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Regular Convolutions on $(\mathbf{Z}^+, \leq_C)$

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Abstract: *Regular Convolutions Care progressions in one – to – one correspondence with sequences $\{\Pi_p\}_{p\in P}$ of decompositions of $\mathbf{N}$ into arithmetical progressions (finite or infinite)and this is represented by writing $C\square\ \{\Pi_p\}_{p\in P}$. In this paper we proved that any regular convolution C gives rise to a structure of meet semilattice on $(\mathbf{Z}^+, \leq_C)$ and the convolution C is completely characterized by certain lattice theoretic properties of $(\mathbf{Z}^+, \leq_C)$. In particular we prove that the only regular convolution including a lattice structure on $(\mathbf{Z}^+, \leq_C)$ is the Dirichlit's convolution.*

Key words: *convolution , Dirichlit's convolution, arithmetical progressions, relation isomorphism*

I. Introduction

G.Birkhoff [1] Introduced the notation of Lattice Theory 1967.Since then a host of researcher'shave studied and contributed a lot to the theoretical aspects of this topic. Among them,[2],[3],[4]and[5] are mentionable. A mapping $C : \mathbf{Z}^+ \to P(\mathbf{Z}^+)$ is called a convolution if C(n) is a nonempty set of positive divisors of n for each n ∈ $\mathbf{Z}^+$. In general convolutions may not induce a lattice structure on $\mathbf{Z}^+$

II. Preliminaries

in this paper $\mathbf{N}$ ={0,1,….} and a covered by b denoted by a $-\!<$ b.

2.1Theorem: For any convolution C, the relation$\leq_C$ is a partial order on $\mathbf{Z}^+$ If and only if $n \in C(n)$ and $\bigcup_{m\in C(n)} C(m) \subseteq C(n)$ for all n $\in \mathbf{Z}^+$.[6]

2.2Theorem: Let C be a convolution and define $\theta : (\mathbf{Z}^+, \leq_C) \to \sum_P N$ by$\theta(n)(p) =$ The largest a in $\mathbf{N}$ such that p^a divides n, for all n $\in \mathbf{Z}^+$ and p ∈ P.Then C is a multiplicative if and only if θ is a relation isomorphism of $(\mathbf{Z}^+, \leq_C)$ onto $(\sum_P N, \leq_C)$.[7]

2.3Theorem: Let $\leq_C$ be the partial on $\mathbf{Z}^+$ induced by aconvolutionC, and for any prime p,let $\leq_C^p$ be the partial order on $\mathbf{N}$ induced by C.

1. If $(\mathbf{Z}^+, \leq_C)$ is a meet (join) semilattice, then so is $(\mathbf{N}, \leq_C^p)$ for any prime p.
2. If $(\mathbf{Z}^+, \leq_C)$ is a lattice, then so is $(\mathbf{N}, \leq_C^p)$ for any prime p.[8]

2.4Theorem: Let C be a multiplicative convolution such that $(\mathbf{Z}^+, \leq_C)$ is a meet (join) semilattice and let F be a filter of $(\mathbf{Z}^+, \leq_C)$. Then F is a prime filter if and only if there exists unique prime number p such that θ(F)(p) is a prime filter of $(\mathbf{N}, \leq_C^p)$ and θ(F)(p)= $\mathbf{N}$ for all q ≠ p in Pand , in this case
F = { n $\in \mathbf{Z}^+$:θ(n)(p) ∈ θ(F)(p)}.

2.5Theorem: Let (S, ∧) be a meet semi lattice with smallest element 0 and satisfying the descending chain condition. Also suppose that every proper filter of S is prime then the following are equivalent to each other.

1. Any two incomparable filters of S are comaximal.
2. For any x and y in S, x$\|$ $\|$y implies $x \wedge y = 0$
3. S $-\{0\}$ is a disjoint union of maximal chains.

2.6 Theorem: Let C be a multiplicative convolution such that $(\mathbf{Z}^+, \leq_C)$ is a meet (join) meet semilattice. Then any two incomparable prime filters of $(\mathbf{Z}^+, \leq_C)$ are comaximal if and only if any two incomparable prime filters of ($\mathbf{N}$, $\leq_C^p$) are comaximal for each prime number p.

2.7 Theorem: Let p be a prime number.Then every proper filter ($\mathbf{N}$, $\leq_C^p$) is prime if and only if $[p^a)$ is aprime filter in $(\mathbf{Z}^+, \leq_C)$ for all $a > 0$.

2.8 Theorem: Let C be any convolution. Then C is regular if and only if the following conditions are satisfied for any positive integers m, nand d:

(1) C is a multiplicative convolution; that is $C(mn) = C(m)C(n)$ whenever (m, n)=1.
(2) $d \in C(n) \Rightarrow \frac{n}{d} \in C(n)$
(3) $1, n \in C(n)$
(4) $d \in C(m)$and$d \in C(n) \Rightarrow d \in C(n)$ and $\frac{m}{d} \in C(\frac{n}{d})$
(5) For any prime number p and positive integera, there exist positive integers r and t such that $rt = a$ and

$$C(p^a) = \{1, p^t, p^{2t}, \dots, p^{rt}\}$$

$p^t \in C(p^{2t})$,$p^{2t} \in C(p^{3t})$,…,$p^{(r-1)t} \in C(p^{rt})$.

2.9Theorem: Let **D**be the set of all decompositions of the set $\mathbf{N}$ of non-negative integers into arithmetic progression (finite and infinite) each containing 0 and no two progressions belonging to same decomposition have a positive integer in common. Let usassociate with each prime number, a member $\prod_p$ of**D.** For any $n = p_1^{a_1} p_2^{a_2} \dots . p_r^{a_r}$ where$p_1 p_2 \dots p_r$are distinct primes and $a_1, a_2, \dots, a_r \in \mathbf{N}$, define $C(n) = \{ p_1^{b_1} p_2^{b_2} \dots . p_r^{b_r} : b_i \leq a_iandb_i$ and a_i belong to the same progression in $\prod_{p_i}$ }.

Then C is a regular convolution and, conversely every regular convolution can be obtained as above. In this case, we write C $\{\Pi_p\}_{p \in P}$.

III. Regular Convolutions

3.1 Definition: Let$(X, \leq)$ be a partially ordered set and a and b are elements of X. Then a is said to be covered by b (or b is a cover of a) if $a < b$ and there is no element c in X such that $a < c < b$. In this case, we express it symbolically by$a -< b$.

3.2 Theorem: Let $(X, \leq)$ be a poset satisfying the descending chain condition. Then any element of X is either maximal element or covered by another element of X.

Proof: Let $a \in X$. suppose a is not maximal. Then there exists $x \in X$ such that $a < x$. Now consider the set

$$A = \{x \in X : a < x\}.$$

Then A is nonempty subset of X. Since X satisfies descending chain condition, A has a minimal element, say b. Then, since $b \in A$, we have $a < b$. Also since b is minimal in A, there cannot be any element c such that $a < c < b$. Thus a is covered by b; That is$a -< b$.

3.3 Corollary: For any convolution C, any positive integer is either maximal in $(\mathbf{Z}^+, \leq_c)$ or covered by some integer in $(\mathbf{Z}^+, \leq_c)$.

Proof: This is immediate consequence of the theorem 5.4.2, since $\left(\mathbf{Z}^{+}, \leq_{c}\right)$ satisfies the descending chain conditionfor any convolution C.

We observe that there is a bijection $\theta : \mathbf{Z}^{+} \rightarrow \sum_{p} \mathbf{N}$ defined by

$$\theta(n)(p) = \text{largest } a \text{ in } \mathbf{N} \text{ such that } p^{a} \text{ devides } n, \text{for any } n \in \mathbf{Z}^{+} \text{ and any prime number } p.$$

3.4 Theorem: Let C be a convolution and $\leq_{C}$ be the binary relation on $\mathbf{Z}^{+}$ induced by C. Then C is a regular convolution if and only if the following properties are satisfied:

(1) $\theta : \left(\mathbf{Z}^{+}, \leq_{C}\right) \rightarrow \left(\sum_{p \in P} N, \leq_{C}^{p}\right)$ is a relation isomorphism.

(2) $\left(\mathbf{Z}^{+}, \leq_{C}\right)$ is a meet semilattice.

(3) Any two incomparable prime filters of $\left(\mathbf{Z}^{+}, \leq_{C}\right)$ are comaximal.

(4) F is a prime filter of $\left(\mathbf{Z}^{+}, \leq_{C}\right)$ if and only if $F = \left[p^{a}\right)$ for some prime number p and $a \in \mathbf{Z}^{+}$.

(5) $m, n \in \mathbf{Z}^{+}, m -\!\!< _C n \Rightarrow 1 -\!\!<_C \frac{n}{m} \leq_C n$

Proof: Suppose that the properties (1) through (5) are satisfied. Then by (4), $F = \left[p^{a}\right)$ is a prime filter of $\left(\mathbf{Z}^{+}, \leq_{C}\right)$ for all $p \in P$ and $a \in \mathbf{Z}^{+}$. Hence by theorem2.6,every proper filter $(\mathbf{N}, \leq_{C}^{p})$ is prime for each $p \in P$.

By (3), any two incomparable prime filter of $\left(\mathbf{Z}^{+}, \leq_{C}\right)$ are comaximal, Hence by theorems 2.5and 2.6, it follows that $(\mathbf{N} - \{0\}, \leq_{C}^{P})$ is a disjoint union ofmaximal chains. Let prime number p be fixed, then

$$\mathbf{N} - \{0\} = \bigcup_{i \in I} Y_{i} \text{ (disjoint union); where I is an index set}$$

here Y_i is a maximal chain in $(\mathbf{N} - \{0\}, \leq_{C}^{P})$ such that, for any $i \neq j \in I$, $Y_{i} - Y_{j} = \phi$ and each element of Y_i isincomparable with each element of Y_j for $i \neq j$.

Now, we shall prove that Y_i is an arithmetical progression(finite or infinite).

Fix $i \in j$. Since $\mathbf{N}$ is countable, Y_i must be countable. Also, since $(\mathbf{N}, \leq_{C}^{p})$ satisfies descending chain condition, we can express

$$Y_i = \{a_1 -\!\!<_C^p a_2 -\!\!<_C^p a_3 -\!\!<_C^p \ldots.\}$$

We shall use induction on r to prove that $a_{r} = r a_{1}$, for all r. clearly this is true for $r = 1$. Assume that $r > 1$ and $a_{s} = s a_{1}$ for all $1 \leq s < r$. since

$$(r-1)a_1 = a_{r-1} -\!\!< a_r \text{ in } (\mathbf{N}, \leq_{C}^{p}),$$

we have in $p^{a_{r-1}} -\!\!<_C p^{a_r}$ in $\left(\mathbf{Z}^{+}, \leq_{C}\right)$ and hence, by (5),

$$1 -\!\!<_C p^{a_r - a_{r-1}} \leq_C p^{a_r}$$

Therefore $0 \neq a_{r} - a_{r-1} \leq_{C}^{p} a_{r}$ and hence $a_{r} - a_{r-1} \in Y_{i}$ (since $a_{r} \in Y_{i}$). Also since $0 -\!\!< a_{r} - a_{r-1}$ in $(\mathbf{N}, \leq_{C}^{p})$, we get that

$$a_{r} - a_{r-1} = a_{1}$$

and hence $a_{r} = a_{r-1} + a_{1} = (r-1)a_{1} + a_{1} = r a_{1}$.

Therefore, for any prime p and $a \in \mathbf{Z}^{+}$,

$$C\left(p^{a}\right)=\left\{1, p^{t}, p^{2t}, p^{3t}, \ldots, p^{st}\right\}, \quad st=a$$

for some positive integer t and s and

$$p^{t} \in C\left(p^{2t}\right), p^{2t} \in C\left(p^{3t}\right), \ldots, p^{(s-1)t} \in C\left(p^{a}\right).$$

The other conditions in theorem2.8 are clearly satisfied.Thus by theorem 2.8, we see that C is a regular convolution.

Conversely, suppose that C is a regular convolution. Then by theorem 2.9, $C \sim \left\{\Pi_{p}\right\}_{p \in P}$, where each Π_{p} is a decomposition of $\mathbf{N}$ into arithmetical progressions(finite or infinite) in which each progression contains 0 and no positive integer belongs to two distinct progressions. For any $a, b \in \mathbf{N}$ and $p \in P$, let us write, for convenience,

$$(a, b) \in \Pi_{P} \Leftrightarrow \text{ a and b belong to the same progression in } \Pi_{P}.$$

Since C is a regular convolution, C satisfies all the properties (1) through (5) of theorem 5.3.4. From (3) and (4) of theorem 5.3.4 and corollary 2.1[1], it follows that $\leq_{C}$ is a partial order relation on $\mathbf{Z}^{+}$. Since C is multiplicative, it follows from theorem 2.2[1]that

$$\theta:\left(\mathbf{Z}^{+}, \leq_{C}\right) \rightarrow\left(\sum_{p \in P} \mathbf{N}, \leq_{C}^{P}\right)$$

In an order isomorphism. Therefore the property (1) is satisfied.

For simplicity and convenience, let us write $\underline{n}$for $\theta(n)$. Recall that, for any $n \in \mathbf{Z}^{+}$ and $p \in P$,

$\underline{n}(p)=\theta(n)(p)$ = the largest a in $\mathbf{N}$ such that p^{a} divides n.

The partial order relations $\leq_{C}$ on $\mathbf{Z}^{+}$ and $\leq_{C}^{p}$ on $\mathbf{N}$ are defined by

$$m \leq_{C} n \Leftrightarrow m \in C(n), \text{ for any } m, n \in \mathbf{Z}^{+}$$

and

$$a \leq_{C}^{p} b, \text{ for any } a, b \in \mathbf{N}.$$

Now for any $m, n \in \mathbf{Z}^{+}$, the element $m \wedge n$ of $\mathbf{Z}^{+}$ be defined by for all $p \in P$.

$$(m \wedge n)(p)=\begin{cases}0 & , \text{ if } (m(p), n(p)) \notin \Pi_{p} \\ \text{minimum of } \{m(p), n(p)\}, & \text{otherwise.}\end{cases}$$

Again for all $p \in P$, if $(\underline{m}(p), \underline{n}(p)) \in \Pi_{p}$, then

$$\underline{m}(p) \leq_{C}^{p} \underline{n}(p) \quad \text{or} \quad \underline{n}(p) \leq_{C}^{p} \underline{m}(p).$$

Thus for all $p \in P$,

$$(\underline{m \wedge n})(p) \leq \underline{m}(p) \text{ and } (\underline{m \wedge n})(p) \leq \underline{n}(p).$$

Therefore $m \wedge n$ is a lower bound of m and n in $\left(\mathbf{Z}^{+}, \leq_{C}\right)$.

Let k be any other lower bound of m and n. For any $p \in P$, if $(\underline{m}(p), \underline{n}(p)) \in \Pi_{p}$, then

$$\underline{k}(p) \leq_{C}^{p} \operatorname{Min}\{\underline{m}(p), \underline{n}(p)\}=(\underline{m \wedge n})(p)$$

and if $(\underline{m}(p), \underline{n}(p)) \notin \Pi_{p}$, then

$$\underline{k}(p)=0=(\underline{m \wedge n})(p).$$

Therefore $\underline{k}(p) \leq(\underline{m \wedge n})(p)$, for all $p \in P$ and hence $k \leq_{C} m \wedge n$.

Thus $m \wedge n$ is the greatest lower bound of m and n in $\left(\mathbf{Z}^{+}, \leq_{C}\right)$.

So $(\mathbf{Z}^{+}, \leq_{C})$ is a meet semilattice and hence the property (2) is satified.

To prove (3), it is enough, to show that any two incomparable prime filters of ($\mathbf{N}$, $\leq_{C}^{p}$) are comaximal for all $p \in P$.

By theorem 2.6, wesee that for any positive integer a and b, if a and b are incomparable in ($\mathbf{N}$, $\leq_{C}^{p}$), then $(a, b) \notin \Pi_{P}$, and hence a and b have no upper bound in ($\mathbf{N}$, $\leq_{C}^{p}$) and therefore $a \vee b$ does not exist in ($\mathbf{N}$, $\leq_{C}^{p}$). Also each progression in Π_{P} is a maximal chain in ($\mathbf{N}$, $\leq_{C}^{p}$)and for any a and b in $\mathbf{N}$ are comparable if and only if $(a, b) \in \Pi_{P}$. Therefore ($\mathbf{N}$ -- $\{0\}$, $\leq_{C}^{p}$) is a disjoint union of maximal chains. Thus by theorem 2.5, any two incomparable prime filters of ($\mathbf{N}$, $\leq_{C}^{p}$) are comaximal. Therefore by theorem 2.6, any two incomparable prime filters of ($\mathbf{Z}^{+}, \leq_{C}$) are comaximal, which proves (3).

The property (4) is a consequence of the theorems 2.4 and 2.6.

Finally we prove(5): Let $m, n \in \mathbf{Z}^{+}$ such that m $-\!\!<_{C}$ n . By theorem 2.8(2), we get $\frac{m}{n} \leq_{C} n$.

Let us write

$$m = p_1^{a_1} p_2^{a_2} \ldots\ldots\ldots p_r^{a_r} \text{ and } n = p_1^{b_1} p_2^{b_2} \ldots\ldots\ldots p_r^{b_r}$$

where $p_1 \, p_2 \ldots\ldots\ldots p_r$ are distinct prime numbers and $a_i, b_i > 0$ such that

$$0 \leq_{C}^{p} a_i \leq_{C}^{p} b_i \text{ for } 1 \leq i \leq r.$$

Since m $\neq$ n, so there exists i such that $a_i \leq_{C}^{p} b_i$. Now if $a_j \leq_{C}^{p_j} b_j$ for $j \neq i$, then the element

$$k = p_1^{c_1} p_2^{c_2} \ldots\ldots\ldots p_r^{c_r} \text{, where } c_s = \begin{cases} a_s & \text{if } s \neq i \\ b_s & \text{if } s = i \end{cases}$$

Will be in between m and n (that is $m \leq_{C} k \leq_{C} n$), which is contradiction to the supposition that m $-\!\!<_{C}$ n.Therefore $a_j = b_j$ for all $j \neq i$ and hence $\frac{n}{m} = p_i^{b_i - a_i}$.

Since $(a_i, b_i) \in \Pi_{p_i}$, there exists t> 0 such that

$$b_t = ut \quad \text{and} \quad a_t = vt$$

for some u and v with $v < u$. Also $vt,\ (v+1)t,\ \ldots\ldots\ldots,\ ut$ are all in the same progression.

Since m $-\!\!<_{C}$ n we get that $u = v + 1$ and hence $\frac{n}{m} = p_i^{t}$. Again since $0 -\!\!<_{C} t$ in ($\mathbf{N}$ -, $\leq_{C}^{p_i}$), it follows that

$$1 -\!\!<_{C} p_i^{t} = \frac{n}{m} \leq_{C} n.$$

Thus the property (5) is satisfied. This completes the proof of the theorem.

We observe that ($\mathbf{Z}^{+}, \leq_{C}$) is a meet semilattice for any convolution C. Note that ($\mathbf{Z}^{+}, \leq_{C}$) need not be lattice. For example, consider the unitary convolution U defined by

$$U(n) = \{d \in \mathbf{Z}^{+} : d \text{ divides } n \text{ and } (d, \frac{n}{d}) = 1\}.$$

Then U is a regular convolution and ($\mathbf{Z}^{+}, \leq_{C}$) is not a lattice. In this context we have the following:

3.5 Theorem : The following are equivalent to each other for any regular convolution

(i) ($\mathbf{Z}^{+}, \leq_{C}$) Is a lattice.

(ii) ($\mathbf{N}, \leq_{C}^{p}$) Is a lattice for each $p \in P$.

(iii) ($\mathbf{N}, \leq_{C}^{p}$) Is a totally ordered set for each $p \in P$.

(iv) $C(n)$ is the set of positive divisors of $n \in \mathbf{Z}^{+}$.

Proof: Let C be a regular convolution and $C \sim \{\Pi_{p}\}_{p \in P}$ as in theorem 2.9.[5].

(i) $\Rightarrow$ **(ii)**: follows from theorem 2.3(2).[6]

(ii) $\Rightarrow$ **(iii):** Suppose that $(\mathbf{N}, \leq_{C}^{p})$ is a lattice for each $p \in P$. Fix $p \in P$ and $a, b \in \mathbf{N}$. Then we can choose $c \in \mathbf{N}$ such that

$$a \leq_{c}^{p} c \quad \text{and} \quad b \leq_{c}^{p} c.$$

Then $(a, c) \in \Pi_{P}$ and $(a, c) \in \Pi_{P}$, which means that a and c belong to the same progression in Π_{P}. Therefore a, b and c should all be in the same progression and hence $a \leq_{c}^{p} b$.

Thus $(\mathbf{N}, \leq_{C}^{p})$ is a totally ordered set.

(iii) $\Rightarrow$ **(iv):** Suppose that $(\mathbf{N}, \leq_{C}^{p})$ is a totally ordered set for each $p \in P$. Then any two elements of $\mathbf{N}$ must be in same progression in Π_{P} for each $p \in P$. This amounts to saying that Π_{P} has only one progression; that is

$$\Pi_{P} = \{\{0, 1, 2, 3, \ldots \quad \ldots\}\}.$$

Therefore for any $a, b \in \mathbf{N}$;

$$a \leq_{c}^{p} b \text{ If and only if } a \leq b.$$

This means that $\leq_{C}^{p}$ coincides with the usual order in $\mathbf{N}$, for each $p \in P$.

Thus for any $m, n \in \mathbf{Z}^{+}$,

$$m \in C(n) \Leftrightarrow m \leq_{C} n \Leftrightarrow m \text{ divides } n.$$

Therefore $C(n)$ = the set of all positive divisors of n, for any $n \in \mathbf{Z}^{+}$.

(iv) $\Rightarrow$ **(i):** From (iv) we get that C is precisely the Dirichlit's convolution D and for any $m, n \in (\mathbf{Z}^{+}, \leq_{D})$

$$n \wedge m = g.c.d\{n, m\} \quad \text{and} \quad n \vee m = l.c.m\{n, m\}.$$

Hence $(\mathbf{Z}^{+}, \leq_{C})$ is a lattice.

IV. Conclusion

The Dirichlit's convolution is the only regular convolution which induces a lattice structure on $\mathbf{Z}^{+}$.

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