : V (Z) $\to$ V(Q)
z $\to$ [(z , 1)]

Clearly p_1 and p_2 are homomorphisms of graphs by (3).

Moreover for all x $\in$ V(X)

$$P_1^* f^*(x) = [(f^*(x) , 0)]$$

$$= [(g^*(x) , 1)] \text{ by (2)}$$

$$= p_2^* g^*(x)$$

and so $p_1 f = p_2 g$ by (Lemma 1.6 in [3]).

Step: 3 Suppose there exists a graph W with $\beta_1 : Y \to W$ *and* $\beta_2 : Z \to W$ and such that $\beta_1 f = \beta_2 g$ …….(4)

Define a homomorphism $\gamma : Q \to W$ as follows.

$\gamma^* [(y , 0)] = \beta_1^* y$ *and* γ^*[(z,1)] = β_2^* (z) for y $\in$ Y and $z \in$ Z .

Clearly γ^* is well defined, since

[(y_1 , 0)] = [(y_2 , 0)] implies y_1 = y_2

so that $\gamma^* [(y_1 , 0)] = \beta_1^* (y)$

$$\beta_2^* (y_2) = \gamma^* [(y_2 , 0)]$$

Similarly [(z_1 ,1)] = [(z_2 ,1)] implies that z_1 = z_2 and so β_1^* (z_1)= β_2^* (z_2)

Also [(y , 0)] = [(z, 1)] implies there exists $x \in X$ such that y = f(x) and z = g(x) and

hence $\gamma^*[(y,0)]=\beta_1^*(y)=\beta_1^* f(x)=\beta_2^* g(x)=\beta_2^*(z)=\gamma^*[(z,1)]$ and so γ^* is well defined.
Moreover γ preserves edges as β_1 and β_2 does so.
Now for all $y \in V(Y)$
$\gamma^* p_1^*(y)=\beta_1^*(y)$ by definition implies that $\gamma p_1 = \beta_1$. Similarly $\gamma p_2 = \beta_2$.
Finally to prove the uniqueness of γ; Suppose there exists $\delta: Q \rightarrow W$ such that $\delta p_1 = \beta_1$.
$\delta p_2 = \beta_2$. Then for all $y \in V(Y)$ and $z \in V(Z)$.

$\gamma^*[(y,0)] = \gamma^* p_1^*(y) = \beta_1^*(y)$

$= \delta^* p_1^*(y) = \delta^*[(y,0)]$.

Similarly $\gamma^*[(z,1)] = \delta^*[(z,1)]$.

Hence $\gamma^* = \delta^*$ and so $\gamma = \delta$ proving the uniqueness of γ [3]. Thus Q is a push out.

IV. Conclusion

The above discussions show that the representation of homomorphism between graphs as a pair of functions $(f^*, \tilde{f})$ is useful in proving some properties in the category of graphs.

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