Complex dynamical behaviour of Disease Spread in a Plant-Herbivore System with Allee Effect

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Abstract: *This paper discusses the complex dynamical behaviour of a communicable disease in a plant-herbivore system with Allee effect. It is assumed that (a) Disease has no vertical transmission but it is untreatable and causes additional mortality in infected plant; (b) Allee effects built in the reproduction of susceptible plant while infected plant has no reproduction; (c) Herbivore captures susceptible and infected plant at the same rate but the consumption of infected plant has less benefits or even causes harm to herbivore; (d) Disease transmission follows the law of mass action; (e) Two predation response functions of Holling-Type II are used for both healthy and infected plants. The feasibility and stability conditions of the equilibrium points of the system were analyzed.Finally we performed numerical simulations to verify the theoretical results and to investigate further the properties of the system.*

Keywords: *Allee effect, Holling-Type II, susceptible plant -infective plant, vertical transmission, law of mass action, stability analysis.*

I. Introduction

Plant-Herbivore interaction is one of basic interspecies relations for ecological and social models. The first differential equation model of predator-prey type, called Lotka-Volterra equation, was formulated by Alfred Lotka and Vito Volterra in 1920's, when attempts were first made to find ecological laws of nature [6, 11]. These equations expressed the relationship between two or more species. Many modifications and extensions have been made on the equations. One of them involves dividing one or two species into Susceptible and Infected class.

Mathematical modeling has been a great tool for understanding species' interactions as well as the disease dynamics, which allow us to obtain useful biological insights and enable us to make correct policies to maintain the diversity in nature [7, 12]. Eco-epidemiology is comparatively a new branch in mathematical biology which simultaneously considers the ecological and epidemiological processes [13]. Hadeler and Freedman first introduced an eco-epidemiological model regarding predator-prey interactions with both prey and predator subject to disease [2, 5].

Many mathematical models have been used to understand the impacts of Allee effects on species' abundance and persistence especially in the presence of disease [14-17]. Allee effects have been recently studied intensively in population dynamics. The Allee effect phenomenon was introduced by Warden Clyde Allee in 1931 and was widely publicized by the recent book by Courchamp *et al* [1, 4].Allee proposed that the per capita birth rate declines at low population densities. At the time, Allee's ideas were revolutionary since most of the biology literature was advocating the negative density dependence principle where higher population density would limit the population growth.

Empirical evidence of Allee effects has been reported in many natural populations including plants, insects, marine invertebrates, birds and mammals. Various mechanisms at low population sizes, such as the need of a minimal group size necessary to successfully raise offspring, produce seeds, forage, and/or sustain herbivore attacks or enhanced genetic inbreeding have been proposed as potential sources of the Allee effect. Former studies have demonstrated that Allee effects can have important dynamical effects on the local stability analysis of population models [8-10]. It may have either a destabilizing role or a stabilizing role in the system. The local stability of an equilibrium point may be changed from stable to unstable or vice versa. It is also possible that for a population subject to an Allee effect, the system may take a much longer time to reach its steady state even though it is stable at an equilibrium point. The Allee effect has been attracting much attention recently owing to its strong potential impact on population dynamics. It is widely accepted that the Allee effect may increase the extinction risk of low-density populations. Therefore the population ecology investigation of the Allee effect is important to conservation biology.

In this paper, we propose a general plant-herbivore model with plant subject to Allee effects and disease. There are three unique features of our assumptions:
(a) Disease has no vertical transmission but it is untreatable and causes additional mortality in infected plant;

(b) Allee effects built in the reproduction of susceptible plant while infected plant has no reproduction;
(c) Herbivore captures susceptible and infected plant at the same rate but the consumption of infected plant has less benefits or even causes harm to herbivore.

These assumptions contribute great impacts on the dynamical outcomes of the proposed model. To explore how interplay among Allee effect, disease and predation affect species' abundance and persistence, we focus on a concrete system with additional two assumptions:

(d) Disease transmission follows the law of mass action;
(e) Plant and Herbivore have Holling-Type II functional responses.

II. Mathematical Model

We illustrate a general plant-herbivore model where plant is subject to Allee effect and disease is given by the following extended model in Figure1.
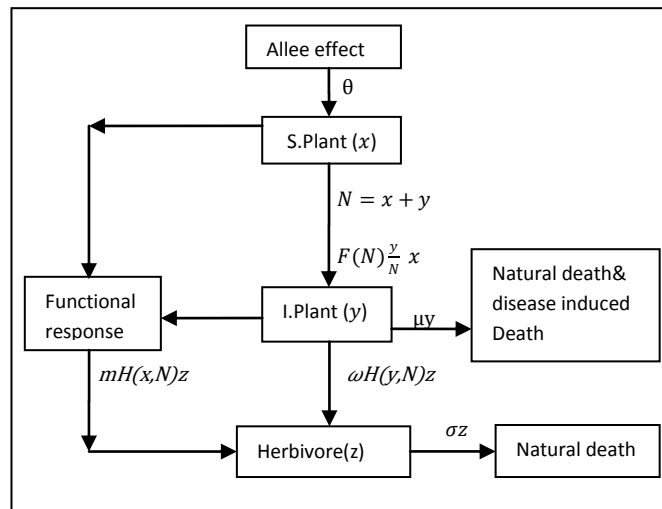

Figure 1: Plant-Herbivore Model where Plant is subject to Allee Effect and Disease

The extended system of Ordinary Differential Equations is

$$\frac{dx}{dt} = rx(x - \theta)(1 - x - y) - F(N)\frac{y}{N}x - H(x, N)z$$
$$\frac{dy}{dt} = F(N)\frac{y}{N}x - H(y, N)z - \mu y \qquad _____(1)$$
$$\frac{dz}{dt} = z[mH(x, N) + \omega H(y, N) - \sigma]$$

The parameters were described in Table 1

Variables/Parameters	Description
x	Density of Susceptible Plant
y	Density of Infected Plant
z	Density of Herbivore
$N = x + y$	Total population of Plant
r	Maximum birth-rate of species, which can be scaled to be 1 by altering the time scale
θ	Allee threshold; $0 < \theta < 1$
μ	Death rate of infected plant; $\mu > \theta$
m	Conversion efficiency on susceptible plant; $0 < m \leq 1$
σ	Natural death rate of herbivore
F(N)	Disease transmission function that can be either density-dependent (that is $F(N) = \lambda N$ which is also referred to the law of mass action) or frequency-dependent (that is $F(N) = \lambda$)
λ	Rate of infection
ω	Conversion efficiency on infected plant; $-\infty < \omega < m$
$\omega < 0$	Consumption of infected plant increases the death rate of the herbivore
$\omega > 0$	Consumption of susceptible plant increases the growth rate of the herbivore
$H(x, N) = \frac{ax}{k+x+y}$; $H(y, N) = \frac{ay}{k+x+y}$	The functional responses can take the form of Holling-Type II
k	Half-saturation constant
a	Attack rate of herbivore
$b = am$	The total effect to herbivore by consuming susceptible plant; $0 < b \leq a$
$\alpha = a\omega$	The total effect to herbivore by consuming infected plant; $-\infty < \alpha \leq b$
$x(x - \theta)(1 - x - y)$	Net reproduction rate of newborns, a term that accounts for Allee effects due to mating limitations as well as reductions in fitness due to the competition for resource from infective

Table 1: Variables and Parameters used Model (1) and (2)

In the presence of predation, we assume that herbivore consume susceptible and infected plants at the rate of $H(x, N) = \frac{ax}{k+x+y}$ and $H(y, N) = \frac{ay}{k+x+y}$ respectively, where infected plant has less or negative contribution to the growth rate of herbivore in comparison to susceptible plant. Based on the assumptions, the mathematical model is formulated as follows.

$$\frac{dx}{dt} = x(x - \theta)(1 - x - y) - \lambda xy - \frac{axz}{k + x + y}$$
$$\frac{dy}{dt} = \lambda xy - \frac{ayz}{k + x + y} - \mu y \qquad \qquad \_\_\_\_\_(2)$$
$$\frac{dz}{dt} = \frac{bxz}{k + x + y} + \frac{\alpha yz}{k + x + y} - \sigma z$$

III. Dynamics of submodels

In order to understand the full dynamics of system (2), we should have a complete picture of the dynamics of the following two submodels:

3.1 Sub-model: I

The plant-herbivore model in the absence of the disease in (2) is represented as

$$\frac{dx}{dt} = x(x - \theta)(1 - x) - \frac{axz}{k + x}$$
$$\frac{dz}{dt} = \frac{bxz}{k + x} - \sigma z \qquad \qquad \_\_\_\_\_(3)$$

For convenience, we introduce a disease-free demographic reproduction number for herbivore $R_0^z = \frac{b}{\sigma}$ which gives the expected number of offspring b of an average individual herbivore in its lifetime $\frac{1}{\sigma}$. The reproduction number R_0^z is based upon the assumptions that the susceptible plant is at unit density and the disease is absent. The value of $R_0^z < 1$ indicates that the herbivore cannot invade while the value of $R_0^z > 1$ indicates that the herbivore may invade.

Equilibrium points

The system (3) has four equilibrium points which are
  i) $E_1^z = (0,0)$
  ii) $E_2^z = (\theta, 0)$
  iii) $E_3^z = (1,0)$
  iv) $E_4^z = (x^*, z^*)$ where $x^* = \frac{k}{R_0^z - 1}$ and $z^* = \frac{k}{a}\left(\frac{k}{R_0^z - 1} - \theta\right)\left(1 - \frac{k}{R_0^z - 1}\right)\left(1 + \frac{1}{R_0^z - 1}\right)$

Dynamical behaviour

In this section, we study the local behaviour of the system (3) about each equilibrium points. The stability of the system (3) is carried out by computing the Jacobian matrix corresponding to each equilibrium point. The Jacobian matrix is given by

$$J^z(x, z) = \begin{bmatrix} (x - \theta)(1 - x) + x(1 - 2x + \theta) - \frac{azk}{(k + x)^2} & \frac{-ax}{k + x} \\ \frac{bzk}{(k + x)^2} & \frac{bx}{k + x} - \sigma \end{bmatrix} \qquad _____(4)$$

Proposition: 1
The equilibrium point $E_1^z = (0,0)$ is locally asymptotically stable if Det $(J^z(0,0)) > 0$ and Tr $(J^z(0,0)) < 0$.
Proof:
The Jacobian matrix evaluated at E_1^z is given by

$$J^z(0,0) = \begin{bmatrix} -\theta & 0 \\ 0 & -\sigma \end{bmatrix}$$

The eigenvalues of $J^z(0,0)$ are

$\lambda_1 = -\theta; \lambda_2 = -\sigma$. Beltrami conditions that, if $Det(J) > 0$ and $Tr(J) < 0$, then the equilibrium state of the model is locally asymptotically stable [3]. Here, $Det(J^z(0,0)) = \sigma\theta > 0$ and $Tr(J^z(0,0)) = (-\theta - \sigma) < 0$. Hence the conditions of Beltrami were satisfied.

Therefore, we concluded that the equilibrium state $E_1^z = (0,0)$ is locally asymptotically stable.

Proposition: 2

The equilibrium point $E_2^z = (\theta, 0)$ is a

   (a) Saddle if $R_0^z < \frac{k+\theta}{\theta}$
   (b) Source if $R_0^z > \frac{k+\theta}{\theta}$

Proof:

The Jacobian matrix evaluated at E_2^z is given by

$$J^z(\theta, 0) = \begin{bmatrix} \theta(1-\theta) & \dfrac{-a\theta}{k+\theta} \\ 0 & \dfrac{b\theta}{k+\theta} - \sigma \end{bmatrix}$$

The eigenvalues of $J^z(\theta, 0)$ are

$$\lambda_1 = \theta(1-\theta) > 0 \, \& \, \lambda_2 = \frac{b\theta}{k+\theta} - \sigma = \frac{\sigma\theta}{k+\theta}\left(R_0^z - \frac{k+\theta}{\theta}\right) = \begin{cases} < 0 & if \, R_0^z < \frac{k+\theta}{\theta} \\ > 0 & if \, R_0^z > \frac{k+\theta}{\theta} \end{cases}$$

Therefore, $E_2^z = (\theta, 0)$ is a saddle if $R_0^z < \frac{k+\theta}{\theta}$ and is a source if $R_0^z > \frac{k+\theta}{\theta}$

Proposition: 3

The equilibrium point $E_3^z = (1,0)$ is a

   (a) Stable if $R_0^z < (k+1)$
   (b) Saddle if $R_0^z > (k+1)$

Proof:

The Jacobian matrix evaluated at E_3^z is given by

$$J^z(1,0) = \begin{bmatrix} \theta - 1 & \dfrac{-a}{k+1} \\ 0 & \dfrac{b}{k+1} - \sigma \end{bmatrix}$$

The eigenvalues of $J^z(1,0)$ are

$$\lambda_1 = (\theta - 1) < 0; \lambda_2 = \frac{b}{k+1} - \sigma = \frac{\sigma}{k+1}\left(R_0^z - (k+1)\right) = \begin{cases} < 0 & if \, R_0^z < (k+1) \\ > 0 & if \, R_0^z > (k+1) \end{cases}$$

Therefore, $E_3^z = (1,0)$ is locally asymptotically stable if $R_0^z < (k+1)$ and is a saddle if $R_0^z > (k+1)$

Proposition: 4

The equilibrium point E_4^z exists and is locally asymptotically stable if $(k+1) < R_0^z < min\left\{1 + \frac{k}{\theta}, 1 + \frac{2k}{(1+\theta)}\right\}$

Proof:

The unique interior equilibrium $E_4^z = (x^*, z^*) = \left(\frac{k}{R_0^z - 1}, \frac{k}{a}\left(\frac{k}{R_0^z - 1} - \theta\right)\left(1 - \frac{k}{R_0^z - 1}\right)\left(1 + \frac{1}{R_0^z - 1}\right)\right)$ exists only if

$(k+1) < R_0^z < \left(1 + \frac{k}{\theta}\right)$. The Jacobian matrix evaluated at E_4^z is given by

$$J^z(x^*, z^*) = \begin{bmatrix} \dfrac{1}{R_0^z}\left(\dfrac{k}{R_0^z - 1} - \theta\right)\left(1 - \dfrac{k}{R_0^z - 1}\right) + \dfrac{k}{R_0^z - 1}\left(1 - \dfrac{2k}{R_0^z - 1} + \theta\right) & \dfrac{-a}{R_0^z} \\ \dfrac{\frac{b}{a}\left(\frac{k}{R_0^z - 1} - \theta\right)\left(1 - \frac{k}{R_0^z - 1}\right)}{\left(1 + \frac{1}{R_0^z - 1}\right)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A & -B \\ C & 0 \end{bmatrix}$$

Whose characteristic equation is given by $\lambda^2 - A\lambda + BC = 0$ where $BC > 0$ and $\begin{cases} A < 0 \, if \, R_0^z < 1 + \frac{2k}{(1+\theta)} \\ A > 0 \, if \, R_0^z > 1 + \frac{2k}{(1+\theta)} \end{cases}$

This indicates that the eigenvalues of $J^z(x^*, z^*)$ are $\lambda_1 = \frac{A - \sqrt{A^2 - 4BC}}{2}$ and $\lambda_2 = \frac{A + \sqrt{A^2 - 4BC}}{2}$ when $A^2 > 4BC$

(or) $\lambda_1 = \frac{A - i\sqrt{4BC - A^2}}{2}$ and $\lambda_2 = \frac{A + i\sqrt{4BC - A^2}}{2}$ when $A^2 < 4BC$. Therefore, $E_4^z = (x^*, z^*)$ exists and is locally

asymptotically stable if $(k + 1) < R_0^z < min\left\{1 + \frac{k}{\theta}, 1 + \frac{2k}{(1+\theta)}\right\} = 1 + \frac{2k}{(1+\theta)}$.

3.2Sub-model: II

The plant-herbivore model in the absence of predation in (2) is represented as

$$\frac{dx}{dt} = x(x - \theta)(1 - x - y) - \lambda xy$$
$$\frac{dy}{dt} = \lambda xy - \mu y \qquad \qquad \text{_____(5)}$$

We introduce the basic reproductive ratio $R_0^y = \frac{\lambda}{\mu}$ whose numerator denotes the number of secondary infections per unit of time and denominator denotes the inverse of the average infectious period μ. The value of $R_0^y < 1$ indicates that the infection cannot invade while the value of $R_0^y > 1$ indicates that the disease can invade.

Equilibrium points

The system (5) has four equilibrium points which are

i) $E_5^y = (0,0)$
ii) $E_6^y = (\theta, 0)$
iii) $E_7^y = (1,0)$

iv) $E_8^y = (x^*, y^*)$ where $x^* = \frac{1}{R_0^y}$ and $y^* = \frac{\left(\frac{1}{R_0^y} - \theta\right)\left(1 - \frac{1}{R_0^y}\right)}{\frac{1}{R_0^y} + \lambda - \theta}$

Dynamical behaviour

In this section, we study the local behaviour of the system (5) about each equilibrium points. It is carried out by computing the Jacobian matrix corresponding to each equilibrium point. The Jacobian matrix is given by

$$J^y(x, y) = \begin{bmatrix} (x - \theta)(1 - x - y) + x(1 - x - y) - x(x - \theta) - \lambda y & -x(x - \theta) - \lambda x \\ \lambda y & \lambda x - \mu \end{bmatrix} \qquad \text{_____(6)}$$

Proposition: 5

The equilibrium point $E_5^y = (0,0)$ is locally asymptotically stable if $Det\ (J^y(0,0)) > 0$ and $Tr\ (J^y(0,0)) < 0$.

Proof:

The Jacobian matrix evaluated at E_5^y is given by

$$J^y(0,0) = \begin{bmatrix} -\theta & 0 \\ 0 & -\mu \end{bmatrix}$$

The eigenvalues of $J^y(0,0)$ are $\lambda_1 = -\theta$ and $\lambda_2 = -\mu$.

Beltrami conditions that, if the $Det\ (J) > 0$ and $Tr\ (J) < 0$ then the equilibrium state of the model is locally asymptotically stable. Here, $Det\ (J^y(0,0)) = \mu\theta > 0$ and $Tr\ (J^y(0,0)) = -\theta - \mu < 0$. Hence the conditions of Beltrami were satisfied. Therefore, we concluded that the equilibrium state $E_5^y = (0,0)$ is locally asymptotically stable.

Proposition: 6

The equilibrium point $E_6^y = (\theta, 0)$ is a

(c) Saddle if $R_0^y < \frac{1}{\theta}$
(d) Source if $R_0^y > \frac{1}{\theta}$

Proof:

The Jacobian matrix evaluated at E_6^y is given by

$$J^y(\theta, 0) = \begin{bmatrix} \theta(1 - \theta) & -\lambda\theta \\ 0 & \lambda\theta - \mu \end{bmatrix}$$

The eigenvalues of $J^y(\theta, 0)$ are $\lambda_1 = \theta(1 - \theta) > 0$ and $\lambda_2 = (\lambda\theta - \mu) = \mu\theta\left(R_0^y - \frac{1}{\theta}\right) = \begin{cases} < 0 \; if R_0^y < \frac{1}{\theta} \\ > 0 \; if R_0^y > \frac{1}{\theta} \end{cases}$

Therefore, $E_6^y = (\theta, 0)$ is a saddle if $R_0^y < \frac{1}{\theta}$ and is a source if $R_0^y > \frac{1}{\theta}$

Proposition: 7

The equilibrium point $E_7^y = (1,0)$ is a
   (c) Stable if $R_0^y < 1$
   (d) Saddle if $R_0^y > 1$

Proof:

The Jacobian matrix evaluated at E_7^y is given by

$$J^y(1,0) = \begin{bmatrix} \theta - 1 & (\theta - 1) - \lambda \\ 0 & \lambda - \mu \end{bmatrix}$$

The eigenvalues of $J^y(1,0)$ are $\lambda_1 = (\theta - 1) < 0$ and $\lambda_2 = (\lambda - \mu) = \mu(R_0^y - 1) = \begin{cases} < 0 \; if R_0^y < 1 \\ > 0 \; if R_0^y > 1 \end{cases}$

Therefore, $E_7^y = (1,0)$ is locally asymptotically stable if $R_0^y < 1$ and is a saddle if $R_0^y > 1$.

Proposition: 8

The equilibrium point E_8^y exists and is locally asymptotically stable if $1 < R_0^y < min\left\{\frac{1}{\theta}, \frac{(\lambda - \theta) + \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}\right\}$

Proof:

The unique interior equilibrium $E_8^y = (x^*, y^*) = \left(\frac{1}{R_0^y}, \frac{\left(\frac{1}{R_0^y} - \theta\right)\left(1 - \frac{1}{R_0^y}\right)}{\frac{1}{R_0^y} + \lambda - \theta}\right)$ exists only if $1 < R_0^y < \frac{1}{\theta}$.

Since the condition $0 < \theta < 1, \mu > \theta, 0 < b \leq a \; and \; -\infty < \alpha \leq b$,

We have, $\frac{1}{R_0^y} + \lambda - \theta = \frac{\mu}{\lambda} + \lambda - \theta > \frac{\theta^2}{4\lambda} + \lambda - \theta = \frac{(\theta - 2\lambda)^2}{4\lambda} \geq 0$. The Jacobian matrix evaluated at E_3^y is given by

$$J^y(x^*, y^*) = \begin{bmatrix} \frac{1}{R_0^y}\left(1 - \frac{2}{R_0^y} - \frac{\left(\frac{1}{R_0^y} - \theta\right)\left(1 - \frac{1}{R_0^y}\right)}{\frac{1}{R_0^y} + \lambda - \theta} + \theta\right) & -\frac{1}{R_0^y}\left(\frac{1}{R_0^y} + \lambda - \theta\right) \\ \frac{\lambda\left(\frac{1}{R_0^y} - \theta\right)\left(1 - \frac{1}{R_0^y}\right)}{\frac{1}{R_0^y} + \lambda - \theta} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} A & -B \\ C & 0 \end{bmatrix}$$

Whose characteristic equation is given by $\lambda^2 - A\lambda + BC = 0$, where $BC > 0$ and

$$A = \frac{1}{R_0^y}\left(1 - \frac{2}{R_0^y} - \frac{\left(\frac{1}{R_0^y} - \theta\right)\left(1 - \frac{1}{R_0^y}\right)}{\frac{1}{R_0^y} + \lambda - \theta} + \theta\right)$$

$$= \frac{1}{\left(R_0^y\right)^2\left(1 + R_0^y(\lambda - \theta)\right)(\lambda + \lambda\theta - \theta^2)}\left(R_0^y - \frac{(\lambda - \theta) + \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}\right)\left(R_0^y - \frac{(\lambda - \theta) - \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}\right)$$

Thus, we have A > 0 if $R_0^y > \frac{(\lambda - \theta) + \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}$ and A < 0 if $R_0^y < \frac{(\lambda - \theta) + \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}$

This indicates that the eigenvalues of $J^z(x^*, z^*)$ are $\lambda_1 = \frac{A - \sqrt{A^2 - 4BC}}{2}$ and $\lambda_2 = \frac{A + \sqrt{A^2 - 4BC}}{2}$ when $A^2 > 4BC$ (or) $\lambda_1 = \frac{A - i\sqrt{4BC - A^2}}{2}$ and $\lambda_2 = \frac{A + i\sqrt{4BC - A^2}}{2}$ when $A^2 < 4BC$. Therefore, $E_8^y = (x^*, y^*)$ exists and is locally asymptotically stable if $1 < R_0^y < min\left\{\frac{1}{\theta}, \frac{(\lambda - \theta) + \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}\right\} = \frac{(\lambda - \theta) + \sqrt{\lambda^2 - \lambda\theta + \lambda}}{(\lambda + \lambda\theta - \theta^2)}$.

IV. Disease/predation-driven extinctions of submodels

In this section, we focus on the disease/predation-driven extinctions of both submodels. First, we have the following theorem regarding the extinction of one or both species:

Theorem:

If the submodels I and II with condition that $0 < \theta < 1, \mu > \theta, 0 < b \le a, -\infty < \alpha < b$ then

(a) If $R_0^z \le (k+1)$, then the population of herbivore in the submodel (3) goes extinction for any initial condition takes in $\Re_+^2$.

(b) If $R_0^y \le 1$, then the population of infective in the submodel (5) goes extinction for any initial condition taken in $\Re_+^2$

(c) If $R_0^z \ge \frac{k+\theta}{\theta}$, then system (3) converges to (0, 0) for any initial condition taken in the interior of $\Re_+^2$, which is predation-driven extinction.

(d) If $R_0^y \ge \frac{1}{\theta}$, then system (5) converges to (0, 0) for any initial condition taken in the interior of $\Re_+^2$, which is disease - driven extinction.

(e) If $x(0) < \theta$, then all species in both submodels (3) and (5) converge to (0, 0).

Proof:

Case: 1 If $R_0^z \le (k+1)$ or $R_0^z \ge \frac{k+\theta}{\theta}$, then the submodel I only has three boundary equilibrium $(0,0,0),(\theta,0,0),(1,0,0)$, where E_θ^z is a saddle when $R_0^z < \frac{k+\theta}{\theta}$ and E_1^z is locally asymptotically stable when $R_0^z < (k+1)$ while E_θ^z is a source when $R_0^z > \frac{k+\theta}{\theta}$ and E_1^z is a saddle when $R_0^z > (k+1)$. For $R_0^z = (k+1)$, E_1^z is non-hyperbolic with one zero eigenvalues and the other negative while E_θ^z remains saddle. For $R_0^z = \frac{k+\theta}{\theta}$, E_θ^z is non-hyperbolic with one zero eigenvalues and the other positive while E_1^z remains saddle.

Case: 2 If $R_0^y \le 1$ or $R_0^y \ge \frac{1}{\theta}$, then the submodel II only has three boundary equilibrium $(0,0,0),(\theta,0,0),(1,0,0)$, where E_θ^y is a saddle when $R_0^y < \frac{1}{\theta}$ and E_1^y is locally asymptotically stable when $R_0^y < 1$ while E_θ^y is a source when $R_0^y > \frac{1}{\theta}$ and E_1^y is a saddle when $R_0^y > 1$. For $R_0^y = 1$, E_1^y is non-hyperbolic with one zero eigenvalues and the other negative while E_θ^y remains saddle. For $R_0^y = \frac{k+\theta}{\theta}$, E_θ^y is non-hyperbolic with one zero eigenvalues and the other positive while E_1^y remains saddle.

V. Dynamics of a general plant-herbivore model where plant is subject to Allee effects and disease

In this section, we study the local behaviour of the system (2) about each equilibrium points. The stability of the system (2) is carried out by computing the Jacobian matrix corresponding to each equilibrium point. The Jacobian matrix is given by

$$J^z(x,y,z) = \begin{bmatrix} (x-\theta)(1-x-y)+x(1-x-y)-x(x-\theta)-\lambda y-\frac{az(k+y)}{(k+x+y)^2} & -x(x-\theta)-\lambda x+\frac{axz}{(k+x+y)^2} & \frac{-ax}{k+x+y} \\ \lambda y & \lambda x-\mu-\frac{az(k+z)}{(k+y+z)^2} & -\frac{ay(k+y)}{(k+y+z)^2} \\ \frac{bz(k+y)}{(k+x+y)^2}-\frac{\alpha yz}{(k+x+y)^2} & -\frac{bxz}{(k+x+y)^2}+\frac{az(k+x)}{(k+x+y)^2} & \frac{bx}{k+x+y}+\frac{\alpha y}{k+x+y}-\sigma \end{bmatrix} \text{--- (7)}$$

Proposition: 9

The equilibrium point $E_9 = (0,0,0)$ is locally asymptotically stable.

Proof:

The Jacobian matrix evaluated at E_9 is given by

$$J(0,0,0) = \begin{bmatrix} -\theta & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\sigma \end{bmatrix}$$

The eigenvalues of $J(0,0,0)$ are $\lambda_1 = -\theta$, $\lambda_2 = -\mu$ and $\lambda_3 = -\sigma$. It is always locally asymptotically stable since its eigenvalues are negative.

Proposition: 10

The equilibrium point $E_{10} = (\theta,0,0)$ is unstable.

Proof:

The Jacobian matrix evaluated at E_{10} is given by

$$J(\theta,0,0) = \begin{bmatrix} \theta(1-\theta) & -\lambda\theta & \dfrac{-a\theta}{k+\theta} \\ 0 & \lambda\theta-\mu & 0 \\ 0 & 0 & \dfrac{b\theta}{k+\theta}-\sigma \end{bmatrix}$$

The equilibrium point $E_{10} = (\theta,0,0)$ is always unstable since the eigenvalues are $\lambda_1 = \theta(1-\theta) > 0$,

$\lambda_2 = \lambda\theta - \mu = \mu\theta\left(R_0^y - \dfrac{1}{\theta}\right) = \begin{cases} < 0 \ if R_0^y < \dfrac{1}{\theta} \\ > 0 \ if R_0^y > \dfrac{1}{\theta} \end{cases}$ and $\lambda_3 = \dfrac{b\theta}{k+\theta} - \sigma = \dfrac{\sigma\theta}{k+\theta}\left(R_0^z - \dfrac{k+\theta}{\theta}\right) = \begin{cases} < 0 \ if R_0^z < \dfrac{k+\theta}{\theta} \\ > 0 \ if R_0^z > \dfrac{k+\theta}{\theta} \end{cases}$

Proposition: 11

The equilibrium point $E_{11} = (1,0,0)$ is locally asymptotically stable if $R_0^z < (k+1)$ and $R_0^y < 1$.

Proof:

The Jacobian matrix evaluated at E_{11} is given by

$$J(1,0,0) = \begin{bmatrix} \theta-1 & (\theta-1)-\lambda & \dfrac{-a}{k+1} \\ 0 & \lambda-\mu & 0 \\ 0 & 0 & \dfrac{b}{k+1}-\sigma \end{bmatrix}$$

The equilibrium point $E_{11} = (1,0,0)$ is locally asymptotically stable if $R_0^z < (k+1)$ and $R_0^y < 1$ since the eigenvalues of $J(1,0,0)$ are $\lambda_1 = (\theta-1) < 0$, $\lambda_2 = (\lambda-\mu) = \mu(R_0^y - 1) = \begin{cases} < 0 \ if R_0^y < 1 \\ > 0 \ if R_0^y > 1 \end{cases}$

and $\lambda_3 = \dfrac{b}{k+1} - \sigma = \dfrac{\sigma}{k+1}(R_0^z - (k+1)) = \begin{cases} < 0 \ if R_0^z < (k+1) \\ > 0 \ if R_0^z > (k+1) \end{cases}$

Proposition: 12

The equilibrium point $E_{12} = (x^*,0,z^*) = \left(\dfrac{k}{R_0^z-1}, 0, \dfrac{k}{a}\left(\dfrac{k}{R_0^z-1}-\theta\right)\left(1-\dfrac{k}{R_0^z-1}\right)\left(1+\dfrac{1}{R_0^z-1}\right)\right)$ is locally asymptotically stable.

Proof:

The Jacobian matrix evaluated at E_{12} is given by

$$J(x^*,0,z^*) = \begin{bmatrix} (x-\theta)(1-x)+x(1-x)-x(x-\theta)-\dfrac{azk}{(k+x)^2} & -x(x-\theta)-\lambda x+\dfrac{axz}{(k+x)^2} & \dfrac{-ax}{k+x} \\ 0 & \lambda x-\mu-\dfrac{az}{(k+z)} & 0 \\ \dfrac{bzk}{(k+x)^2} & -\dfrac{bxz}{(k+x)^2}+\dfrac{\alpha z}{(k+x)} & \dfrac{bx}{k+x}-\sigma \end{bmatrix} \text{——(8)}$$

The equilibrium point $E_{12} = (x^*,0,z^*) = \left(\dfrac{k}{R_0^z-1}, 0, \dfrac{k}{a}\left(\dfrac{k}{R_0^z-1}-\theta\right)\left(1-\dfrac{k}{R_0^z-1}\right)\left(1+\dfrac{1}{R_0^z-1}\right)\right)$ is locally

asymptotically stable if it is locally asymptotically stable in the submodel I and $\dfrac{1}{y}\dfrac{dy}{dt} = \lambda x - \mu - \dfrac{az}{(k+z)}$ .At $E_{12}, \dfrac{1}{y}\dfrac{dy}{dt}$

$= \dfrac{R_0^y k}{R_0^z-1} - 1 - \dfrac{k\left(\frac{k}{R_0^z-1}-\theta\right)\left(1-\frac{k}{R_0^z-1}\right)\left(1+\frac{1}{R_0^z-1}\right)}{\mu(k+z)} < 0$ if and only if $\dfrac{R_0^y}{R_0^z-1} < \dfrac{1}{k} + \dfrac{\left(\frac{k}{R_0^z-1}-\theta\right)\left(1-\frac{k}{R_0^z-1}\right)\left(1+\frac{1}{R_0^z-1}\right)}{\mu(k+z)}$ which indicates

that disease is not able to invade at E_{12}.

Therefore, we can conclude that the equilibrium point $E_{12} = (x^*,0,z^*)$ is locally asymptotically stable if

$\dfrac{R_0^y}{R_0^z-1} < \dfrac{1}{k} + \dfrac{\left(\frac{k}{R_0^z-1}-\theta\right)\left(1-\frac{k}{R_0^z-1}\right)\left(1+\frac{1}{R_0^z-1}\right)}{\mu(k+z)}$ and $(k+1) < R_0^z < 1 + \dfrac{2k}{(1+\theta)}$.

Proposition: 13

The equilibrium point $E_{13} = (x^*,y^*,0) = \left(\dfrac{1}{R_0^y}, \dfrac{\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\frac{1}{R_0^y}+\lambda-\theta}, 0\right)$ is locally asymptotically stable.

Proof:

The Jacobian matrix evaluated at E_{13} is given by

$$J^z(x^*, y^*, 0) = \begin{bmatrix} (x-\theta)(1-x-y)+x(1-x-y)-x(x-\theta)-\lambda y & -x(x-\theta)-\lambda x & \dfrac{-ax}{k+x+y} \\ \lambda y & \lambda x - \mu & -\dfrac{ay}{(k+y)} \\ 0 & 0 & \dfrac{bx}{k+x+y}+\dfrac{\alpha y}{k+x+y}-\sigma \end{bmatrix} \text{—(9)}$$

The equilibrium point$E_{13} = (x^*, y^*, 0) = \left(\dfrac{1}{R_0^y}, \dfrac{\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\frac{1}{R_0^y}+\lambda-\theta}, 0 \right)$ is locally asymptotically stable if it is locally

asymptotically stable in the submodel II and$\dfrac{1}{z}\dfrac{dz}{dt} = \dfrac{bx}{k+x+y} + \dfrac{\alpha y}{k+x+y} - \sigma$.

At the equilibrium point $E_{12} = (x^*, 0, z^*)$, $\dfrac{1}{z}\dfrac{dz}{dt} = \dfrac{\left(\frac{R_0^z}{R_0^y}\right)}{\left[k+\frac{1}{R_0^y}+\frac{\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\frac{1}{R_0^y}+\lambda-\theta}\right]} + \dfrac{\left[\frac{\left(\frac{\alpha}{\sigma}\right)\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\left(\frac{1}{R_0^y}+\lambda-\theta\right)}\right]}{\left[k+\frac{1}{R_0^y}+\frac{\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\frac{1}{R_0^y}+\lambda-\theta}\right]} - 1 < 0$if and only if

$\left(\dfrac{R_0^z}{R_0^y}\right) < \left[k+\dfrac{1}{R_0^y}+\dfrac{\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\frac{1}{R_0^y}+\lambda-\theta}\right] - \left[\dfrac{\left(\frac{\alpha}{\sigma}\right)\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\left(\frac{1}{R_0^y}+\lambda-\theta\right)}\right]$ which indicates that herbivore is not able to invade at

E_{13}.Therefore, we can conclude that the equilibrium point $E_{12} = (x^*, 0, z^*)$ is locally asymptotically stable

if$\left(\dfrac{R_0^z}{R_0^y}\right) < \left[k+\dfrac{1}{R_0^y}+\dfrac{\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\frac{1}{R_0^y}+\lambda-\theta}\right] - \left[\dfrac{\left(\frac{\alpha}{\sigma}\right)\left(\frac{1}{R_0^y}-\theta\right)\left(1-\frac{1}{R_0^y}\right)}{\left(\frac{1}{R_0^y}+\lambda-\theta\right)}\right]$ and $1 < R_0^y < \dfrac{(\lambda-\theta)+\sqrt{\lambda^2-\lambda\theta+\lambda}}{(\lambda+\lambda\theta-\theta^2)}$.

VI. Numerical Analysis

In this section, we illustrate the analytical findings with the help of numerical simulations performed with MATLAB programming. We present time series plots for system (3) and (5) to confirm the theoretical results with hypothetical set of data. They show interesting complex dynamical behaviours.

Figure: 1 (See Appendix) shows the time series graph for Susceptible Plant and Herbivore with Allee effect when$\theta = 0.3$, $a = 6$, $k = 6.8$, $b = 4.5$ and $\sigma = 0.6$

Figure: 2 (See Appendix) shows the time series graph for Susceptible Plant and Infected Plant with Allee effect when$\theta = 0.3$, $\lambda = 1.5$ and $\mu = 1$

VII. Conclusion

A mathematical model describing the spread of a disease in a plant-herbivore system with Allee effect has been discussed. An infectious disease is assumed to spread only among plants and herbivore consumes both healthy and infected plants with two different predation responses. In the presence of Allee effects, species are prone to extinction and initial condition plays an important role on the surviving of plant as well as its corresponding herbivore and in the presence of Allee effects, disease may be able to save plant from the predation-driven extinction and leads to the coexistence of susceptible and infected plant population while herbivore cannot save the disease-driven extinction. In this investigation, the stability of different equilibrium points has been thoroughly examined. Numerical simulations were done to observe the effect of the disease and Allee effect on the species and diagrams were presented which are supporting our results. All these findings may have potential applications in conservation biology.

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Appendix:

Figure: 1	**Figure: 2**
Time series graph for susceptible plant and Herbivore with Allee Effect	**Time series graph for susceptible plant and Infected plant with Allee Effect**
(θ = 0.3, a = 6, k = 6.8, b = 4.5, σ = 0.6)	(θ = 0.3, λ = 1.5, μ = 1)