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On elements of deterministic chaos and cross links in non- linear dynamical systems

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Abstract: *In this paper we examine the existing definitions of deterministic chaos and the characterisation of its various ingredients. We then make use of some classical examples to provide cross links between the different chaotic behaviour of some simple but interesting maps which are then explained in a precise manner.*

I. Introduction

The late twentieth century non-linear dynamical systems terminology 'chaos' traces its origin to Li and York [9]. It asserts that for map on the real line which has a point with period three, there exists an uncountable scrambled set. This contrasts sharply with the definition given by Devaney [6] whose main ingredients are topological transitivity, density of periodic points and sensitive dependence on initial conditions. In fact it was widely accepted that sensitive dependence on initial conditions was the main ingredients in Devaney's definition of chaos but Banks et al [2], has proved that it was a redundant hypothesis since it was implied from the other two conditions. Other famous notions of chaos are those due to Wiggins and Lyapunov and these, together with Devaney's chaos, will form the basis for the analysis of cross links between maps exhibiting them. But for a thorough treatise on the fundamental elementary notions from basic point set topology and dynamical systems, we refer the interested reader to [1]

II. Ingredients Of Chaos

Here under, we provide precise definitions as well as clear explanations of the main ingredients of chaos; topological transitivity, density of periodic points and sensitive dependence on initial conditions.

Definition 2.1 Given the metric space X and the continuous map$f: X \to X$. We say that f is topologically transitive if for every pair of non empty open sets U and V in X, there exists $k > 0$ such that $f^k(U) \cap V \neq \phi$. Another famous definition of transitivity is the following:

Definition 2.2 The map f is said to be topologically transitive if there exists $x \epsilon X$ such that its orbits $\{f^n(x): n \geq 0$ is dense in X.
Intuitively these two definitions of topological transitivity are clearly not equivalent as can be seen from the following example.

Example 2.3 Consider the continuous map $f: X \to X$ where $X = \{0\} \cup \left\{\frac{1}{n} : n \epsilon N\right\}$ equipped with the metric $d = |x - y|\ \forall x, y \epsilon X$. Then clearly f is defined by $f(0) = 0$ and $f(\frac{1}{n}) = \frac{1}{n+1}$ for $n = 1,2, \ldots$. Then by choosing $U = \left\{\frac{1}{2}\right\}$ and $V = \{1\}$, f does not satisfy Definition 2.1 but then the point $x = 1$ has dense orbit in X so Definition 2.2 is satisfied and so is not equivalent to Definition 2.1. However, by [10], these two definitions are equivalent when X is a compact metric space.

Definition 2.4 Given the metric space X equipped with the metric d and the continuous map $f: X \to X$. We say that f exhibits sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \epsilon X$ any open neighbourhood $N_\varepsilon(x)$ of x for some $\varepsilon > 0$ there exists a point $y \epsilon N_\varepsilon(x)$ and $n \geq 0$ such that $d\big(f^n(x), f^n(y)\big) \geq \delta$.

Remark 2.5 a) A map which has sensitive dependence on initial conditions has points in $N_\varepsilon(x)$ which eventually separate from x by at least a distance δ under iteration.

b) From Definition 2.4 we can see that not all points in the open neighbourhood $N_\varepsilon(x)$ of x eventually separate from x under iteration but there is at least one such point in every open neighbourhood.

c) It is noteworthy that sensitivity is a metric property since it depends on the metric of the space.

d) The sensitivity constant δ neither depends on x nor on ε but only on the dynamical system (see also, [5]).

III. Characterization Of Devaney's, Wiggin's And Lyapunov's Chaos

We examine in the sequel the different perspectives of chaos such as Devaney's, Wiggin's and Lyapunov's chaos. We state however, that, conventionally in the rest of what follows, "transitivity" will always mean "topological transitivity" and "sensitivity" will always mean sensitive dependence on initial conditions".

Definition 3.1 (Devaney's) Let $f: X \to X$ be a continuous map and X is a metric space. Then f is said to be chaotic according to Devaney, hence forth, D- chaotic if:

i) f is topologically transitive
ii) The periodic points of f are dense in X
iii) f exhibits sensitive dependence on initial conditions.

Banks et al [2] has since faulted this definition by proving that sensitivity is indeed a redundant hypothesis because it is implied by transitivity and density of periodic points. It has also been proved in [4] that a continuous map with dense periodic points and sensitive dependence on initial conditions doesn't need to be transitive. We illustrate this with a counter example.

Counter example 3.2 Consider the continuous map $f: Y \to Y$, on $Y = S \times [0,1]$ is a metric equipped with the "taxi cab" metric $d\big((x_1, y_1), (x_2, y_2)\big) = |x_2 - x_1| + |y_2 - y_1|$ for every pair (x_1, y_1) and $(x_2, y_2) \in Y$. We then define f by $f\big(e^{i\theta}, t\big) = (e^{2i\theta}, t)$ and we see clearly that a point $z = (e^{i\theta}, t)$ will be a periodic point of f when $e^{i\theta}$ is the root of unity of order $2^n - 1$ for some n, so the periodic points of f are dense in Y. On the other hand if we take two sets, A and B, where $A = S' \times [0, \frac{1}{2})$ and $B = S' \times \left(\frac{1}{2}, 1\right]$, so that, $f^n(A) \cap B = A \cap B = \emptyset$, $\forall n \in N$. Consequently, f is not transitive. However if we equip the space with the "arc length" metric, then it becomes obvious that the map is sensitive and so f is not D-chaotic and also not W- chaotic**.**

Definition 3.3 (Wiggin's chaos [11]) Let $f: X \to X$ be a continuous map on a metric space X. Then the map f is said to be chaotic in the sense of Wiggin or W- chaotic if;

i) f is topologically transitive
ii) f exhibits sensitive dependence on initial conditions

If one may then ask "when does D- chaotic imply W- chaotic or vice versa in the dynamics of non- linear maps? To be able to answer this question favourably, we make haste to afford a third definition.

Definition 3.4 (Lyapunov's chaos) Consider $X \in \mathbb{R}$ and $f: X \to X$ be a continuous and differentiable map, then f is said to be Lyapunov chaotic or L- chaotic if:

i) f is topologically transitive
ii) f has positive Lyapunov exponent λ

Note that for the mapping f and for all $x \in X \in \mathbb{R}$ we define the Lyapunov exponent of x by;

$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)|$ for all $x_i \in \mathbb{R}$. This is a measure of the exponential rate at which nearby points are moving apart under iteration. However, in a set of positive measure the Lyapunov exponent can always be found from the relation

$\lambda(x) = \int \log|f'(x)|\, \rho(x)dx$, where $\rho(x)$ is the invariant measure and is always unique if f is ergodic (see e g, [7]).

It is therefore eminent from the definitions that transitivity is a common notion. Consequently, we are motivated to isolate it and seek the overbearing effect of a positive Lyapunov exponent on a continuous and differentiable map against the background of the absence of sensitive dependence on initial conditions in the definition.

The motivation for this search arose from the following consideration of the definition of the Lyapunov exponent.

Consider the iterative scheme

$x_{n+1} = f(x_n)$ and let the points x_0 and x'_0 be initially displaced by

$\delta = |x'_0 - x_0|$. Then after n- iterations, we get

$$\delta_{x_n} = |x'_n - x_n| = |f^n(x_0 + \delta) - f^n(x_0)| = \delta e^{n\lambda(x_0)} \qquad 3.1$$

Solving for $\lambda(x_0)$ we get in the limit,

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\delta\to 0} \frac{1}{n} \left|\frac{f^n(x_0+\delta) - f^n(x_0)}{\delta}\right| = \lim_{n\to\infty} \frac{1}{n} \log\left|\frac{df^n(x_0)}{dx}\right|$$
$$= \lim_{n\to\infty} \frac{1}{n} \log\left|\prod_{i=0}^{n-1} f'(x_i)\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)| \qquad 3.2$$

This therefore leads us to the following proposition which is part of the main thrust of this paper.

Proposition 3.5 Let $f: X \to X \epsilon \mathbb{R}$ be a continuous and differentiable map which has a positive Lyapunov exponent, then f has also sensitive dependence on initial conditions.

Proof: Consider an initial point $x_0 \epsilon \mathbb{R}$ .such that$x'_0 \epsilon N_\varepsilon(x_0)$, then, by 3.1, we have

$$\delta_{x_n} = |f^n(x_0+\delta) - f^n(x_0)| = |x'_n - x_n| = \delta_{x_0} e^{n\lambda x_0} = \delta \qquad 3.3$$

Where $\delta = |x'_0 - x_0|$ and $x'_0 = x_0 + \delta \Rightarrow e^{n\lambda x_0} = \frac{\delta}{\delta_{x_0}} \Rightarrow n = \frac{1}{\lambda_{x_0}} = log\left|\frac{\delta}{\delta_{x_0}}\right|$

Therefore after $m > n$ iterations, we get using 3.3, that

$|f^m(x'_0) - f^m(x_0)| = \delta_{x_0} e^{m\lambda x_0} = \delta_{x_0} e^{(m-n)\lambda x_0} e^{n\lambda x_0} = e^{(m-n)\lambda x_0}\delta > \delta$, showing that f has sensitive dependence on initial conditions. This then establishes the proof of the proposition. Intuitively this result shows that every real expanding map has sensitive dependence on initial conditions, see also [3] & [8].

IV. Cross Links between D-Chaotic, W-Chaotic and L-Chaotic Maps

In this section we provide cross links between the three types of chaotic behaviour highlighted in the preceding sections of some interesting classical maps.

Proposition 4.1 The Bernoulli Shift map $B(x): [0,1) \to [0,1)$ defined by;

$$B(x) = 2xmod1 = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} \le x \le 1 \end{cases}$$

is D-chaotic, W-chaotic and L- chaotic.

Proof: We first prove that $B(x)$ is transitive being the common ingredient using symbolic dynamics. We let Σ be the metric space of all infinite sequences containing 0's and 1's equipped with the metric;

$\rho(s,t) = \frac{1}{2^i}|s_i - t_i|$, for all $S = (s_0 s_1 s_2 \ldots)$ and T= $(t_0 t_1 t_2 \ldots) \epsilon \Sigma$.

Define $\sigma: \Sigma \to \Sigma$ by $\sigma(s_0 s_1 s_2) = \sigma(s_1 s_2 s_3)$, then there exists a point say, $x = (0100011011000001 \ldots)$ created by blocks of 0's and 1's, which has a dense orbit. By a similar argument as in [8] , σ is transitive and so $B(x)$ is transitive. It remains to show that $B(x)$ has a dense orbit. We have however, that Fix$B(x) = Per_1 B(x) = \{0\} \Rightarrow |Per_1 B| = 1 = 2^1 - 1$.

The period two map $B^2(x)$ is then given by;

$B^2(x) = 4xmod1$, so $Per_2(B) = \left\{0, \frac{1}{3}, \frac{2}{3}\right\} \Rightarrow |Per_2 B| = 3 = 2^2 - 1$.

Generalising this, an inductive argument really, we see that the nth iterated map is given by

$B^n(x) = 2^n xmod1$, so $Per_n B = \left\{0, \frac{1}{2^n-1}, \frac{2}{2^n-1}, \ldots, \frac{2^n-2}{2^n-1}\right\} \Rightarrow |Per_n B| = 2^n - 1$

And certainly, $lim_{n\to\infty}|Per_n(B)| = \infty$ so $\forall x \epsilon [0,1)$ and $\varepsilon > 0$, $N_\varepsilon(x)$ will contain at least a periodic point, hence the periodic points of B are dense.

Also, since $B(x) = 2xmod1$, then $B'(x) = 2\ \forall x \epsilon [0,1)$ except for $x = \frac{1}{2}$ where the derivative is not defined, implying that $\lambda(x) = log|B'(x)| = log2 > 0 \Rightarrow B(x)$ has a positive Lyapunov exponent. So all the conditions have been satisfied hence $B(x)$ is indeed D-chaotic, W-chaotic and L-chaotic. This establishes the proposition.

In the next proposition we give yet another example of a map which is at variance with the requirements of proposition 4.1.

The quadratic map $F: [0,1] \to [0,1]$ given by $F(x) = 4x(1-x)$ is known to be D-chaotic as well as W-chaotic(see eg, [6]). We shall use this result to investigate the chaotic behaviour of the map $G: D(0,1) \to U \subset D(0,1)$ in the next proposition.

Proposition 4.2 The map $G: D(0,1) \to U \subset D(0,1)$ defined by $G: (r,\theta) = (4r(1-r), \theta + 1)$ is only W-chaotic but not L-chaotic.

Proof: Using the polar coordinates(r,θ), we define the map G on the disk D(0,1)= $\{x\epsilon\mathbb{R}^2: \|x\| \leq 1\}$. We note that after a finite number of iterations, the image of a small disk in D(0,1) will contain an open set $U \subset D(0,1)$ with a full radius. Similarly a rotation of 1 radian will spread U totally over D(0,1) after a finite number of iterations, so G is transitive on D(0,1).

Now since the quadratic map F is sensitive on [0,1] by [6], then G is also sensitive on D(0,1). Consequently, G has only a fixed point in the origin and so does not have any periodic points of period $p > 1$. Basically G shrinks or stretches the distance of every point of D(0,1) from the origin while rotating by an angle of 1 radian. Since $\frac{1}{\pi}$ is irrational, no point x_n that belongs to the orbit of x_0 can return to the same ray which containsx_0. Hence G has no dense periodic points. So G is W-chaotic but not L-chaotic and consequently by [10], is also not D-chaotic.
We give yet another example of a different kind in proposition 4.3

Proposition 4.3 The continuous map $f: X \to X$ defined by $f\left(e^{i\theta}\right) = e^{2i\theta}$ and $X = S^1 \backslash \left\{e^{\frac{2\pi pi}{q}: p,q\epsilon\mathbb{Z}, q\neq 0}\right\}$ a metric space endowed with the arc length metric ρ is L-chaotic but not D-chaotic or W- chaotic.

Proof: Observe that every non empty subset of X is eventually expanded under iteration to coverX, so f is transitive. Also by defining in this way the set X we let out all the periodic points of$f$, so f has no dense periodic points. Finally for any given two points in$X$, say $e^{i\theta}$ and $e^{i\varphi}$ such that, 0< $|\theta - \varphi| < \pi$, we can choose n that satisfies$2^n|\theta - \varphi| \leq |\theta - \varphi| \leq \pi < 2^{n+1}|\theta - \varphi| \Rightarrow$ f is sensitive with sensitivity constant $\frac{\pi}{2}$ since $\rho(f^n\left(e^{i\theta}\right), f^n\left(e^{i\varphi}\right)) > \frac{\pi}{2}$ so the map f is L-chaotic, and not D-chaotic or W- chaotic.

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