IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 5 Ver. 5 (Sep – Oct 2015), PP 59-66
www.iosrjournals.org

On Relationships for Moments of k-th Record Values from Lomax Distribution

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Abstract
In this paper, we establish some recurrence relations satisfied by single and product moments of k-th record values from Lomax distribution.

Keywords & Phrases*:Order Statistics, Single moments, product moments, record times, k-th record values, recurrence relations, Lomax distribution.*

I. Introduction

Let $\{X_n, n\geq 1\}$ be a sequence of i.i.d. random variables with c.d.f. F(x)= P[X≤x] and p.d.f. f(x). For a fixed k≥1 we define the sequence $\{U_n^{(k)}, n\geq 1\}$ of k- th upper record times of $\{X_n, n\geq 1\}$ as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{u_n}^{(k)} : U_n^{(k)} + k - 1\}$$

For K=1 and n=1,2,…,we write $U_n^{(1)} = U_n$. Then $\{U_n, n\geq 1\}$ is the sequence of record times of $\{X_n, n\geq 1\}$. The sequence $\left\{y_n^{(k)}, n \geq 1\right\}$,where $y_n^{(1)} = X_{u_n}^{(k)}$ is called the sequence of k-th upper record values of $\{X_n, n\geq 1\}$.Chandler (1952) introduced record values and record value times. Feller(1966) gave some examples of record values with respect to gambling problems.Properties of record values of i.i.d. r.v.'s have been extensively studies in the literature for example see Nagaraja(1988), Nevzorov(1987),Arnold,Balakrishnan and Nagaraja (1992), Balakrishnan and Ahsanullah (1994) and Ahsanullah(1995).For convenience ,we shall also take $y_0^{(k)} = 0$. Note that for k=1 we have $y_n^{(1)} = X_{u_n}^{\backslash}$,n≥1,which are the record values of $\{X_n, n\geq 1\}$.Moreover ,we see that $y_1^{(k)} = \min(X_1, X_2, \ldots X_k) = X_{1:k}$.

Then the pdf of $y_n^{(k)}$ and $(y_m^{(k)}, y_n^{(k)})$ are as follows:

$$f_{y_n}^{(k)}(x) = \frac{k^n}{(n-1)!}[-\ln(1-F(x)]^{n-1}(1-F(x))^{k-1} f(x)dx; n \geq 1, \tag{1.1}$$

$$f_{y_m}^{(k)}, y_n^{(k)}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[\ln(1-F(x)) - \ln(1-F(y))]^{n-m-1}[-\ln(1-F(x))]^{m-1}.\frac{f(x)}{1-F(x)}[1-F(y)]^{k-1}.f(y)$$

$$x < y; 1 \leq m < n, n \geq 2. \tag{1.2}$$

We present recurrence relations for moment of $y_n^{(k)}$ and product moments of $(y_m^{(k)}, y_n^{(k)})$ when X has the Lomax distribution with p.d.f.

$$f(x) = \gamma(1+x)^{-\gamma-1} \ , x \geq 0, \gamma > 0 . \tag{1.3}$$

And the c.d.f.

$$F(x) = 1 - (1+x)^{-\gamma} \ , x \geq 0.$$

It is easy to see that in this case that

$$(1+x)f(x) = \gamma\{1 - F(x)\} \ , \ x \geq 0, y > 0. \tag{1.4}$$

The relation in (1.4) will be used to derive some simple recurrence relations for the single and product moments of k-th record values from Lomax distribution. Using these relations we give characterization of the Lomax distributions.In this paper we will establish some recurrence relations for single and product moments of k-th record values from Lomax distribution. Lomax (1954) used this distribution in analysis of business failure data. Similar recurrence relations from Lomax and other related distribution are also available in the literature (See David (1981),Similar results for linear exponential were derived by Saran and Singh(2008).

Notations

1. $\mu_{(n):k}^{r} = E\left(y_n^{(k)}\right)^r ; r, n = 1,2, ...,$

2. $\mu_{(m,n):k}^{r,s} = E\left(\left(y_m^{(k)}\right)^r \left(y_n^{(k)}\right)^s\right) : 1 \leq m \leq n-1 \ andr, s, m, n = 1,2, ...,$

II. Relations for Single and Product Moments

Theorem 2.1. Fix a positive integer k≥1. For n≥1 and r=0,1,2,…,

$$E(y_n^{(k)})^{r+1} = \frac{r+1}{k\nu - r - 1}E(y_n^{(k)})^r + \frac{\nu}{k\nu - r - 1}E(\gamma_{n-1}^{(k)})^{r+1} \tag{2.1}$$

Proof : For $n \geq 1$ and r=0,1,2,…, we have from (1.1) and (1.4)

$$E\left(y_n^{(k)}\right)^r + E\left(y_n^{(k)}\right)^{r+1} = \frac{k^n}{(n-1)!}\int_0^\infty x^r(1+x).f_{y_n}^{(k)}(x)dx$$

$$=\frac{k^n}{(n-1)!}\int_0^\infty x^r(1+x)(-\ln(1-F(x))^{n-1}(1-F(x))^{k-1}f(x)dx$$

$$=\frac{k^n}{(n-1)!}\nu\int_0^\infty x^r(-\ln(1-F(x))^{n-1}(1-F(x))^k dx$$

Integrating by parts,taking x^ras the part to be integrated rest part under differentiation we get(2.1).

$$=\frac{k^n\nu}{(n-1)!}[\left|\frac{x^{r+1}}{r+1}\{-\ln(1-F(x)\}\}^{n-1}\left[\left(1-F(x)\right)\right]^k\right|_0^\infty - \int\frac{x^{r+1}}{r+1}[(n-1)\{-\ln(1-F(x)\}^{n-2}\frac{f(x)}{1-F(x)}\left(1-F(x)\right)^k + K\left(1-F(x)\right)^{k-1}(-f(x)\{-ln(1-F(x)\}^{n-1}]dx$$

$$=\frac{k^{n+1}\nu}{(n-1)!}\int\frac{x^{r+1}}{r+1}\{-\ln(1-F(x)\}^{n-1}\,[1-F(x)]^{k-1}.f(x)dx - \frac{k^n\nu}{(n-2)!}\int\frac{x^{r+1}}{r+1}\{-\ln(1-F(x)\}^{n-2}\left(1-F(x)\right)^{k-1}f(x)dx$$

$$=\frac{k\nu}{r+1}E\left(y_n^{(k)}\right)^{r+1}-\frac{\nu}{r+1}E\left(y_{n-1}^{(k)}\right)^{r+1}$$

$$E\left(y_n^{(k)}\right)^{r+1}+\left(y_n^{(k)}\right)^{r+1}=\frac{k\nu}{r+1}E\left(y_n^{(k)}\right)^{r+1}-\frac{\nu}{r+1}E\left(y_{n-1}^{(k)}\right)^{r+1}$$

Or $(r+1)E\left(y_n^{(k)}\right)^{r}+(r+1)E\left(y_n^{(k)}\right)^{r+1}=k\nu E\left(y_n^{(k)}\right)^{r+1}-\nu E\left(y_{n-1}^{(k)}\right)^{r+1}$

Or $(k\nu-r-1)E\left(y_n^{(k)}\right)^{r+1}=(r+1)E\left(y_n^{(k)}\right)^{r+1}+\nu E\left(y_{n-1}^{(k)}\right)^{r+1}$

$$\therefore E\left(y_n^{(k)}\right)^{r+1}=\frac{r+1}{k\nu-r-1}E\left(y_n^{(k)}\right)^{r}+\frac{k\nu}{k\nu-r-1}E\left(y_{n-1}^{(k)}\right)^{r+1}$$

Corollary 1: By repeatedly applying recurrence relation in (2.2) we get $n\geq 2, 1\leq m\leq n-1$ and r=0,1,2,…, and $\nu > r+1$

$$E(y_n^{r+1})=\left(\frac{r+1}{k\nu-r-1}\right)\sum_{p=0}^{n-m-1}\left(\frac{k\nu}{k\nu-r-1}\right)^p.E\left(y_{n-p}^{(k)}\right)^r + \left(\frac{k\nu}{k\nu-r-1}\right)^{n-m}.E\left(y_m^{(k)}\right)^{r+1} \quad (2.3)$$

Corollary 2: Write $(r+1)^{(p)} = 1$ for p=0 &$(r+1)^{(p)} = (r+1)-(r-p), forp \geq 1$
By repeated application of the recurrence relation in (2.2), we obtain for $n \geq 2$, r=0,1,2,… and $\nu > r+1$

$$E\left(y_n^{(k)}\right)^{r+1} = \nu \sum_{p=0}^{r+1} \frac{(r+1)^{(p)}}{(k\nu - r - 1 + p)^{(p+1)}} E\left(y_{n-1}^{(k)}\right)^{r+1-p} \quad (2.4)$$

Putting p=1 in (2.1) the results of Balakrishnan and Ahsanullah (1994) easily be established

Theorem 2.2: For $m \geq 1$ and r,s=0,1,2,….

$$E\left[\left(y_m^{(k)}\right)^r \left(y_{m+1}^{(k)}\right)^{s+1}\right] = \frac{k\nu}{k\nu - s - 1} E\left(y_m^{(k)}\right)^{r+s+1} + \frac{s+1}{k\nu - s - 1} E\left[\left(y_m^{(k)}\right)^r \left(y_{m+1}^{(k)}\right)^s\right] \quad (2.5)$$

For $1 \leq m \leq \mathrm{n} - 2$ and r,s= 0,1,2,….,

$$E\left[\left(y_m^{(k)}\right)^r \left(y_n^{(k)}\right)^{s+1}\right] = \frac{k\nu}{k\nu - s - 1} E\left[\left(y_m^{(k)}\right)^r \left(y_n^{(k)}\right)^{s+1}\right] + \frac{s+1}{k\nu - s - 1} E\left[\left(y_m^{(k)}\right)^r \left(y_n^{(k)}\right)^s\right] \quad (2.6)$$

Proof: From (1.2) for $1 \leq m \leq \mathrm{n} - 1$ and r,s=0,1,2,… and using (1.4) we get

$$E\left[\left(y_m^{(k)}\right)^r \left(y_n^{(k)}\right)^s\right] + E\left[\left(y_m^{(k)}\right)^r \left(y_n^{(k)}\right)^{s+1}\right] = \iint_{x<y}^{\infty} (x^r y^s + x^{r+1} y^s) f_{m,n}(x,y) dy dx$$

$$= \frac{k^n}{(m-1)!(n-m-1)!} \int_0^\infty x^r \{-ln(1-F(x))\}^{m-1} \frac{f(x)}{1-F(x)} I(x) dx \quad (2.7)$$

Where $I(x) = \int_x^\infty y^s (1+y) \{-\ln(1-F(y)) + \ln(1-F(x))\}^{n-m-1} f(y) dy$

$$= \nu \int_x^\infty y^s \{-\ln(1-F(y)) + \ln(1-F(x))\}^{n-m-1} \{1-F(y)\}^k dy$$

(Upon using(2.1))

$$= \frac{k\nu}{s+1} \left[\int_x^\infty y^{s+1} f(y) dy - x^{s+1} \{1-F(x)\}\right] \quad , forn = m+1$$

$$= \frac{k\nu}{s+1} [\int_x^\infty y^{s+1} \{-\ln(1-F(y)) + \ln(-1-F(x))\}^{n-m-1} [1-F(y)]^{k-1} f(y) dy$$

$$-(n-m-1)\int_{x}^{\infty}\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-2}\frac{f(y)}{1-F(y)}\left(1-F(y)\right)^{k-1}dy],$$

for$n \geq m+2$

The last two equations are derived by integrating by parts upon substituting the above expression of I(X) in equation (2.7) and simplifying the resulting equations,we obtain

$$E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^s\right]+E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^{s+1}\right]$$
$$=\frac{k^n}{(m-1)!\,(n-m-1)!}\int x^r\{-\ln(1$$
$$-F(x)\}^{m-1}\frac{f(x)}{1-F(x)}\frac{kv}{s+1}[\int_{x}^{\infty} y^{s+1}\{-\ln(1-F(y))$$
$$+\ln(1-F(x))\}^{n-m-1}[1-F(y)]^{k-1}f(y)dy$$
$$-(n-m-1)\int_{x}^{\infty}\{-\ln(1-F(y))+\ln(1-F(x))\}^{n-m-2}$$
$$\frac{f(y)}{1-F(y)}\left(1-F(y)\right)^{k-2}f(y)dy]dx, \qquad for\ n\geq m+2$$

$$=\frac{kv}{s+1}\left\{E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^{s+1}\right]-E\left[\left(y_m^{(k)}\right)^r\left(y_{n-1}^{(k)}\right)^{s+1}\right]\right\}$$

We obtain,when n=m+1 that

$$E\left[\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^s\right]+E\left[\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^{s+1}\right]=\frac{kv}{s+1}\left\{\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^{s+1}-E\left(y_m^{(k)}\right)^{r+s+1}\right\}$$

$$E\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^{s+1}=\frac{kv}{kv-s-1}E\left(y_m^{(k)}\right)^{r+s+1}+\frac{s+1}{kv-s-1}E\left[\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^s\right]$$

Putting k=1 in (2.5) and (2.6) we get the recurrence relations simply by rewriting the above equations

Theorem 2.3: For $m \geq 2$ and r,s=0,1,2,…,

$$E\left[\left(y_m^{(k)}\right)^{r+1}\left(y_{m+1}^{(k)}\right)^s\right]$$
$$=\frac{kv}{r+1}\left\{E\left(y_m^{(k)}\right)^{r+s+1}-E\left[\left(y_{m-1}^{(k)}\right)^{r+1}\left(y_m^{(k)}\right)^s\right]\right\}$$
$$-E\left[\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^s\right] \quad (2.8)$$

And For $1 \leq m \leq$ n − 2 and r, s= 0,1,2,…,

$$E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^s\right]$$
$$=\frac{k\nu}{r+1}\left\{E\left(y_m^{(k)}\right)^{r+1}\left(y_{n-1}^{(k)}\right)^s-E\left(y_{m-1}^{(k)}\right)^{r+1}\left(y_{n-1}^{(k)}\right)^s\right\}-E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^s\right]$$

Proof From (1.2), let us consider for $2\le m\le \text{n}-1$ and r,s=0,1,2,…,

$$E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^s+\left(y_m^{(k)}\right)^{r+1}\left(y_m^{(k)}\right)^s\right]=\iint_{x<y}^{\infty}(x^ry^s+x^{r+1}y^s)f_{m,n}(x,y)dxdy$$
$$=\frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty y^sf(y)J(y)dy \qquad (2.10)$$

Where

$$J(y)=\int_0^y x^r(1+x)\{-\ln(1-f(x)\}^{m-1}\left\{\ln\left(1-F(x)\right)-\ln(1-F(y)\right\}^{n-m-1}\frac{f(x)}{1-F(x)}dx$$

$$=k\nu\int_0^y x^r\{-\ln(1-F(x)\}^{m-1}.\left\{\ln\left(1-F(x)\right)-\ln(1-F(y)\right\}^{n-m-1}dx \quad (Using(1.4))$$

$$=\frac{k\nu}{r+1}\left[y^{r+1}\{-\ln(1-F(y)\}^{m-1}-(m-1)\int_0^y x^{r+1}\{-\ln(1-F(x)\}^{m-2}.\frac{f(x)}{1-F(x)}dx\right]$$

$=\frac{k\nu}{r+1}\Big[(n-m-1)\int_0^y x^{r+1}\{-\ln(1-F(x)\}^{m-1}\frac{f(x)}{1-F(x)}.\{-\ln(1-F(y)+\ln(1-$
*F(x)n−m−2dx−m−1xr+1−*ln*(1−F(x)m−2fx1−Fx*ln*(1−Fx−*ln*(1−F(y)n−m−1dx*
,for *n≥m+2*

As before the last two equations are obtained by integration by parts upon substituting the above expressions of J(y) in equation (2.10) and simplifying the resulting equation

$$=\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty y^sf(y).\frac{k\nu}{r+1}[n-m-1]$$
$$\int_0^y x^{r+1}\{-\ln(1-F(x)\}^{m-1}\frac{f(x)}{1-F(x)}$$
$$\{-\ln(1-F(y)+\ln(1-F(x)\}^{n-m-2}dx-(m-1)\int_0^y x^{r+1}\{-\ln(1-F(x)\}^{m-2}$$
$$\frac{f(x)}{1-F(x)}\{\ln(1-F(x)-\ln(1-F(y)\}^{n-m-1}dx]$$

We obtain when n=m+1 that

$$E\left[\left(y_m^{(k)}\right)^r\left(y_{m+1}^{(k)}\right)^s+\left(y_m^{(k)}\right)^{r+1}\left(y_m^{(k)}\right)^s\right]=\frac{k\nu}{r+1}\left\{E\left(y_m^{(k)}\right)^{r+s+1}-E\left[\left(y_{m-1}^{(k)}\right)^{r+1}\left(y_m^{(k)}\right)^s\right]\right\}$$

And when $n \geq m+2$ that

$$E\left[\left(y_m^{(k)}\right)^r\left(y_n^{(k)}\right)^s+\left(y_m^{(k)}\right)^{r+1}\left(y_n^{(k)}\right)^s\right]$$
$$=\frac{k\nu}{r+1}\left[E\left(y_m^{(k)}\right)^{r+1}\left(y_{n-1}^{(k)}\right)^s-E\left(y_{m-1}^{(k)}\right)^{r+1}\left(y_{n-1}^{(k)}\right)^s\right]$$

The recurrence relations in (2.8) and (2.9) are derived simply by rewriting the above equations

Corollary: By repeated application of the recurrence relations in (2.8) and (2.9) for $m \geq 1$& r,s=0,1,2,...,

$$E\left[\left(y_m^{(k)}\right)^{r+1}\left(y_{m+1}^{(k)}\right)^s\right]$$
$$=\sum_{p=0}^{m-1}(-1)^p\left(\frac{k\nu}{r+1}\right)^p\left\{\frac{k\nu}{r+1}E\left(y_{m-p}^{(k)}\right)^{r+s+1}-E\left(y_{m-p}^{(k)}\right)^r\left(y_{m-p+1}^{(k)}\right)^s\right\}$$

And for $1 \leq m \leq n-2$ and r,s=0,1,2,…,

$$E\left[\left(y_m^{(k)}\right)^{r+1}\left(y_n^{(k)}\right)^s\right]$$
$$=\sum_{p=0}^{m-1}(-1)^p\left(\frac{k\nu}{r+1}\right)^p\left\{\frac{k\nu}{r+1}E\left(y_{m-p}^{(k)}\right)^{r+1}\left(y_{n-p-1}^{(k)}\right)^s-E\left(y_{m-p}^{(k)}\right)^r\left(y_{n-p}^{(k)}\right)^s\right\}$$

III. Applications

For the Lomax distribution, exact distributional results for appropriate pivots useful for relevant inference are not available (Ahsanullah(1991)). The results established in this paper and some similar generalization can be used to compute all moments of order up to four and hence can be utilized to determine mean,varience, skewness and kurtosis etc.

IV. Conclusion

In the study presented above, we demonstrate the recurrence relation for single and product moments of k-th record values from Lomax distribution. These results generalized the corresponding results of Saran and Singh(2008).

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