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Existence, Uniqueness and Stability Solution of Differential Equations with Boundary Conditions

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Abstract: *In this work, we investigate the existence ,uniqueness and stability solution of non-linear differential equations with boundary conditions by using both method Picard approximation and Banach fixed point theorem which were introduced by [6] .These investigations lead us to improving and extending the above method. Also we expand the results obtained by [1] to change the non-linear differential equations with initial condition to non-linear differential equations with boundary conditions.*
Keywords: *Picard approximation method, Banach fixed point theorem, existence, uniqueness, boundary conditions.*

I. Introduction

Many results about the existence , uniqueness and stability solution of non–linear differential equations have been obtained by Picard approximation method and Banach fixed point theorem that were proposed by [6] which had been later applied in many studies [2, 5 ,7,8,9].

Definition1. Let $\{f_m(t)\}_{m=0}^{\infty}$ be a sequence of functions defined on a set. $E \subseteq R^1$ We say that $\{f_m(t)\}_{m=0}^{\infty}$ converges uniformly to the limit function f on E if given $\varepsilon > 0$ there exists a positive integer N such that :-

$$|f_m(t) - f(t)| < \varepsilon, \quad (m \geq N, t \in E).$$

Theorem1. If f is continuous on $[a,b]$ and if $F(x) = \int_a^x f(t)dt$, $a \leq x \leq b$, then $F(x)$ is also continuous on $[a,b]$.

Definition 2. Let f be a continuous function defined on a domain $G = \{(t,x): a \leq t \leq b, c \leq x \leq d$. Then f is said to satisfy a Lipschitz condition in the variable x on G, provided that a constant $L>0$ exists with ythe property that$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|$,
for all $(t,x_1),(t,x_2) \in G$. The constant L is called a Lipschitz constant for f .

Definition 3. A solution $x(t)$ is said to be stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\bar{x}(t)$ which satisfies $\|\bar{x}(t_0) - x(t_0)\| < \delta$ for some t_0 , also satisfies$\|\bar{x}(t) - x(t)\| < \varepsilon$ for all $t \geq t_0$.

Definition 4. Let E be a vector space a real-valued function $\|.\|$ of E into R^1 called a norm if satisfies

I. $\|x\| \geq 0$ for all $x \in E$,
II. $\|x\| = 0$ if and only if $x = 0$,
III. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$,
IV. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in E$ and $\alpha \in R$.

Definition 5. A linear space E with a norm defined on it is called a normed space.

Definition 6. A normed linear space E is called complete if every Cauchy sequence in E converges to an element in E .
Definition 7. A complete normed linear space is a Banach space .

Definition 8. if T maps E into itself and z is a point of E such that $Tz = z$, then z is a fixed point of T .
Definition 9. Let (C [0, T], $\|.\|$) be a norm space if T maps into itself we say that T is a contraction mapping on C [0,T] if there exists $\alpha \in R$ with $0 < \alpha < 1$ such that
$\|Tx - Ty\| \leq \alpha\|x - y\|, (x, y) \in$ C [0, T].
Theorem 2. Let E be a Banach space , if T is a contraction mapping on E then T has one and only one fixed point in E .
(For the definitions and theorems see [6]).
Butris [1] used Picard approximation method for studying the existence and uniqueness solution of the following differential equations

$$\frac{dx}{dt} = f\left(t, x, \frac{dx}{dt}\right)$$

with boundary conditions
$x(0) + x(T) = d.$
where $x \in D \subseteq R^n$, D is a closed and bounded domain, dϵR^1 .
In this paper, we study the existence, uniqueness and stability solution of non-linear differential equations with boundary conditions which has the form:-

$$\left.\begin{aligned} &\frac{dx}{dt} = Ax + f(t, x, y) \\ &x(0) = x_o, x(T) = x_T \\ &\frac{dy}{dt} = By + g(t, x, y) \\ &y(0) = y_0, y(T) = y_T \end{aligned}\right\} \quad \cdots \text{(P)}$$

where
$x_T = x_0 e^{\mathrm{AT}} + \int_0^T e^{A(T-s)} [f(s, x(s, x_0, y_0), y(s, x_0, y_0))] ds$
and
$y_T = y_0 e^{\mathrm{AT}} + \int_0^T e^{A(T-s)} [g(s, x(s, x_0, y_0), y(s, x_0, y_0))] ds$,$0 \leq s \leq t \leq T < \infty$
and $x \in D_1 \subseteq R^n, y \in D_2 \subseteq R^m$, D_1 and D_2 are a compact domains.
The vector functions $f(t, x, y)$, $g(t, x, y)$ are defined and continuous on the domain
$G_{1,2} = \{(t, x, y); t \in R^1, x \in D_1, y \in D_2\}$ . ... (1)
Also $A = \left(A_{ij}\right)$ and $B = (B_{ij})$ are $n \times n$ non-negative matrices.
Suppose that the vector functions $f(t, x, y)$ and $g(t, x, y)$ satisfy the following inequalities

$$\left.\begin{aligned} &\|f(t, x, y)\| \leq M_1, \ \|g(t, x, y)\| \leq M_2, \\ &\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| \\ &\|g(t, x_1, y_1) - g(t, x_2, y_2)\| \leq L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| , \end{aligned}\right\} \quad \ldots (2)$$

for all $t \in R^1$, $x, x_1, x_2 \in D_1$, $y, y_1, y_2 \in D_2$, where M_1, M_2, K_1, K_2 and L_1, L_2 are positive constants, provided that

$$\left\|e^{A(t-s)}\right\| \le \alpha, \quad \left\|e^{B(t-s)}\right\| \le \beta \qquad \cdots (3)$$

where α *and* β are positive conistants, $\|.\| = \max_{t\in[0,T]}|.|$.

We define non-empty sets as follows:-

$$\left.\begin{aligned} D_{1_\alpha} &= D_1 - (T\alpha M_1 + h_1) \\ D_{2_\beta} &= D_2 - (T\beta M_2 + h_1) \end{aligned}\right\} \qquad \dots(4)$$

where

$h_1 = (\|x_0\|(\|E\| + \|e^{AT}\|)$ and $h_2 = (\|y_0\|(\|E\| + \|e^{BT}\|)$.

Furthermore, we suppose that the largest Eigen- value of the matrix

$\Omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}$ does not exceed unity, i.e

$$\frac{1}{2}\left[(\omega_1 + \omega_4) + \sqrt{(\omega_1 + \omega_4) + 4(\omega_1 w_4 - \omega_2\omega_3)}\ \right] \le 1. \qquad \dots(5)$$

where $\omega_1 = K_1\alpha T$, $\omega_2 = K_2\alpha T, \omega_3 = L_1\beta T$ *and* $\omega_4 = L_2\beta T$.

Define a sequence of functions

$\{x_m(t, x_0, y_0), y_m(t, x_0, y_0)\}_{m=0}^{\infty}$ by the following

$$x_{m+1}(t, x_0, y_0) = x_0 e^{At} + \int_0^t e^{A(t-s)}\, [f(s, x_m(s, x_0, y_0), y_m(s, x_0, y_0))ds \qquad \cdots (6)$$

with

$x_0(0, x_0, y_0) = x_0$

and

$$y_{m+1}(t, x_0, y_0) = y_o e^{Bt} + \int_0^t e^{B(t-s)}\, [g(s, x_m(s, x_0, y_0), y_m(s, x_0, y_0))ds \qquad \cdots (7)$$

with

$y_0(0, x_0, y_0) = y_o$

II. Existences Solution Of (P).

The investigation of the existences solution of the problem (P) will be introduced by the following theorem.

Theorem 3. Let the vector functions $f(t, x, y)$ *and* $g(t, x, y)$ are satisfying the inequalities (2), (3) and the conditions (4), (5).Then there exist a sequences of functions (6) and (7) converges uniformly on the domain

$$G_{\alpha\beta} = (t, x_0, y_0) \in [0, T] \times D_{1_\alpha} \times D_{2_\beta} \qquad \cdots (8)$$

to the limit vector function $\begin{pmatrix} x^0(t, x_0, y_0) \\ y^0(t, x_0, y_0) \end{pmatrix}$ which is a continuous on the domain (1.1) and satisfies the following integral equations:-

$$\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix} = \begin{pmatrix} x_0 e^{At} + \int_0^t e^{A(t-s)} \left[f\left(s, x(s,x_0,y_0), y(s,x_0,y_0)\right) \right] ds \\ y_o e^{Bt} + \int_0^t e^{B(t-s)} \left[g(s, x(s,x_0,y_0), y(s,x_0,y_0)) \right] ds \end{pmatrix} \qquad \dots (9)$$

and it's exist solution of the problem (P).

Provided that

$$\begin{pmatrix} \|x^0(t,x_0,y_0) - x_0\| \\ \|y^0(t,x_0,y_0) - y_0\| \end{pmatrix} \le \begin{pmatrix} T\alpha M_1 + h_1 \\ T\beta M_2 + h_2 \end{pmatrix} \qquad \dots (10)$$

and

$$\begin{pmatrix} \| x_m(s,x_0,y_0) - x^0(t,x_0,y_0) \| \\ \| y_m(s,x_0,y_0) - y^0(t,x_0,y_0) \| \end{pmatrix} \le \Omega^m (\text{E-}\Omega)^{-1} \Psi_0 \qquad \dots (11)$$

for all t ∈ [0,T] and $x_0 \in D_{1_\alpha}$, $y_0 \in D_{2_\beta}$,m=1,2,… ,

where $\Psi_0 = \begin{pmatrix} T\alpha M_1 + h_1 \\ T\beta M_2 + h_2 \end{pmatrix}$.

Proof. Setting m=0 in (1.6), we have

$$\|x_1(t,x_0,y_0) - x_0\| \le \left\| \int_0^t e^{A(t-s)} f(s,x_0,y_0)\ ds \right\|$$
$$\le T\alpha M_1 + h_1$$

Hence $x_1(t,x_0,y_0) \in D_{1_\alpha}$ for all t ∈ [0,T]

Then by mathematical induction we can prove that

$$\|x_m(t,x_0,y_0) - x_0\| \le T\alpha M_1 + h_1 \qquad \cdots (12)$$

That is $x_m(t,x_0,y_0) \in D_{1_\alpha}$ for all t ∈ [0,T] .

Similarly, from the sequence of functions (7), when m=0, we get

$$\|y_1(t,x_0,y_0) - y_0\| \le T\beta M_2 + h_2$$

Hence $y_1(t,x_0,y_0) \in D_{2_\beta}$ for all t ∈ [0,T]

and by mathematical induction also we can obtain that

$$\|y_m(t,x_0,y_0) - y_0\| \le T\alpha M_2 + h_2 \qquad \cdots (13)$$

then $y_m(t,x_0,y_0) \in D_{2_\beta}$ for all t ∈ [0,T] .

Next, we shall prove that the sequence of functions (6) and (7) converges uniformly on the domain (1).

Putting m=1 in (6) and by the inequalities (2), (3), we get

$$\|x_2(t,x_0,y_0) - x_1(t,x_0,y_0)\|$$
$$\le \left(\|E\| - \frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \int_0^t \left\|e^{A(t-s)}\right\| \|f(s, x_1(s,x_0,y_0),$$
$$, y_1(s,x_0,y_0)) - f(s,x_0,y_0)\| ds +$$
$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|} \right) \int_t^T \left\|e^{A(t-s)}\right\| \|f(s, x_1(s,x_0,y_0), y_1(s,x_0,y_0)),$$
$$- f(s,x_0,y_0)\|\ ds$$

$$\le t\alpha (K_1 \|x_1(t,x_0,y_0) - x_0\| + K_2 \|y_1(t,x_0,y_0) - y_0\|).$$

Then by mathematical induction we can prove that

$$\|x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0)\| \le t\alpha(K_1\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| + K_2\|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\|) \,. \quad \dots (14)$$

And similarly ,when we use the sequence of functions (7), we have

$$\|y_2(t,x_0,y_0) - y_1(t,x_0,y_0)\| \le t\beta(L_1\|x_1(t,x_0,y_0) - x_0\| + L_2\|y_1(t,x_0,y_0) - y_0\|)$$

And by mathematical induction also we find that

$$\|y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0)\| \le t\beta(L_1\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| + L_2\|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\|) \quad \dots (15).$$

Rewrite (15) and (16) in a vector form , we get

$$\Psi_{m+1}(t) \le \Omega(t)\Psi_m(t) \quad \dots (16)$$

where

$$\Psi_{m+1} = \begin{pmatrix} \|x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0)\| \end{pmatrix},$$

$$\Psi_m = \begin{pmatrix} \|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| \\ \|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\| \end{pmatrix}$$

and

$$\Omega(t) = \begin{pmatrix} K_1\alpha t & K_2\alpha t \\ L_1\beta t & L_2\beta t \end{pmatrix}$$

Now we take the maximum value for the both sides of the inequality (16) we get

$$\Psi_{m+1} \le \Omega\Psi_m \,, \quad \cdots (17)$$

where $\Omega = \max_{t\in[0,T]} \Omega(t)$, $\Omega = \begin{pmatrix} K_1\alpha T & K_2\alpha T \\ L_1\beta T & L_2\beta T \end{pmatrix}$.

By repetition of (17) we find that $\Psi_{m+1} \le \Omega^m \Psi_1$ and also we get

$$\sum_{i=1}^{m} \Psi_i \le \sum_{i=1}^{m} \Omega^{i-1}\Psi_0 \quad , \quad \cdots (18)$$

Using the condition (1.5), thus the sequence of functions (6) and (7) are uniformly convergent, that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Omega^{i-1}\Psi_0 = \sum_{i=1}^{\infty} \Omega^{i-1}\Psi_0 = (E-\Omega)^{-1}\Psi_0 \quad \cdots (19)$$

Let

$$\lim_{m\to\infty} \begin{pmatrix} x_m(t,x_0,y_0) \\ y_m(t,x_0,y_0) \end{pmatrix} = \begin{pmatrix} x^0\,(t,x_0,y_0) \\ y^{\,0}(t,x_0,y_0) \end{pmatrix} \quad \cdots (20)$$

Since the sequence of functions (6) and (7) are defined and continuous in the domain (1) then the limiting vector function $\begin{pmatrix} x^0\ (t,x_0,y_0) \\ y^{\ 0}(t,x_0,y_0) \end{pmatrix}$ is also defined and continuous in the same domain.

By using the same method above, we can proved that the inequalities (10) and (11) will be satisfied for all for all t ∈ [0,T] , $x_0 \in D_{1_\alpha}$, $y_0 \in D_{2_\beta}$ and m=0,1,2,… .

So that the vector function $\begin{pmatrix} x^0\ (t,x_0,y_0) \\ y^{\ 0}(t,x_0,y_0) \end{pmatrix} = \begin{pmatrix} x\ (t,x_0,y_0) \\ y\ (t,x_0,y_0) \end{pmatrix}$ *is* exist and it's a solution of the problem (P).

III. Uniqueness Solution Of (P).

The investigation of the uniqueness solution of the problem (P) will be introduced by the following theorem.

Theorem4. Let all assumptions and conditions of Theorem3 be satisfied.

Then the solution $\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$ is a unique of the problem (P). Let $\begin{pmatrix} \overline{x}(t,x_0,y_0) \\ \overline{y}(t,x_0,y_0) \end{pmatrix}$ be another solution of (P), i. e.

$$\overline{x}(t,x_0,y_0) = x_0 e^{\mathrm{At}} + \int_0^t e^{A(t-s)}\,[f(s,\overline{x}(s,x_0,y_0),\overline{y}(s,x_0,y_0)]ds$$

and

$$\overline{y}(t,x_0,y_0,z_0) = y_o e^{Bt} + \int_0^t e^{B(t-s)}\,[g(s,\overline{x}(s,x_0,y_0),\overline{y}(s,x_0,y_0)\,]ds$$

Now, taking

$$\|x(t,x_0,y_0) - \overline{x}(t,x_0,y_0)\| \le \left\|\int_0^t e^{A(T-s)}\,[f(s,x(s,x_0,y_0),y(s,x_0,y_0)) - f(s,\overline{x}(s,x_0,y_0),\overline{y}(s,x_0,y_0))]ds\ \right\|$$

$$\le \left(\|E\| - \frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|\mathrm{A}\|\mathrm{T}} - \|E\|}\right)\int_0^t \left\|e^{A(t-s)}\right\|\,\|\,f(s,x(s,x_0,y_0),$$
$$,y_1(s,x_0,y_0,z_0)\ - f(s,\overline{x}(s,x_0,y_0),,\overline{y}(s,x_0,y_0))\|ds +$$
$$\left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|\mathrm{A}\|\mathrm{T}} - \|E\|}\right)\int_t^T \left\|e^{A(t-s)}\right\|\|\,f(s,x(s,x_0,y_0),y(s,x_0,y_0)) - f(s,\overline{x}(s,x_0,y_0),\overline{y}(s,x_0,y_0))\,\|ds.$$

So that

$$\|x(t,x_0,y_0,z_0) - \overline{x}(t,x_0,y_0,z_0)\| \le t\alpha(K_1\|x(t,x_0,y_0) - \overline{x}(t,x_0,y_0)\| + K_2\|y(t,x_0,y_0) - \overline{y}(t,x_0,y_0)\|) \quad ...(21)$$

Now similarly

$$\|y(t,x_0,y_0) - \overline{y}(t,x_0,y_0)\| \le t\beta(L_1\|x(t,x_0,y_0) - \overline{x}(t,x_0,y_0)\| + L_2\|y(t,x_0,y_0) - \overline{y}(t,x_0,y_0)\|) \quad ...(22).$$

Rewrite the inequalities (21)and (22) in a vector form, we get

$$\begin{pmatrix} \|x(t,x_0,y_0\ ) - \overline{x}(t,x_0,y_0\ )\| \\ \|y(t,x_0,y_0\ ) - \overline{y}(t,x_0,y_0\ )\| \end{pmatrix}$$

$$\leq\ \Omega \begin{pmatrix} \|x(t,x_0,y_0\ ) - \overline{x}(t,x_0,y_0\ )\| \\ \|y(t,x_0,y_0\ ) - \overline{y}(t,x_0,y_0\ )\| \end{pmatrix} \qquad \ldots (23)$$

By iterating the inequality (23), we have

$$\begin{pmatrix} \|x(t,x_0,y_0\ ) - \overline{x}(t,x_0,y_0\ )\| \\ \|y(t,x_0,y_0\ ) - \overline{y}(t,x_0,y_0\ )\| \end{pmatrix} \leq \Omega^m \begin{pmatrix} \|x(t,x_0,y_0\ ) - \overline{x}(t,x_0,y_0\ )\| \\ \|y(t,x_0,y_0\ ) - \overline{y}(t,x_0,y_0\ )\| \end{pmatrix}$$

Then by the condition (1.5), we fine that

$$\begin{pmatrix} \|x(t,x_0,y_0\ ) - \overline{x}(t,x_0,y_0\ )\| \\ \|y(t,x_0,y_0\ ) - \overline{y}(t,x_0,y_0\ )\| \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x(t,x_0,y_0\ ) \\ y(t,x_0,y_0\ ) \end{pmatrix} = \begin{pmatrix} \overline{x}(t,x_0,y_0\ ) \\ \overline{y}(t,x_0,y_0\ ) \end{pmatrix}.$$

Hence the solution $\begin{pmatrix} x(t,x_0,y_0\ ) \\ y(t,x_0,y_0\ ) \end{pmatrix}$ of the problem (P) is a unique on the domain (1).

V. Stability Solution Of $(\boldsymbol{P}\ )$.

In this section, we study the stability solution of the problem (P) by the following theorem:

Theorem 5. If the inequalities (2) , (3) and the conditions(4),(5) are satisfied and $\begin{pmatrix} \tilde{x}(t,x_0\ ,y_0) \\ \tilde{y}\ (s,x_{0\ ,}y_0) \end{pmatrix}$ which was defined bellow as different solution for the problem (P), then the solution was stabile if satisfy the inequality:-

$$\begin{pmatrix} \|x(t,x_0,y_0\ ) - \tilde{x}(t,x_0\ ,y_0)\| \\ \|y(tx_0,y_0,) - \tilde{y}(t,x_0\ ,y_0)\| \end{pmatrix} \leq \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \qquad \epsilon_1,\epsilon_2 \geq 0$$

where

$$x(t,x_0,y_0\ ) = x_0e^{At} + \int_0^t e^{A(t-s)}\, f(s,x(s,x_{0\ ,}y_0\ ),y(s,x_{0\ ,}y_0))ds,$$

$$\tilde{x}(t,x_0\ ,y_0) = \tilde{x}_0e^{At} + \int_0^t e^{A(t-s)}\, f\left(s,\tilde{x}(s,x_0,y_0),\tilde{y}\ (s,x_{0\ ,}y_0)\right) ds$$

and

$$y(s,x_{0\ ,}y_0)\ = y_0e^{Bt} + \int_0^t e^{B(t-s)}\, g(s,x(s,x_{0\ ,}y_0\ ),y(s,x_{0\ ,}y_0)))ds,$$

$$\tilde{y}(t,x_0\ ,y_0) = \tilde{y}_0e^{Bt} + \int_0^t e^{B(t-s)}\, g(s,\tilde{x}(t,x_0\ ,y_0),\tilde{y}(t,x_0\ ,y_0))ds$$

Taking

$$\begin{aligned} \|x(t,x_0,y_0\ ) -\ \tilde{x}(t,x_0,y_0)\ \| &\leq \\ &\leq \|x_0 - \tilde{x}_0\|\alpha \\ &+ \alpha T\big[k_1\|x(t,x_{0\ ,}y_0) - \tilde{x}(t,x_{0\ ,}y_0)\| \\ &+ k_2\|y(t,x_{0\ ,}y_0) - \tilde{y}(t,x_{0\ ,}y_0)\|\big] \qquad \ldots (24) \end{aligned}$$

and

$$\|y(s,x_{0},y_0) - \tilde{y}(t,x_0,y_0)\| \leq \|y_0-\tilde{y}_0\|\beta + \beta T\big[l_1\|x(t,x_{0},y_0)-\tilde{x}(t,x_{0},y_0)\| + l_2\|y(t,x_{0},y_0)-\tilde{y}(t,x_{0},y_0)\|\big] \quad \dots (25)$$

Rewrite (24) and (25) in a vector form ,that is

$$\begin{pmatrix}\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ \|y(s,x_{0},y_0)-\tilde{y}(t,x_0,y_0)\|\end{pmatrix} \leq \begin{pmatrix}\|x_0-\tilde{x}_0\|\alpha \\ \|y_0-\tilde{y}_0\|\beta\end{pmatrix} + \begin{pmatrix}k_1\alpha T & k_2\alpha T \\ l_1\beta T & l_2\beta T\end{pmatrix}\begin{pmatrix}\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ \|y(t,x_{0},y_0)-\tilde{y}(t,x_0,y_0)\|\end{pmatrix}$$

For $\|x_0-\tilde{x}_0\| \leq \delta_1$, $\|y_0-\tilde{y}_0\| \leq \delta_2$ then

$$\begin{pmatrix}\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ \|y(s,x_{0},y_0)-\tilde{y}(t,x_0,y_0)\|\end{pmatrix} \leq \begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} + \begin{pmatrix}k_1\alpha T & k_2\alpha T \\ l_1\beta T & l_2\beta T\end{pmatrix}\begin{pmatrix}\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ \|y(t,x_{0},y_0)-\tilde{y}(t,x_0,y_0)\|\end{pmatrix}$$

By using the condition(5), we have

$$\begin{pmatrix}\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ \|y(s,x_{0},y_0)-\tilde{y}(t,x_0,y_0)\|\end{pmatrix} \leq \begin{pmatrix}\epsilon_1 \\ \epsilon_2\end{pmatrix}, \qquad \epsilon_1,\epsilon_2 \geq 0.$$

By the definition of the stability we find that

$\begin{pmatrix}\tilde{x}(t,x_0,y_0) \\ \tilde{y}(s,x_{0},y_0)\end{pmatrix}$ is a stable solution of the problem (p).

VI. Existence And Uniqueness Solution Of (P) .

In this section, we prove the existence and uniqueness theorem of the problem $(P)$by using Banach fixed point theorem .

Theorem 6. Let the vector functions $f(t,x,y)$ *and* $g(t,x,y)$ in the problem (P) are defined and continuous on the domain (1) and satisfies assumptions and all conditions of theorem 3, then the problem (P) has a unique continuous solution on the domain (1).

Proof . Let (C [0,T] , $\|.\|$) be a Banach space and T^* be a mapping on C [0,T] as follows :-

$$T^*x(t,x_0,y_0) = x_0e^{At} + \int_0^t e^{A(t-s)}f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds,$$

$$T^*\tilde{x}(t,x_0,y_0) = \tilde{x}_0e^{At} + \int_0^t e^{A(t-s)}f(s,\tilde{x}(s,x_0,y_0),\tilde{y}(s,x_0,y_0))ds$$

and

$$T^*y(t,x_0,y_0) = y_0e^{Bt} + \int_0^t e^{B(t-s)}g\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)ds,$$

$$T^*\tilde{y}(t,x_0,y_0) = \tilde{y}_0e^{Bt} + \int_0^t e^{B(t-s)}g(s,\tilde{x}(s,x_0,y_0),\tilde{y}(s,x_0,y_0))ds$$

Then

$$\|T^*x(t,x_0,y_0)-T^*\tilde{x}(t,x_0,y_0)\| \\ \le \alpha T[k_1\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ +k_2\|y(t,x_0,y_0)-\tilde{y}(t,x_0,y_0)\|] \quad \ldots \quad (26)$$

and

$$\|T^*y(t,x_0,y_0)-T^*\tilde{y}(t,x_0,y_0)\| \\ \le \beta T[l_1\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ +l_2\|y(t,x_0,y_0)-\tilde{y}(t,x_0,y_0)\|] \quad \ldots \quad (27)$$

Rewrite (26`) and (27) in a vector form, that is

$$\begin{pmatrix}\|T^*x(t,x_0,y_0)\ -T^*\tilde{x}(t,x_0,y_0)\| \\ \|T^*y(t,x_0,y_0)-T^*\tilde{y}(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\alpha Tk_1 & \alpha Tk_2 \\ \beta Tl_1 & \beta Tl_2\end{pmatrix}\begin{pmatrix}\|x(t,x_0,y_0)-\tilde{x}(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-\tilde{y}(t,x_0,y_0)\|\end{pmatrix}$$

From the condition (1.5), we get

$\begin{pmatrix}T^*x(t,x_0,y_0) \\ T^*y(t,x_0,y_0)\end{pmatrix}$ is a contraction mapping on $[0,T]$. By using Banach fixed point theorem , there exists a fixed point $\begin{pmatrix}x(t,x_0,y_0) \\ y(t,x_0,y_0)\end{pmatrix}$in C [0,T] such that

$$\begin{pmatrix}T^*x(t,x_0,y_0) \\ T^*y(t,x_0,y_0)\end{pmatrix}=\begin{pmatrix}x(t,x_0,y_0) \\ y(t,x_0,y_0)\end{pmatrix}=\begin{pmatrix}x_0e^{At}+\int_0^t e^{A(t-s)}f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds \\ y_0e^{Bt}+\int_0^t e^{B(t-s)}g(s,x(s,x_0,y_0)(s),y((s,x_0,y_0)s))ds\end{pmatrix}$$

So that

$\begin{pmatrix}T^*x(t,x_0,y_0) \\ T^*y(t,x_0,y_0)\end{pmatrix}$ is exist and it's a unique solution of the problem (p).

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