IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 6 Ver. III (Nov. - Dec. 2015), PP 32-38
www.iosrjournals.org

On Cubic Implicative Hyper BCK-Ideals of Hyper BCK-Algebras

[1]B. Satyanarayana, [2]A.A.A. Agboola1 and [3]U. Bindu Madhavi
[1,3]Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522510, India
[2]Department of Mathematics, Federal University of Agriculture,
Abeokuta, Nigeria

Abstract: *In this paper, we present the notions of cubic (weak) implicative hyper BCK-ideals of hyper BCK-algebras and then we present some results which characterize the above notions according to the level subsets. In addition, we obtain the relationship among these notions, cubic positive implicative hyper BCK-ideals of types-1, 2…8 and obtain some related results.* ***AMS (2010):*** *06F35.*
Key words: *Hyper BCK-algebras, cubic sets, cubic (weak) implicative hyper BCK-ideal.*

I. Introduction

BCI/BCK-algebras are generalizations of the concepts of set-theoretic difference and propositional calculi. These two classes of logical algebras were introduced by Imai and Iseki [11] in 1966. It is well known that the class of MV-algebras introduced by Chang in [7] is a proper subclass of the class of BCK- algebras which in turn is a proper subclass of the class of BCI-algebras. Since the introduction of BCI/BCK-algebras, a great deal of literature has been produced, for example see [8, 10, 12, 13, 14]. For the general development of BCI/BCK-algebras, the ideal theory plays an important role. Hence much research emphasis has been on the ideal theory of BCI/BCK-algebras, see [1, 2, 3, 9].

The hyper structure theory was introduced in 1934 by Marty [24] at the 8th Science Congress of Scandinavian Mathematicians. In the following years, several authors have worked on this subject notably in France, United States, Japan, Spain, Russia and Italy. Hyperstructures have many applications in several sectors of both pure and applied sciences. In [9], Jun et. al. applied hyperstructures to BCK-algebras and introduced the notion of hyper BCK-algebras which is a generalization of a BCK- algebra. After the introduction of the concept of fuzzy sets by Zadeh [28], several researchers have carried out the generalizations of fuzzy sets. The idea of cubic set was introduced by Jun et. al. [21], which contains interval valued membership function and non degree membership function. In [15], Jun et. al. introduced the cubic hyper BCK-ideals and obtained some related results. In [27], Satyanarayana and Bindu introduced the concept of cubic hyper BCK-ideals of hyper BCK-algebras. In the present paper, we introduce the concepts of cubic (weak) implicative hyper BCK-ideals of hyper BCK-algebras and we obtain some results which characterize the concept according to level subsets. Also, we obtain the relationship among these concepts, cubic (strong, weak, reflexive) hyper BCK-ideals, cubic positive implicative hyper BCK-ideals of types - 1,2,...,8 and some related properties are investigated.

II. Prelimianries

Let H be a non-empty set endowed with hyper operation that is, $\circ$ is a function from $H \times H$ to $P^*(H) = P(H)\backslash\{\phi\}$. For any two subsets A and B of H, $A \circ B$ is denoted by $\bigcup_{a\in A, b\in B} a \circ b$. We shall use the $x \circ y$ instead of $x \circ \{y\}, \{x\} \circ y$ or $\{x\} \circ \{y\}$.

Definition 2.1. By a hyper BCK-algebra, we mean a nonempty set H endowed with a hyper operation $\circ$ and a constant 0 satisfying the following axioms:
$(HK-1)(x \circ z) \circ (y \circ z) \ll x \circ y$,
$(HK-2)(x \circ y) \circ z = (x \circ z) \circ y$,
$(HK-3)\ x \circ H \ll \{H\}$,
$(HK-4)\ x \ll y$ and $y \ll x$ implies that $x = y$ for all $x, y, z \in H$.

We can define a relation $\ll$ on H by letting $x \ll y$ if and only if $0 \in x \circ y$ and for every $A, B \subseteq H, A \ll B$ is defined for all $a \in A$ there exists $b \in B$ such that $a \ll b$. In such case, we call the relation $\ll$ the hyper order in H. In the sequel, H will denote hyper BCK-algebra. It should be noted that the condition (HK-3) is equivalently to the condition:
(P1) $x \circ y \ll \{x\}$, for all $x, y \in H$.
In any hyper BCK-algebra H, the following hold:
(P2) $0 \circ 0 = \{0\}$,

(P3) $0 \ll x$,
(P4) $x \ll x$,
(P5) $A \subseteq B$ implies $A \ll B$,
(P6) $0 \circ x = \{0\}$,
(P7) $x \circ 0 = \{x\}$,
(P8) $0 \circ A = \{0\}$,
(P9) $x \in x \circ 0$, for all $x, y, z \in H$ and for any nonempty subsets A and B of H.

Definition 2.2. Let I be a nonempty subset of hyper BCK-algebra H and $0 \in I$. Then
(I-1) I is called a hyper BCK-subalgebra of H if $x \circ y \subseteq I$ for all $x, y \in H$.
(I-2) I is called a weak hyper BCK-ideal of H, if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
(I-3) I is called a hyper BCK-ideal of H, if $x \circ y \ll I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
(I-4) I is called a strong hyper BCK-ideal of H, if $x \circ y \cap I \neq \phi$ and $y \in I$ imply $x \in I$, for all $x, y \in H$.
(I-5) I is said to be reflexive if $x \circ x \subseteq I$, for all $x \in H$.
(I-6) I is said to be S-reflexive, if $(x \circ y) \cap I \neq \phi \Rightarrow x \circ y \ll I$ for all $x, y \in H$.
(I-7) I is said to be closed if $x \ll I$ and $y \in I$ implies that $x \in I$ for all $x, y \in H$.
It is easy to see that every S-reflexive subset of H is reflexive.

Definition 2.3. A hyper BCK-algebra H is called a positive implicative hyper BCK-algebra if for all $x, y, z \in H$, we have $(x \circ y) \circ z = (x \circ z) \circ (y \circ z)$.
If I is nonempty subset of H and $0 \in I$, then I is called a weak implicative hyper BCK-ideal of H if $(x \circ z) \circ (y \circ x) \subseteq I$and $z \in I$ imply that $x \in I$ for all $x, y, z \in H$.
I is said to be an implicative hyper BCK-ideal of H, if $(x \circ z) \circ (y \circ x) \ll I$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in H$.

Definition 2.4. [25] Let H be a hyper BCK-algebra. A nonempty subset I of H is called a strong implicative hyper BCK-ideal if it satisfies:
(i) $0 \in I$,
(ii) $\big((x \circ z) \circ (y \circ x)\big) \cap I \neq \emptyset$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in H$.

Example 1[25] Let $H = \{0,1,2\}$. Consider the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

It is clear that $(H,\circ)$ is a Hyper BCK-algebra. Putting $I = \{0,2\}$, it can seen that I is a strong implicative Hyper BCK-ideal of H.

A fuzzy set in set X is a function $\mu: X \to [0,1]$. Let μ and λ be the fuzzy sets of X. For $s, t \in [0,1]$ the set $U(\mu_A; s) = \{x \in X / \mu_A(x) \geq s\}$ is called upper s-level cut of μ and set $L(\lambda_A; t) = \{x \in X / \lambda_A(x \leq t)\}$ is called lower t-level cut of λ.

Let μ be a fuzzy subset of H and $\mu(0) \geq \mu(x)$, for all $x \in H$. Then μ is called:
(i) a fuzzy weak implicative hyper BCK-ideal of H if $\mu(x) \geq \min\{\inf_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z)\}$
(ii) a fuzzy implicative hyper BCK-ideal of H if $\mu(x) \geq \min\{\sup_{a \in (x \circ z) \circ (y \circ x)} \mu(a), \mu(z)\}$for all $x, y, z \in H$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. In [5] Biswas described a method to find max/sup and min/inf between two intervals or set of intervals. By the interval number D we mean an interval $[a^-,a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. For interval numbers $D_1 = [a_1^-,b_1^+]$, $D_2 = [a_2^-, b_2^+]$.

We define the following
$\min(D_1, D_2) = D_1 \cap D_2 = \min([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$.
$\max(D_1, D_2) = D_1 \cup D_2 = \max([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$.
$D_1 + D_2 = [a_1^- + a_2^- - a_1^-.a_2^-,\ b_1^+ + b_2^+ - b_1^+.b_2^+]$.
We put

$$D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^- \text{ and } b_1^+ \leq b_2^+$$
$$D_1 = D_2 \Leftrightarrow a_1^- = a_2^- \text{ and } b_1^+ = b_2^+$$
$$D_1 < D_2 \Leftrightarrow D_1 \leq D_2 \text{ and } D_1 \neq D_2$$
$$mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+], \text{ where } 0 \leq m \leq 1.$$

An interval valued fuzzy set A over X is an object having the form $A = \{(x, \tilde{\mu}_A) : x \in X\}$ where $\tilde{\mu}_A(x) : X \longrightarrow D[0,1]$ is the set of all subintervals of [0, 1]. The interval $\tilde{\mu}_A(x)$ denotes the intervals of the degree of membership of element x to the set A where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ for all $x \in X$.

Definition 2.5. [5] Consider two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$ then $rmin(D_1, D_2) = [\min(a_1^-, a_1^+), \min(a_2^-, a_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus if $D_i = [a_i^-, a_i^+] \in D[0,1]$ for i = 1,2,3 ... then we define $r\sup_i(D_i) = [\sup_i(a_i^-), \sup_i(a_i^+)]$ that is $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. Now we call $D_1 \geq D_2$ if and only if $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.
Based on interval valued fuzzy sets, Jun et.al.[21] introduced the notion of (internal, external) cubic sets and investigated several properties.

Definition 2.6. [27] Let X be a nonempty set. A cubic set A in X is a Structure $A = \{(x, \tilde{\mu}_A(x), \lambda_A(x))\,; x \in X\}$ which is briefly denoted by $A = (\tilde{\mu}_A, \lambda_A)$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval valued fuzzy set in X and λ_A is fuzzy set in X.

Definition 2.7. [27] A cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in H is called a cubic hyper BCK-ideal of H if it satisfies the following conditions:
(K1) $x \ll y$ implies that $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$,
(K2) $\tilde{\mu}_A(x) \geq rmin\{\inf_{a \in x \circ y} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\}$,
(K3) $\lambda_A(x) \leq \max\{\sup_{b \in x \circ y} \lambda_A(b), \lambda_A(y)\}$, for all $x, y \in H$.

Definition 2.7. [27] A cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in H is called a cubic strong hyper BCK-ideal of H if it satisfies the following conditions:
(SH1) $\inf_{a \in x \circ x} \tilde{\mu}_A(a) \geq rmin\{\sup_{b \in x \circ y} \tilde{\mu}_A(b), \tilde{\mu}_A(y)\}$ and
(SH2) $\sup_{c \in x \circ x} \lambda_A(c) \leq \max\{\inf_{d \in x \circ y} \lambda_A(d), \lambda_A(y)\}$, for all $x, y \in H$.

Definition 2.9. [27] A cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in H is called a cubic s-weak hyper BCK-ideal of H if it satisfies the following conditions:
(S1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(y)$ and $\lambda_A(0) \leq \lambda_A(y)$, for all $x, y \in H$.
(S2) for every $x, y \in H$ there exists $a, b \in x \circ y$, $\tilde{\mu}_A(x) \geq rmin\{\tilde{\mu}_A(a), \tilde{\mu}_A(y)\}$ and
$\lambda_A(x) \leq \max\{\lambda_A(b), \lambda_A(y)\}$.

Definition 2.10 [27] A cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in H is called a cubic weak hyper BCK-ideal of H if it satisfies the following conditions:
(W1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \geq rmin\{\inf_{a \in x \circ y} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\}$ and
(W2) $\lambda_A(0) \leq \lambda_A(x) \leq \max\{\sup_{b \in x \circ y} \lambda_A(b), \lambda_A(y)\}$, for all $x, y \in H$.

Definition 2.11. [27] A cubic set $A = (\tilde{\mu}_A, \lambda_A)$ in H is said to satisfy the "sup-inf" property if for any subset T of H there exist $x_0, y_0 \in T$ such that

$$\tilde{\mu}_A(x_0) = \sup_{x \in T} \tilde{\mu}_A(x) \text{ and } \lambda_A(y_0) = \inf_{y \in T} \lambda_A(y).$$

Theorem 2.12. [27] Every cubic strong hyper BCK-ideal is a cubic hyper BCK-ideal.

III. Cubic Implicative Hyper Bck-Ideals Of Hyper Bck-Algebra

Definition 3.1. Let $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic subset of H and $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x), \lambda_A(0) \leq \lambda_A(x)$ for all $x, y \in H$. Then $A = (\tilde{\mu}_A, \lambda_A)$ is called a cubic positive implicative hyper BCK-ideal of :

(i) Type 1, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \text{rmin}\left\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\right\}$$

$$\lambda_A(t) \leq \max\{\sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d)\}$$

(ii) Type 2, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \text{rmin}\left\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\right\}$$

$$\lambda_A(t) \leq \max\{\inf_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d)\}$$

(iii) Type 3, if for all $t \in x \circ z$,

$$\tilde{\mu}_A(t) \geq \text{rmin}\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\}$$

$$\lambda_A(t) \leq \max\{\inf_{c \in (x \circ y) \circ z} \lambda_A(c), \inf_{d \in y \circ z} \lambda_A(d)\}$$

(iv) Type 4, if for all $t \in x \circ z$,

$\tilde{\mu}_A(t) \geq \text{rmin}\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\}$,

$\lambda_A(t) \leq \max\{\sup_{c \in (x \circ y) \circ z} \lambda_A(c), \inf_{d \in y \circ z} \lambda_A(d)\}$ for all $x, y, z \in H$.

Definition 3.2 Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic subset of H. Then $A = (X, \tilde{\mu}_A, \lambda_A)$ is called a cubic positive implicative hyper BCK-ideal of:

(i) Type 5, if there exists $t \in x \circ z$ such that
$\tilde{\mu}_A(t) \geq \text{rmin}\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\}$ $\lambda_A(t) \leq \max\{\sup_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d)\}$.

(ii) Type 6, if there exists $t \in x \circ z$ such that
$\tilde{\mu}_A(t) \geq \text{rmin}\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\}$,
$\lambda_A(t) \leq \max\{\inf_{c \in (x \circ y) \circ z} \lambda_A(c), \inf_{d \in y \circ z} \lambda_A(d)\}$.

(iii) Type 7, if there exists $t \in x \circ z$ such that $\tilde{\mu}_A(t) \geq \text{rmin}\{\inf_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \sup_{b \in y \circ z} \tilde{\mu}_A(b)\}$,
$\lambda_A(t) \leq \max\{\sup_{c \in (x \circ y) \circ z} \lambda_A(c), \inf_{d \in y \circ z} \lambda_A(d)\}$.

(iv) Type 8, if there exists $t \in x \circ z$ such that
$\tilde{\mu}_A(t) \geq \text{rmin}\{\sup_{a \in (x \circ y) \circ z} \tilde{\mu}_A(a), \inf_{b \in y \circ z} \tilde{\mu}_A(b)\}$,
$\lambda_A(t) \leq \max\{\inf_{c \in (x \circ y) \circ z} \lambda_A(c), \sup_{d \in y \circ z} \lambda_A(d)\}$ for all $x, y, z \in H$.

Definition 3.3. Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic subset of H and $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, $\lambda_A(0) \leq \lambda_A(x)$ for all $x, y, z \in H$. Then $A = (\tilde{\mu}_A, \lambda_A)$ is called:

(i) a cubic weak implicative hyper BCK-ideal of H, if $\tilde{\mu}_A(x) \geq \text{rmin}\{\inf_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(z)\}$,
$\lambda_A(x) \leq \max\{\sup_{b \in (x \circ z) \circ (y \circ x)} \lambda_A(b), \lambda_A(z)\}$.

(ii) a cubic implicative hyper BCK-ideal of H, if
$\tilde{\mu}_A(x) \geq \text{rmin}\{\inf_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(z)\}$,
$\lambda_A(x) \leq \max\{\inf_{b \in (x \circ z) \circ (y \circ x)} \lambda_A(b), \lambda_A(z)\}$ for all $x, y, z \in H$.

Theorem 3.4 Every cubic implicative hyper BCK-ideal of H is a cubic weak implicative hyper BCK-ideal.

Proof: Suppose $A = (\tilde{\mu}_A, \lambda_A)$is a cubic implicative hyper BCK-ideal of H. Then for all $x, y, z \in H$. We have $\inf_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a) \leq \sup_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a)$ and also, $\inf_{b \in (x \circ z) \circ (y \circ x)} \lambda_A(b) \leq \sup_{b \in (x \circ z) \circ (y \circ x)} \lambda_A(b)$. Hence,

$$\begin{aligned}\tilde{\mu}_A(x) &\geq \text{rmin}\{\sup_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(z)\} \\ &\geq \text{rmin}\{\inf_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(z)\},\end{aligned}$$

$$\begin{aligned}\lambda_A(x) &\leq \max\left\{\inf_{b \in (x \circ z) \circ (y \circ x)} \lambda_A(b), \lambda_A(z)\right\} \\ &\leq max\{sup_{b \in (x \circ z) \circ (y \circ x)} \lambda_A(b), \lambda_A(z)\}.\end{aligned}$$

Accordingly, $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic weak implicative hyper BCK-ideal of H.

Example 2. Let$H = \{0, a, b\}$. Consider the following cayley table

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

It follows from [4] that $(H,\circ)$ is a hyper BCK-algebra.
Define a cubic set $A = (\tilde{\mu}_A, \lambda_A)$ on H by $\tilde{\mu}_A(0) = [1, 0.8]$, $\tilde{\mu}_A(a) = [0, 0.2], \tilde{\mu}_A(b) = [0.6, 0.7]$ and $\lambda_A(0) = 0, \lambda_A(a) = 1, \lambda_A(b) = 0.4$. Then $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic weak implicative hyper BCK-ideal of H but it is not a cubic implicative hyper BCK-ideal of H, since
$\tilde{\mu}_A(a) = [0,0.2] < [1,0.8] = \tilde{\mu}_A(0) = rmin\{sup_{t\in(a\circ 0)\circ(a\circ a)} \tilde{\mu}_A(t), \tilde{\mu}_A(0)\}$,
$\lambda_A(a) = 1 > 0 = \lambda_A(0) = max\{inf_{t\in(a\circ 0)\circ(a\circ a)} \lambda_A(t), \lambda_A(0)\}$.

Theorem 3.5.
(i) Every cubic implicative hyper BCK-ideal of H is a cubic strong hyper BCK-ideal.
(ii) Every cubic weak implicative hyper BCK-ideal of H is a cubic weak hyper BCK-ideal.

Proof. (i) Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic hyper implicative hyper BCK-ideal of H. Putting $y = 0$ and $z = y$ in Definition 2.8(SH2), we obtain

$$\begin{aligned}\tilde{\mu}_A(x) &\geq rmin\left\{\sup_{a\in(x\circ y)\circ(0\circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\}\\ &= rmin\{sup_{a\in x\circ y}\, \tilde{\mu}_A(a), \tilde{\mu}_A(y)\} \text{ and}\\ \lambda_A(x) &\leq max\left\{\inf_{b\in(x\circ y)\circ(0\circ x)} \lambda_A(b), \lambda_A(y)\right\}\\ &= max\{inf_{b\in(x\circ y)}\, \lambda_A(b), \lambda_A(y)\} \qquad (1)\end{aligned}$$

First we show that for $x, y \in H, if\ x \ll y$ implies that $\tilde{\mu}_A(x) \geq \tilde{\mu}_A(y)$ and $\lambda_A(x) \leq \lambda_A(y)$.
For this, let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$ and from (1), we get

$$\begin{aligned}\tilde{\mu}_A(x) &\geq rmin\left\{\sup_{a\in x\circ y} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\}\\ &= rmin\{\tilde{\mu}_A(0), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(y) \text{ and}\\ \lambda_A(x) &\leq max\left\{\inf_{b\in(x\circ y)} \lambda_A(b), \lambda_A(y)\right\}\\ &= max\{\lambda_A(0), \lambda_A(y)\} = \lambda_A(y) \qquad (2).\end{aligned}$$

Let $x \in H$ and $a \in x \circ x$. Since $x \circ x \ll x$, then $a \ll x$ for all $a \in x \circ x$ and so by (2), we have $\tilde{\mu}_A(a) \geq \tilde{\mu}_A(x)$ and $\lambda_A(a) \leq \lambda_A(x)$ for all $a \in x \circ x$. Hence

$$inf_{a\in x\circ x}\, \tilde{\mu}_A(a) \geq \tilde{\mu}_A(x) \text{ and } sup_{a\in x\circ x}\, \lambda_A(a) \leq \lambda_A(x) \qquad (3)$$

Combining (1) and (3), we obtain

$$inf_{a\in x\circ x}\, \tilde{\mu}_A(a) \geq \tilde{\mu}_A(x) \geq rmin\{sup_{b\in x\circ y}\, \tilde{\mu}_A(b), \tilde{\mu}_A(y)\} \text{ and}$$
$$sup_{c\in x\circ x}\, \lambda_A(c) \leq \lambda_A(x) \leq max\{inf_{d\in(x\circ y)}\, \lambda_A(d), \lambda_A(y)\} \text{ for all } x, y \in H.$$

Thus $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic strong hyper BCK-ideal of H.
(ii) Let $A = (\tilde{\mu}_A, \lambda_A)$ be a cubic weak implicative hyper BCK-ideal of H. Putting $y = 0$ and $z = y$ in Definition 2.8(SH1), we obtain

$$\begin{aligned}\tilde{\mu}_A(x) &\geq rmin\left\{\inf_{a\in(x\circ y)\circ(0\circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\right\}\\ &= rmin\{inf_{a\in x\circ y}\, \tilde{\mu}_A(a), \tilde{\mu}_A(y)\} \text{ and}\\ \lambda_A(x) &\leq max\left\{\sup_{b\in(x\circ y)\circ(0\circ x)} \lambda_A(b), \lambda_A(y)\right\}\\ &= max\{sup_{b\in(x\circ y)}\, \lambda_A(b), \lambda_A(y)\}.\end{aligned}$$

Therefore,

$$\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \geq rmin\{inf_{a\in x\circ y}\tilde{\mu}_A(a), \tilde{\mu}_A(y)\} \text{ and}$$
$$\lambda_A(0) \leq \lambda_A(x) \leq max\{sup_{b\in(x\circ y)}\lambda_A(b), \lambda_A(y)\} \text{ for all } x, y \in H.$$

Thus $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic weak implicative hyper BCK-ideal of H.

Theorem 3.6. Let $A = (\tilde{\mu}_A, \lambda_A)$ be cubic subset of H, then the following statements hold:
(i) If A is a cubic weak implicative hyper BCK-ideal of H then for all $\tilde{s}, t \in [0,1]$,
$U(\tilde{\mu}_A; \tilde{s}) \neq \phi \neq L(\lambda_A; t)$ are weak implicative hyper BCK-ideals of H.
(ii) If A is a cubic implicative hyper BCK-ideal of H, then for all $\tilde{s}, t \in [0,1]$,
$U(\tilde{\mu}_A; \tilde{s}) \neq \phi \neq L(\lambda_A; t)$ are implicative hyper BCK-ideals of H.

Proof. (i) Suppose that $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic weak implicative hyper BCK-ideal of H.
Let $\tilde{s}$, $t \in [0,1]$ and $x, y, z \in H$ be such that $(x \circ z) \circ (y \circ x) \subseteq U(\tilde{\mu}_A; \tilde{s})$ and $z \in U(\tilde{\mu}_A; \tilde{s})$.
Then $a \in U(\tilde{\mu}_A; \tilde{s})$ for all $a \in (x \circ z) \circ (y \circ x)$ which implies that, $\tilde{\mu}_A(a) \geq \tilde{s}$ and $\tilde{\mu}_A(z) \geq \tilde{s}$ implies that $inf_{a\in(x\circ z)\circ(y\circ x)}\tilde{\mu}_A(a) \geq \tilde{s}$ and $\tilde{\mu}_A(z) \geq \tilde{s}$. Thus by hypothesis,
$\tilde{\mu}_A(x) \geq rmin\{inf_{a\in(x\circ z)\circ(y\circ x)}\tilde{\mu}_A(a), \tilde{\mu}_A(z)\} \geq rmin\{\tilde{s}, \tilde{s}\} = \tilde{s}$ and thus
$x \in U(\tilde{\mu}_A; \tilde{s})$. Now let $x, y, z \in H$ be such that $(x \circ z) \circ (y \circ x) \subseteq L(\lambda_A; t)$ and $z \in L(\lambda_A; t)$. Then $b \in L(\lambda_A; t)$, and $b \in (x \circ z) \circ (y \circ x)$ implies $\lambda_A(b) \leq t$ and $\lambda_A(z) \leq t$ implies that $sup_{b\in(x\circ z)\circ(y\circ x)}\lambda_A(b) \leq t$ and $\lambda_A(z) \leq t$. Thus by hypothesis, $\lambda_A(x) \leq max\{sup_{b\in(x\circ z)\circ(y\circ x)}\lambda_A(b), \lambda_A(z)\} \leq max\{t, t\} = t$ which implies that $x \in L(\lambda_A; t)$.

Hence, $U(\tilde{\mu}_A; \tilde{s})$ and $L(\lambda_A; t)$ are weak implicative hyper BCK-ideals of H.
(ii) Suppose $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic implicative hyper BCK-ideal of H. Then by theorem 3.6(i), $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic strong hyper BCK-ideal of H and so it is a cubic hyper BCK-ideal of H. For all $\tilde{s}, t \in [0,1]$, $U(\tilde{\mu}_A; \tilde{s}) \neq \phi \neq L(\lambda_A; t)$ are hyper BCK-ideals of H. By theorem 4.6(ii) [25], it is enough to show that for $x, y, z \in H$, $x \circ (y \circ x) \ll U(\tilde{\mu}_A; \tilde{s})$ and $x \circ (y \circ x) \ll L(\lambda_A; t)$, $x \in U(\tilde{\mu}_A; \tilde{s}) \cap L(\lambda_A; t)$. To this end, let $x \circ (y \circ x) \ll U(\tilde{\mu}_A; \tilde{s})forx, y \in H$. Then for all $a \in x \circ (y \circ x)$ there exists $b \in U(\tilde{\mu}_A; \tilde{s})$ such that $a \ll b$and we have $\tilde{\mu}_A(a) \geq \tilde{\mu}_A(b) \geq \tilde{s}$ which implies that $\tilde{\mu}_A(a) \geq \tilde{s}$ for all $a \in x \circ (y \circ x)$ which implies that $sup_{a\in x\circ(y\circ x)}\tilde{\mu}_A(a) \geq \tilde{s}$.

Hence by hypothesis, $\tilde{\mu}_A(x) \geq rmin\{sup_{a\in(x\circ 0)\circ(y\circ x)}\tilde{\mu}_A(a), \tilde{\mu}_A(0)\} = sup_{a\in x\circ(y\circ x)}\tilde{\mu}_A(a) \geq \tilde{s}$.
That is $x \in U(\tilde{\mu}_A; \tilde{s})$. Let $x \circ (y \circ x) \ll L(\lambda_A; t)$for $x, y \in H$. Then for all $c \in x \circ (y \circ x)$ there exists $d \in L(\lambda_A; t)$ such that $c \ll d$. Hence we have $\lambda_A(c) \leq \lambda_A(d) \leq t$ which implies that $\lambda_A(c) \leq t$ for all $c \in x \circ (y \circ x)$ which implies that $inf_{c\in x\circ(y\circ x)}\lambda_A(c) \leq t$. Hence by hypothesis, $\lambda_A(x) \leq max\{inf_{c\in(x\circ 0)\circ(y\circ x)}\lambda_A(c), \lambda_A(0)\} = inf_{c\in x\circ(y\circ x)}\lambda_A(c) \leq t$. That is $x \in L(\lambda_A; t)$. Hence $U(\tilde{\mu}_A; \tilde{s})$ and $L(\lambda_A; t)$ are implicative hyper BCK-ideals of H.

Theorem 3.7. Let H be a positive implicative hyper BCK-algebra. If $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic weak implicative hyper BCK-ideal of H, then A is a cubic positive implicative hyper BCK-ideal of type1.

References

[1]. Agboola A.A.A. and Davvaz B., Introduction to Neutrosophic BCI/BCK-algebras,Int. J. Math. Math. Sci. Article ID 370267, (2015).
[2]. Agboola A.A.A. and Davvaz B., On Neutrosophic ideals of Neutrosophic BCIalgebras,Critical Review, 10 (2015), 101-111.
[3]. Aslam M. and Thaheem A.B., On ideals of BCK-algebras, Demonstratio Math.,27 (1994), No 3-4, 635-640.
[4]. Bakhshi, M. Zahedi and Borzooei, Fuzzy (Positive, Weak) hyper BCK-ideals,Iranian Journal of Fuzzy Systems, 1(2)(2004), 63-79.
[5]. Biswas R., Rosenfeld's fuzzy subgroup with interval valued membership function,Fuzzy Sets and Systems, 63 (1994), 87-90.
[6]. Borzooei R.A. and Bakshi, Some results on hyper BCK-algebras, Quasi groups andrelated systems, 11 (2004), 9-24.
[7]. Chang C.C., Algebraic analysis of many valued logics, Transactions of the AmericanMathematical society, 88(2) (1958), 467-490.
[8]. Huang Y.S., BCI-algebra, Science Press, Beijing, China, (2006).
[9]. Hoo C.S., Closed ideals and p-semisimple BCI-algebras, Math. Japon., 23 (1990),1103-1112.
[10]. Hoo C.S., A survey of BCK and BCI-algebras, Southeast Asian Bull. Math., 12(1988), 1-9.
[11]. Imai Y, and Is´eki K., On axiom systems of propositional calculi, XIV, Proc. JapanAcademy, 42 (1966), 19-22.
[12]. Iseki K and Tanaka S., An introduction to the theory of BCK-algebras, Math.Japonica, 23(1) (1978), 1-26.
[13]. Iseki K., An algebra related with a propositional calculus, Japan Acad., 42 (1966)26-29.
[14]. Iseki K., On BCI-algebras, Math. Sem. Notes, 8 (1980), 125-130.
[15]. Jun Y.B. and Shim W.H., Fuzzy implicative hyper BCK-ideals of hyper BCKalgebras,Int. J. Math. and Math. Sci., 29(2) (2002), 63-70.
[16]. Jun Y.B. and Xin X.L., Scalar elements and hyper atoms of hyper BCK-algebras,Scientiae Mathematicae Japonica, 2(3) (1999), 303-309.

[17]. Jun Y.B. and Xin X.L., Fuzzy hyper BCK-ideals of hyper BCK-algebras, Scientiae Mathematicae Japonica, 53(2) (2001), 353-360.
[18]. Jun Y.B. and Xin X.L., Roh E.H. and Zahedi M.M., Strong hyper BCK-idealsof hyper of hyper BCK-algebras, Scientiae Mathematicae Japonica, 51(3) (2000),493-498.
[19]. Jun Y.B., Zahedi M.M., Xin X.L. and Borzooei R.A., On hyper BCK-algebras,Italian J. of Pure and Applied Math. 8 (2000), 127-136.
[20]. Jun Y.B. and Lee K.J., Closed cubic ideals and cubic o-subalgebras in BCK/BCIalgebras,Applied Mathematical Sciences, 4(68) (2010), 3395-3402.
[21]. Jun Y.B., Kim C.S. and Yang K.O., Cubic sets, Ann. Fuzzy Math. Inform., 4(1)(2012), 83-98.
[22]. Jun Y.B., Jung S.T. and Kim M.S., Cubic subgroups, Ann. Fuzzy Math. Inform.,2(1) (2011), 600-615.
[23]. Jun Y.B., Jung S.T. and Kim M.S., Cubic subalgebras and ideals of BCK/BCIalgebras,Far East J. Math. Sc., 44(2) (2010), 239-250.
[24]. Marty F., Sur une generalization de la notion de groupe, 8th Congress Math. Scandinaves, Stockholm, 1934, 45-49.
[25]. Roh E.H., Zhang Q. and Jun Y.B., Some results in hyper BCK-algebras, Scientiae Mathematicae Japonica, 5 (2001), 265-272.
[26]. Satyanarayana B., Krishna L. and Durga Prasad R., On intuitionistic fuzzy implicative hyper BCK-ideals of hyper BCK-algebras, Int. J. Math. and Stat. In., 2(2) (2014), 77-85.
[27]. Satyanarayana B. and Bindu Madhavi U., On cubic hyper BCK-ideals of hyperBCK-algebras, (submitted for publication).
[28]. Zadeh L.A., Fuzzy sets, Information and Control, 8 (1965), 338-353.