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On Certain Classes of Multivalent Functions

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Abstract: *In this paper, we defined certain analytic p-valent function with negative type denoted by τ_p. We obtained sharp results concerning coefficient bounds, distortion theorem belonging to the class τ_p.*
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I. Introduction

Let A (p) denote the class of f normalized univalent functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad ,(a_{n+p} \geq 0, p \in \mathbb{N} = \{1,2,3,\dots\}) \qquad (1.1)$$

analytic and p-valent in the unit disc $E = \{z : z \in C; |z| < 1\}$.

A function f(z) Є A (p) is said to in the class of $S_p^*(\alpha)$ p-valently starlike function of order α $(0 \leq \alpha < p)$ if it satisfies, for $z \in E$, the condition

$$Re\left(\frac{z\, f'(z)}{f(z)}\right) > \alpha \qquad (1.2)$$

Furthermore, a function $f(z) \in A\,(p)$ is said to in the class $\mathcal{K}_p(\alpha)$ of p-valently convex function of order α $(0 \leq \alpha < p)$ if it satisfies, for $z \in E$, the condition

$$Re\left(1+\frac{z\, f''(z)}{f'(z)}\right) > \alpha \qquad (1.3)$$

It follows from the definition (1.2) and (1.3) that

$$f(z) \in \mathcal{K}_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha) \qquad (0 \leq \alpha < p) \qquad (1.4)$$

whose special case, when α = 0 is the familiar Alexander theorem (see for example [1] p.43, Theorem 2.12). We also note that

$$\mathcal{K}_p(\alpha) \subset S_p^*(\alpha) \qquad (0 \leq \alpha < p)$$

$$S_p^*(\alpha) \subseteq S_p^*(0) \equiv S_p^* \qquad (0 \leq \alpha < p)$$

and

$$\mathcal{K}_p(\alpha) \subseteq \mathcal{K}_p(0) \equiv \mathcal{K}_p \qquad (0 \leq \alpha < p)$$

Where S_p^* and $\mathcal{K}_p$ denote the subclasses of A (p) consisting of p-valently starlike and convex functions in unit disk E respectively.

Let $\tau_p(\alpha, \beta)$ denote the subclass of A (p) consisting of functions analytic and p-valent which can be express in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad , a_{n+p} \geq 0$$

The subclass $\tau_p(\alpha, \beta)$ of p-valent functions with negative coefficients is studied by H. M. Srivastava and M. K. Auof [2].

Following S. Owa [3], we say that a function $f(z) \in \tau_p$ is in the subclass $\tau_p(\alpha, \beta)$ if and only if

$$\left|\frac{f'(z) - pz^{1-p}}{f'(z) + pz^{1-p}(1-2\alpha)}\right| < \beta$$

The subclass $\tau_p(\alpha,\beta)$ was studied by Goel and Sohi [4]. Moreover S. Owa studied several interesting results on radius of convexity for p-valent function with negative coefficients. In this present paper we investigate sharp results concerning coefficient inequalities, distortion theorem and radius of convexity for class the $\tau_p(\alpha,\beta)$.

II. Main Result

Theorem 2.1 A function

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad , a_{n+p} \geq 0$$

is in the class $\tau_p(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} (n+p)(1+\beta)|a_{n+p}| \leq 2\beta(1-\alpha)p \tag{2.1}$$

The result is sharp.

Proof: Assume (2.1) holds. We show that $f(z) \in \tau_p(\alpha,\beta)$.

Let $|z| = 1$. We have,

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \tag{2.2}$$

$$f'(z) = pz^{p-1} - \sum_{n=1}^{\infty} a_{n+p}\,(n+p) z^{n+p-1} \tag{2.3}$$

Now,

$$|f'(z) - pz^{p-1}| = \left|pz^{p-1} - \sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1} - pz^{p-1}\right|$$

$$= \left|-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}\right|$$

Also,

$\beta|f'(z) + pz^{p-1}(1-2\alpha)|$

$$= \left|\beta pz^{p-1} - \beta\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1} + \beta pz^{p-1}(1-2\alpha)\right|$$

$$= \left|-\beta\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1} + 2\beta pz^{p-1} - 2\alpha\beta pz^{p-1}\right|$$

Then,

$|f'(z) - pz^{p-1}| - \beta|f'(z) + pz^{p-1}(1-2\alpha)|$

$$= \left|-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}\right| - \left|-\beta\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1} + 2\beta pz^{p-1} - 2\alpha\beta pz^{p-1}\right|$$

since $|z| = 1$

$|f'(z) - pz^{p-1}| - \beta|f'(z) + pz^{p-1}(1-2\alpha)|$

$$\leq \sum_{n=1}^{\infty}(n+p)|a_{n+p}| + \beta\sum_{n=1}^{\infty}(n+p)|a_{n+p}| - 2\beta p + 2\alpha\beta p$$

$$\leq \sum_{n=1}^{\infty}(1+\beta)(n+p)|a_{n+p}| - 2\beta p + 2\alpha\beta p$$

$$\leq \sum_{n=1}^{\infty}(1+\beta)(n+p)|a_{n+p}| - 2\beta(1-\alpha)p$$

≤ 0

Hence by maximum modulus theorem, $f(z) \in \tau_p(\alpha,\beta)$.
Conversely, suppose that

$$\left|\frac{f'(z)-pz^{p-1}}{f'(z)+pz^{p-1}(1-2\alpha)}\right| = \left|\frac{pz^{p-1}-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}-pz^{p-1}}{pz^{p-1}-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}+pz^{p-1}(1-2\alpha)}\right|$$

$$= \left|\frac{-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}}{2z^{p-1}(1-\alpha)p-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}}\right|$$

since $|Re(z)| \le |z|$for all z , we have

$$Re\left[\frac{\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}}{2z^{p-1}(1-\alpha)p-\sum_{n=1}^{\infty} a_{n+p}\,(n+p)z^{n+p-1}}\right] < \beta \qquad (2.4)$$

Choose value of z on real axis so that $f'(z)$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1$ through real values, we have

$$\sum_{n=1}^{\infty}(n+p)|a_{n+p}| \le 2\beta(1-\alpha)p-\beta\sum_{n=1}^{\infty}(n+p)|a_{n+p}|$$

$$\sum_{n=1}^{\infty}(n+p)|a_{n+p}| + \beta\sum_{n=1}^{\infty}(n+p)|a_{n+p}| \le 2\beta(1-\alpha)p$$

$$\sum_{n=1}^{\infty}(n+p)(1+\beta)|a_{n+p}| \le 2\beta(1-\alpha)p$$

This completes the proof.

III. Distortion Theorem

Theorem 3.1 If $f(z) \in \tau_p(\alpha,\beta)$, then

$$r^p-\frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)}r^{p+1} \le |f(z)| \le r^p+\frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)}r^{p+1} \quad ,|z|=r \qquad (3.1)$$

and

$$pr^{p-1}-\frac{2\beta(1-\alpha)p}{(1+\beta)}r^p \le |f'(z)| \le pr^{p-1}+\frac{2\beta(1-\alpha)p}{(1+\beta)}r^p \quad ,|z|=r \qquad (3.2)$$

The result is sharp.

Proof: from Theorem 1, we have

$$\sum_{n=1}^{\infty}(n+p)(1+\beta)|a_{n+p}| \le 2\beta(1-\alpha)p$$

$$(1+p)(1+\beta)\sum_{n=1}^{\infty}|a_{n+p}| \le \sum_{n=1}^{\infty}(n+p)(1+\beta)|a_{n+p}| \le 2\beta(1-\alpha)p$$

This implies that

$$\sum_{n=1}^{\infty}|a_{n+p}| \le \frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)}$$

Hence

$$|f(z)| \le |z|^p+\sum_{n=1}^{\infty}|a_{n+p}|\,|z|^{n+p}$$

$$\le |r|^p+\sum_{n=1}^{\infty}|a_{n+p}|\,|r|^{n+p} \qquad \because\ |z|=r$$

$$\le r^p+\frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)}r^{p+1} \qquad (3.3)$$

and

$$|f(z)| \ge |z|^p-\sum_{n=1}^{\infty}|a_{n+p}|\,|z|^{n+p}$$

$$\geq |r|^p - \sum_{n=1}^{\infty} |a_{n+p}| \, |r|^{n+p} \qquad \because \ |z| = r$$

$$\geq \ r^p - \frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)} r^{p+1} \tag{3.4}$$

From (3.3) and (3.4) we get,

$$r^p - \frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)} r^{p+1} \leq \ |f(z)| \leq r^p + \frac{2\beta(1-\alpha)p}{(1+p)(1+\beta)} r^{p+1} \quad , |z| = r$$

Thus (3.1) holds.
Also

$$|f'(z)| \leq p|z|^{p-1} + \sum_{n=1}^{\infty} |a_{n+p}| (n+p) \, |z|^{n+p-1}$$

$$\leq r^{p-1} \left(p + r \sum_{n=1}^{\infty} |a_{n+p}| \, (n+p) \right) \qquad \because \ |z| = r$$

$$\leq r^{p-1} \left(p + r \frac{2\beta(1-\alpha)p}{(1+\beta)} \right)$$

$$\leq pr^{p-1} + \frac{2\beta(1-\alpha)p}{(1+\beta)} r^p \tag{3.5}$$

Also,

$$|f'(z)| \geq p|z|^{p-1} - \sum_{n=1}^{\infty} |a_{n+p}| (n+p) \, |z|^{n+p-1}$$

$$\geq r^{p-1} \left(p - r \sum_{n=1}^{\infty} |a_{n+p}| \, (n+p) \right) \qquad \because \ |z| = r$$

$$\geq r^{p-1} \left(p - r \frac{2\beta(1-\alpha)p}{(1+\beta)} \right)$$

$$\geq pr^{p-1} - \frac{2\beta(1-\alpha)p}{(1+\beta)} r^p \tag{3.6}$$

Thus from (3.5) and (3.6) we get,

$$pr^{p-1} - \frac{2\beta(1-\alpha)p}{(1+\beta)} r^p \leq \ |f'(z)| \leq \ pr^{p-1} + \frac{2\beta(1-\alpha)p}{(1+\beta)} r^p \quad , |z| = r$$

Thus (3.1) holds.
This completes the proof.

IV. Radius of convexity

Theorem 4.1 If $f(z) \in \tau_p(\alpha, \beta)$ is p-valently convex in the disc then

$$|z| \leq \left[\frac{(1+\beta)p}{2\beta(1-\alpha)(n+p)} \right]^{\frac{1}{n}} \quad , n = 1, 2, 3, \ldots \tag{4.1}$$

The result is sharp.
Proof: Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

Then,

$$f'(z) = pz^{p-1} - \sum_{n=1}^{\infty} a_{n+p} \ (n+p) z^{n+p-1}$$

$$f''(z) = p(p-1) z^{p-2} - \sum_{n=1}^{\infty} a_{n+p} \ (n+p)(n+p-1) z^{n+p-2}$$

Now,

$$1+\frac{zf''(z)}{f'(z)}$$

$$= 1+\frac{p(p-1)z^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)(n+p-1)z^{n+p-1}}{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}$$

$$= \frac{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}+p(p-1)z^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)(n+p-1)z^{n+p-1}}{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}$$

$$=\frac{p^2z^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)^2z^{n+p-1}}{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}$$

To prove the theorem it is sufficient to show,

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\le p$$

Now,

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|$$

$$=\left|\frac{p^2z^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)^2z^{n+p-1}}{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}-p\right|$$

$$=\left|\frac{p^2z^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)^2z^{n+p-1}-p^2z^{p-1}+p\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}\right|$$

$$=\left|\frac{-\sum_{n=1}^{\infty}a_{n+p}\,n\ (n+p)z^{n+p-1}}{pz^{p-1}-\sum_{n=1}^{\infty}a_{n+p}\ (n+p)z^{n+p-1}}\right|$$

$$\le\frac{\sum_{n=1}^{\infty}|a_{n+p}\ |n\ (n+p)|z|^n}{p-\sum_{n=1}^{\infty}|a_{n+p}|\ (n+p)|z|^n}$$

Thus

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\le p$$

if

$$\frac{\sum_{n=1}^{\infty}|a_{n+p\ |}n\ (n+p)|z|^n}{p-\sum_{n=1}^{\infty}|a_{n+p}|\ (n+p)|z|^n}\le p$$

$$\sum_{n=1}^{\infty}n\ (n+p)\left|a_{n+p}\right||z|^n\le\ p^2-\sum_{n=1}^{\infty}p\ (n+p)\left|a_{n+p}\right||z|^n$$

$$\sum_{n=1}^{\infty}n\ (n+p)+p\ (n+p)\left|a_{n+p}\right||z|^n\le\ p^2$$

$$\sum_{n=1}^{\infty}(n^2+2np+p^2)\left|a_{n+p}\right||\mathrm{z}|^{\mathrm{n}}\le\ \mathrm{p}^2$$

$$\sum_{\mathrm{n}=1}^{\infty}(\mathrm{n}+\mathrm{p})^2\left|\mathrm{a}_{\mathrm{n}+\mathrm{p}}\right||\mathrm{z}|^{\mathrm{n}}\le\ \mathrm{p}^2$$

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^2 |a_{n+p}||z|^n \leq 1$$

But from Theorem 1, we get

$$\sum_{n=1}^{\infty} \frac{(n+p)(1+\beta)|a_{n+p}|}{2\beta(1-\alpha)p} \leq 1$$

hence f(z) is p-valently convex if

$$\left(\frac{n+p}{p}\right)^2 |a_{n+p}||z|^n \leq \frac{(n+p)(1+\beta)|a_{n+p}|}{2\beta(1-\alpha)p}$$

$$\left(\frac{n+p}{p}\right)^2 |z|^n \leq \frac{(n+p)(1+\beta)}{2\beta(1-\alpha)p}$$

or

$$|z| \leq \left[\frac{(1+\beta)p}{2\beta(1-\alpha)(n+p)}\right]^{\frac{1}{n}} \quad , n = 1, 2, 3, \ldots$$

This completes the proof.

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