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A Coupled Thermoelastic Problem of A Half – Space Due To Thermal Shock on the Bounding Surface.

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I. Introduction:

This paper is concerned with the determination of temperature and displacement of a half space bounding surface due to thermal shock. This paper deals with the place boundary of the half-space is free of stress and is subjected to a thermal shock. Moreover , the perturbation method is employed with the thermoelastic coupling facter ε as the perturbation parameter. The Laplace transform and its inverse with very small thermoelastic coupling facter ε are used. The deformation field is obtained for small values of time. *Paria*[7] has formulated different types of thermal boundary condition problems.

II. Formulation Of The Problem: Governing Equations

Let the elastic half-space be x ≥ 0 with surface plane x = 0 , free of tractions for all time. The solid is assumed to be mechanically constrained so that the displacement components $u_x = u\,(x,t\,), u_y = u_z = 0$ and the temperature distribution is of the form T= T (x , t) , x and t denoting respectively the space- coordinate and time.
The coupled thermo- elastic differential *equations*[12] are

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} = \rho\,\frac{\partial^2 u}{\partial t^2} + \boldsymbol{\alpha}\,(3\boldsymbol{\lambda} + 2\boldsymbol{\mu})\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}} \quad \textbf{(1)}$$

$$K\frac{\partial^2 T}{\partial x^2} = \boldsymbol{\rho c_\theta}\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{t}} + \boldsymbol{\alpha}\,(3\boldsymbol{\lambda} + 2\boldsymbol{\mu})\boldsymbol{\tau_0}\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}\partial \boldsymbol{t}} \quad \textbf{(2)}$$

The initial conditions are taken to be

$$\mathbf{U\,(x,0)} = \frac{\partial \boldsymbol{u}(\boldsymbol{x},\mathbf{0})}{\partial \boldsymbol{t}} = \mathbf{0}$$

The boundary condition is that the normal stress $\sigma_{xx}\ (0,t) = 0$. Introducing dimensionless variables

$$x_1 = \frac{ax}{K}\ ,\ t_1 = \frac{a^2 t}{K}\ ,\ \sigma = \frac{\sigma_{xx}}{\beta T_\phi}$$

$$T_1 = \frac{T}{T_\phi}\ , \qquad U_1 = \frac{a(\lambda + 2\mu)}{K\beta T_\phi}\,u\,,$$

$$K = \frac{K}{\rho c_\varphi}\ ,\ a^2 = \frac{\lambda + 2\mu}{\rho}\ ,\ \beta = (3\boldsymbol{\lambda} + 2\boldsymbol{\mu})\boldsymbol{\alpha} \qquad \textbf{(3)}$$

Equations (1) and (2) reduce to

$$\frac{\partial^2 \boldsymbol{u_1}}{\partial \boldsymbol{x_1^2}} = \frac{\partial \boldsymbol{T_1}}{\partial \boldsymbol{x_1}} + \frac{\partial^2 \boldsymbol{u_1}}{\partial \boldsymbol{t_1^2}} \quad \textbf{(4)}$$

$$\frac{\partial^2 \boldsymbol{T_1}}{\partial \boldsymbol{x_1^2}} = \frac{\partial \boldsymbol{T_1}}{\partial \boldsymbol{t_1}} + \in \frac{\partial^2 \boldsymbol{u_1}}{\partial \boldsymbol{x_1}\partial \boldsymbol{t_1}} \quad \textbf{(5)}$$

With the initial conditions

$$\boldsymbol{U_1},\ \boldsymbol{T_1}\frac{\partial \boldsymbol{U_1}}{\partial \boldsymbol{t_1}} = \mathbf{0} \quad \textbf{at t = 0 for all} \quad \boldsymbol{x_1} > 0 \quad \textbf{(6)}$$

The boundary conditions

(a) $\sigma = \frac{\partial_{xx}}{\beta T_\phi} = \frac{\partial u_1}{\partial x_1} - T_1 = 0\,,\ for\ x_1 = 0$ **(7)**

(b) $T_1 = T_0\,\delta\,(t_1)$ for all $t_1 \geq 0$

Due to instantaneousthermal shock on $x_1 = 0$ where T_0 is a constant and $\partial(t_1)$ is the well known Dirac – delta function.
Besides, the regularity conditions require

$$U_1, T_1, \frac{\partial U_1}{\partial x_1}\ ,\ \frac{\partial T_1}{\partial x_1} \to 0\ as\ x_1 \to \infty\ \textbf{(8)}$$

The constant ε represents the thermoelastic coupling factor and is given by

$$\varepsilon = \frac{\beta^2 T_0}{\rho c_\vartheta(\lambda + 2\mu)}$$

III. Solution Of The Problem

Let Laplace transform of $U_1(x_1, t_1)$, $T_1(x_1, t_1)$ be

$$\bar{U}_1(x_1, p) = \int_0^\infty U_1(x_1, t_1) e^{-pt_1}\, dt_1$$

$\bar{T}_1(x_1, p) = \int_0^\infty T_1(x_1, t_1) e^{-pt_1}\, dt_1$ **(9)**

Laplace transform of (4) and (5) give

$\frac{d^2\bar{U}_1}{\partial x_1^2} - p^2 \bar{U}_1 = \frac{d\bar{T}_1}{dx_1}$ **(10)**

And $\frac{d^2\bar{T}_1}{\partial x_1^2} - p\,\bar{T}_1 = \varepsilon p \frac{d\bar{U}_1}{dx_1}$ **(11)**

With $\bar{U}_1(x_1, p), \bar{T}(x_1, p) \to 0 \; as \; x_1 \to \infty$

$\bar{T}_1 = T_0 \; on \; x_1 = 0$ and $\frac{d\bar{U}_1}{dx_1} = T_0 \; on \; x_1 = 0$ **(12)**

Elimination of $\bar{T}_1$ from (10) and (11) gives

$\frac{d^4\bar{U}_1}{dx_1^4} - p(1 + p + \varepsilon)\frac{d^2\bar{U}_1}{dx_1^2} + p^3 \bar{U}_1 = 0$ **(13)**

Eliminating $\bar{U}_1$ from (10) and (11) we get the same equation in terms of $\bar{T}_1$

Solving these two equations satisfying first conditions of (12)

$$\bar{U}_1 = Ae^{-m_1 x_1} + Be^{-m_2 x_1}$$

$\bar{T}_1 = A_1 e^{-m_1 x_1} + B_1 e^{-m_2 x_1}$ **(14)**

Where $m_1^2 \; and \; m_2^2$ are the roots of the quadratic equation

$x^4 - p(1 + p + \varepsilon)x^2 + p^3 = 0$ **(15)**

Hence $m_1^2 + m_2^2 = p(1 + p + \varepsilon)$, $m_1^2 m_2^2 = p^3$

And $m_1 m_2 = \frac{\sqrt{p}}{2}(\alpha_0 \pm \beta_0)$

$$\alpha_0 = \sqrt{(1 + p + \varepsilon + 2\sqrt{p})}$$
$$\beta_0 = \sqrt{(1 + p + \varepsilon - 2\sqrt{p})}$$

Now $(\bar{T}_1)_{x_1=0} = T_0 \; gives \; B_1 = T_0 - A_1$

And $\left(\frac{d\bar{U}_1}{dx_1} - T_0\right)_{x_1=0} = 0 \; gives \; B = -\frac{T_0 + Am_1}{m_2}$

Therefore $\bar{T}_1 = A_1 e^{-m_1 x_1} + (T_0 - A_1)e^{-m_2 x_1}$

$\overline{U_1} = A_1 e^{-m_1 x_1} - \frac{T_0 + Am_1}{m_2} e^{-m_2 x_1}$ **(16)**

Now, substituting the solutions (16) into (10),

$$A(m_1^2 - p^2)e^{-m_1 x_1} + \left\{\frac{T_0 + Am_1}{m_2}p^2 - (T_0 + Am_1)m_2\right\}e^{-m_2 x_1}$$
$$= - A_1 m_1 e^{-m_1 x_1} + (A_1 - T_0)m_2 e^{-m_2 x_1}$$

This is satisfied if

$A_1 m_1 = Ap^2 - Am_1^2$

And $m_2(T_0 - A_1) = m_2(T_0 + Am_1) - \frac{p^2(T_0 + Am_1)}{m_2}$

Solving for A and A_1

$A = \frac{m_1}{m_2^2 - m_1^2} T_0$, $A_1 = \frac{m_1^2 - p^2}{m_1^2 - m_2^2} T_0$

Hence , $\bar{T}_1 = T_0\left[\frac{m_1^2 - p^2}{m_1^2 m_2^2} e^{-m_1 x_1} - \frac{m_2^2 - p^2}{m_1^2 - m_2^2} e^{-m_2 x_1}\right]$ **(17)**

$\overline{U_1} = T_0\left[\frac{m_2}{m_1^2 - m_2^2} e^{-m_1 x_1} - \frac{m_1}{m_1^2 - m_2^2} e^{-m_2 x_1}\right]$ **(18)**

Since ε is generally very small, we expand in ascending powers of ε and retain terms up to its first power.

Hence

$\alpha_0 = \sqrt{\left(\sqrt{p} + 1\right)^2 + \varepsilon} = \left(\sqrt{p} + 1\right)\left\{1 + \frac{\varepsilon}{2}\frac{1}{(\sqrt{p}+1)^2}\right\} = \left(\sqrt{p} + 1\right) + \frac{\varepsilon}{2}\frac{1}{\sqrt{p}+1}$

$\beta_0 = \left(\sqrt{p}-1\right)+\frac{\varepsilon}{2}\frac{1}{\sqrt{p}-1}$, $m_1 = p\left(1+\frac{\varepsilon}{2}\frac{1}{p-1}\right)$, $m_2 = \sqrt{p}\left(1-\frac{\varepsilon}{2}\frac{1}{p-1}\right)$ **(19)**

$$\frac{1}{m_1^2-m_2^2} = \frac{1}{p(p-1)}\left\{1-\varepsilon\frac{p+1}{(p-1)^2}\right\}$$

Also, $m_1^2-p^2 = \frac{\varepsilon p^2}{p-1}$, $m_2^2-p^2 = \left(1-\frac{\varepsilon}{p-1}-p\right)$

Using these approxemations

$$\frac{\overline{T_1}(x_1,p)}{T_0} = e^{-x_1\sqrt{p}}+\varepsilon\left[\frac{e^{-px_1}}{p-1}+\frac{e^{-px_1}}{(p-1)^2}+\frac{x_1}{2}\frac{e^{-\sqrt{p}x_1}}{\sqrt{p}+1}-\frac{2-x_1}{2}\frac{e^{-x_1\sqrt{p}}}{2}-\frac{e^{-x_1\sqrt{p}}}{(p-1)^2}\right] \quad \textbf{(20)}$$

$$\frac{\overline{U_1}(x_1,p)}{T_0} = \frac{e^{-x_1\sqrt{p}}}{\sqrt{p}(p-1)}-\frac{e^{-x_1p}}{p-1}+\varepsilon\left[\frac{p+1}{p-1}e^{-p_1x_1}+\frac{x_1}{2}\frac{p}{\sqrt{p}(p-1)^2}e^{-px_1}+\frac{x_1}{2}\frac{e^{-x_1\sqrt{p}}}{(p-1)^2}-\frac{1}{2}\frac{e^{-x_1\sqrt{p}}}{(p-1)^2}-\frac{1}{2}\frac{e^{-x_1\sqrt{p}}}{\sqrt{p}(p-1)^2}-\frac{p+1}{\sqrt{p}(p-1)^3}e^{-x_1\sqrt{p}}\right]$$

(21)

IV. Temperature Field

Using the $table^{(14)}$, the temperature field is given by

$$\begin{aligned}\frac{T(x_1,t_1)}{T_0} = {} & \frac{x_1}{2\sqrt{\pi t_1^3}}e^{-\frac{x_1^2}{4t_1}}+\varepsilon[e^{t_1-x_1}H(t_1-x_1)]+(t_1-x_1)e^{(t_1-x_1)}H(t_1-x_1)\\ & +\frac{x_1}{2}\left\{\frac{1}{\sqrt{\pi t_1}}e^{-\frac{x_1^2}{4t_1}}-e^{x_1+t_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)-\left(1-\frac{x_1}{2}\right)\frac{e^{t_1}}{2}\right\}\\ & \times\left\{e^{-x_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}-\sqrt{t_1}\right)+e^{x_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}-\\ & \frac{t_1e^{t_1}}{2}\left\{e^{-x_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}-\sqrt{t_1}\right)+e^{x_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}\\ & +\frac{x_1}{4}e^{t_1}\left\{e^{-x_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}-\sqrt{t_1}\right)-e^{x_1}erfc\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}\end{aligned}$$

$$= \frac{x_1}{2\sqrt{\pi t_1^3}}e^{-\frac{x_1^2}{4t_1}}+\varepsilon F(x_1,\ t_1) \quad \textbf{(22)}$$

Where

$$F((x_1,t_1) = e^{t_1-x_1}H(t_1-x_1)+(t_1-x_1)e^{t_1-x_1}H(t_1-x_1)+\frac{x_1}{2}\left\{\frac{1}{\sqrt{\pi t_1}}e^{-x_1^2/4t_1}-------\right\}$$

In thetemperature distribution(22), the first term on the right-hand side represents the solution of the classical heat conduction equation while the $F(x_1,t_1)$ is the perturbation due to the thermoelastic coupling coefficient ε . $F(x_1,t_1)$ is the perturbation function for temperature. It is seen that perturbation function is zero when t_1 is zero for all values of$x_1 > 0$.

V. Deformation Field

Thermoelastic deformation for small values of time is calculated when parameter p is large and the expansions are inverse powers of p. Hence using approximations (19) and expanding for large p, keeping terms upto$p^{-\frac{7}{2}}$, we obtain from (21)

$$\begin{aligned}\frac{\overline{U}(x_1,p)}{T_0} = {} & \left(\frac{\varepsilon x_1}{2}+2\varepsilon-1\right)\frac{e^{-x_1p}}{p}+(x_1\varepsilon+2\varepsilon-1)\frac{e^{-x_1p}}{p^2}+\left(\frac{3\varepsilon x_1}{2}+\varepsilon-1\right)\frac{e^{-x_1p}}{p^3}+\left(x_1\frac{\varepsilon}{2}-\frac{\varepsilon}{2}\right)\frac{e^{-x_1\sqrt{p}}}{p^2}\\ & +(x_1\varepsilon-\varepsilon)\frac{e^{-x_1\sqrt{p}}}{p^3}+\frac{e^{-x_1\sqrt{p}}}{p^{\frac{3}{2}}}+\left(1-\frac{3\varepsilon}{2}\right)\frac{e^{-x_1\sqrt{p}}}{p^{\frac{5}{2}}}+(1-5\varepsilon)\frac{e^{-x_1\sqrt{p}}}{p^{\frac{7}{2}}}\end{aligned}$$

Taking the inversion, we get, for small values of time, the deformation field

$$\frac{U_1(x_1,t_1)}{T_0} = \left(\frac{\varepsilon x_1}{2} + 2\varepsilon - 1\right) H(t_1 - x_1) + (x_1\varepsilon + 2\varepsilon - 1) H(t_1 - x_1)(t_1 - x_1)$$
$$+ \left(\frac{3\varepsilon x_1}{2} + \varepsilon - 1\right) H(t_1 - x_1) \frac{(t_1 - x_1)^2}{2} + \left(\frac{x_1\varepsilon}{2} - \frac{\varepsilon}{2}\right) 4t_1 i^2 erfc\left(\frac{x_1}{2\sqrt{t_1}}\right)$$
$$+ (x_1\varepsilon - \varepsilon)(4t_1)^2 i^4 erfc\left(\frac{x_1}{2\sqrt{t_1}}\right) + 2\sqrt{t_1 ierfc\left(\frac{x_1}{2\sqrt{t_1}}\right)} + \left(1 - 3\frac{\varepsilon}{2}\right)(4t_1)^{\frac{3}{2}} i^3 erfc\left(\frac{x_1}{2\sqrt{t_1}}\right)$$
$$+ (1 - 5\varepsilon)(4t_1)^{\frac{5}{2}} i^5 erfc\left(\frac{x_1}{2\sqrt{t_1}}\right)$$

Where $i^n erfc(x)$ denote the associated complementary error function of the n^{th} degree and H(η) is the Heavy side unit function defined by

H(η) = 1, $\eta > 0$

= 0, $\eta < 0$

Where $x_1 = 0$ the surface displacement is

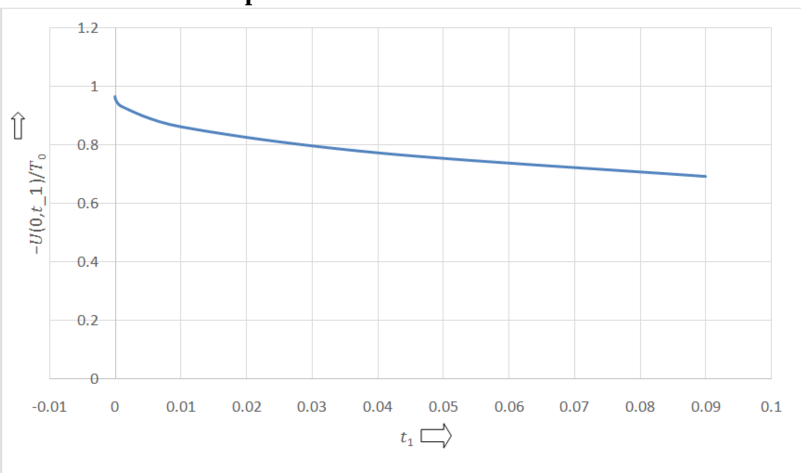
$$\frac{U(0,t)}{T_0} = (2\varepsilon - 1)H(t_1) + (2\varepsilon - 1)H(t_1)t_1 + (\varepsilon - 1)\mathrm{H}(t_1)\frac{t_1^2}{2} - 2\varepsilon t_1 i^2 erfc(0) + 6t_1^2 \varepsilon i^4 erfc(0)$$
$$+ 2\sqrt{t_1}\, i\, erfc(0) + \left(1 - \frac{3\varepsilon}{2}\right) 8t_1^{\frac{3}{2}} i^3\, erfc(0) + (1 - 5\varepsilon)\, 32\, t_1^{\frac{5}{2}} i^5\, erfc(0)$$

When the material of the half space is copper, ε =0.0168. The values of the surface displacement for small values of time are shown in the table:

TABLE

t_1	0	0.000001	0.0001	0.001	0.01	0.04	0.09
$-\frac{U(0,t_1)}{T_0}$	0.9653	0.9653	0.9552	0.9317	0.8626	0.7738	0.6932

Graphical representation of surface displacement for small values of time:



Surface displacement for small values of time

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